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GLOBAL STABILITY AND BIFURCATION ANALYSIS OF A DISCRETE TIME SIR EPIDEMIC MODEL

ÖZLEM AK GÜMÜŞ, QIANQIAN CUI, GEORGE MARIA SELVAM, AND ABRAHAM VIANNY

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Abstract. In this paper, we study the complex dynamical behaviors of a discrete-time SIR epidemic model. Analysis of the model demonstrates that the Diseases Free Equilibrium (DFE) point is globally asymptotically stable if the basic reproduction number is less than one while the Endemic Equilibrium (EE) point is globally asymptotically stable if the basic reproduction number is greater than one. The results are further substantiated visually with numerical simulations. Furthermore, numerical results demonstrate that the discrete model has more complex dynamical behaviors including multiple periodic orbits, quasi-periodic orbits and chaotic behaviors. The maximum Lyapunov exponent and chaotic attractors also confirm the chaotic dynamical behaviors of the model.

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1. INTRODUCTION

For centuries, infectious diseases have ranked with wars and famine as major challenges to human progress and survival [8]. The Black Death of fourteen century and the 1918-1920, the SARS of 2003 and the global outbreak of covid-19 recently, which have a huge impact on the world economy and people's health without doubt. To prevent and control the spread of infectious disease, we must understand its pathogenesis. Epidemic dynamical model is one of the most useful tools to understand the pathogenesis of diseases. Kermack and McKendrick pioneered compartment model to predict the spread of disease. Hence then, various epidemic models are studied to control infectious disease [3],[4],[5],[6],[7],[11],[12],[15].

Generally, epidemic dynamical models have two kinds: one is continuous-time models described by differential equations, another is discrete-time models described by difference equations. When the size of population is rarely small or the population has no overlapping generation, the discrete-time models are more appropriate than the continuous ones. In reality, most fish and insect populations have no overlap between successive generations, and thus their population evolves in discrete-time steps [1],

[13],[17]. In theory, discrete-time system has much more richer dynamical behaviors than the continuous system.

Recently, many authors studied the dynamical behaviors of discrete epidemic model. Hu et al. [19] discussed the dynamical behaviors of a discrete SIR epidemic model, which derived by applying Euler scheme to corresponding continuous SIR epidemic model. Cao et al. [9] investigated a discrete SIR epidemic model with bilinear incidence rate and constant recovery. These all shows that the stability of discrete model determined by the basic reproduction number and, discrete model undergoes bifurcation and exhibit complex dynamical behaviors, such as the period-doubling bifurcation in period-2,4,8, quasi-periodic orbits and chaotic sets.

In this paper, we mainly study the stability and bifurcation of the following discrete SIR epidemic model by including vaccination to the model given by [16]

$$\begin{aligned} S_{t+1} =& (1-p)S_t - \frac{\alpha}{N} I_t S_t + \beta(R_t + I_t), \\ I_{t+1} =& \frac{\alpha}{N} I_t S_t + (1-\beta-\gamma) I_t, \\ R_{t+1} =& (1-\beta)R_t + \gamma I_t + p S_t, \end{aligned} \tag{1.1}$$

with initial conditions S_0, I_0 and R_0, $(S_0 + I_0 + R_0 = N)$, which are positive real numbers. Here $0 < p + \alpha < 1$ and $0 < \beta + \gamma < 1$. Also, β is the probability of death which is equal to the probability of birth, γ is the probability of recovery, p is the proportion vaccinated, α is the contact rate, and N is the total population size.

The organization of this paper is as follows. We first give the existence and stability of equilibria in section 2. Then, the numerical simulations are stated in section 3. In section 4, we simulate the complex dynamical behaviors including multiple periodic orbits, quasi-periodic orbits and chaotic behaviors. Finally, we give a discussion in the last section.

2. Existence and stability of equilibria

Since $S_n + I_n + R_n = N$ is a constant, then the dynamical behaviors of model (1.1) is equivalent to the dynamical behaviors of the following model

$$\begin{aligned} S_{t+1} =& (1-p)S_t - \frac{\alpha}{N} I_t S_t + \beta(N - S_t), \\ I_{t+1} =& \frac{\alpha}{N} I_t S + (1-\beta-\gamma) I_t, \end{aligned} \tag{2.1}$$

where p, α, β, and γ are positive parameters.

On examining system (2.1), the equilibrium points $P_0 = \left(\frac{\beta N}{\beta+p}, 0\right)$ and $P_1 = \left(\frac{(\beta+\gamma)N}{\alpha}, \frac{N(\alpha\beta-(\beta+\gamma)(p+\beta))}{\alpha(\beta+\gamma)}\right)$ has been obtained in [14] by using $S_t = S_{t+1} = S^*$ and $I_t = I_{t+1} = I^*$.

The condition for the equilibrium to be locally asymptotically stable is presented in the following lemma.

Lemma 1. *[2] Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$ be a system of difference equations and $\overline{X}$ is a steady state point of F. If all eigenvalues of the Jacobian matrix J_F about the steady state $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has absolute value greater than one, then $\overline{X}$ is unstable.*

Definition 1. [10] Suppose x is an equilibrium of the difference equation,

$$x_{t+1} = f(x_t)$$

where $f: [0,a) \to [0,a)$, $0 < a \leq \infty$. Then $\bar{x}$ is said to be globally attractive if for all initial conditions $x_0 \in (0,a), \lim_{t\to\infty} = \bar{x}$ and is said to be globally asymptotically stable if $\bar{x}$ is globally attractive and if $\bar{x}$ is locally stable.

The theorem signifies the result on the roots of the polynomial based on trace and determinant of the jacobian matrix.

Theorem 1. *[18] The characteristic polynomial*

$$F(x) = x^2 + Bx + C,$$

has all its roots inside the unit open disk ($|x| < 1$) if and only if

(i) $F(1) > 0$ *and* $F(-1) > 0$*;*
(ii) $D_1^+ = 1 + C > 0$ *and* $D_1^- = 1 - C > 0$

where $-B$ is the trace of Jacobian matrix and C is the determinant of Jacobian matrix.

Now, we investigate the global stability of these equilibrium points. For analyzing the global stability of disease-free equilibrium $P_0 = \left(\frac{\beta N}{\beta+p}, 0\right)$, we give the following theorems.

Theorem 2. *Assume that $0 < \beta + p < 1$. The disease-free equilibrium (DFE) point $P_0 = \left(\frac{\beta N}{\beta+p}, 0\right)$ of system (2.1) is globally asymptotically stable if,*

$$R_0 = \frac{\alpha\beta}{(\beta+p)(\beta+\gamma)} < 1. \tag{2.2}$$

P_0 is unstable if $R_0 > 1$.

Proof. By considering (2.1), we can get the Jacobian matrix as

$$J_{P_0} = \begin{pmatrix} 1-p-\beta & \frac{-\alpha\beta}{(\beta+p)} \\ 0 & \frac{\alpha\beta}{(\beta+p)} + (1-\beta-\gamma) \end{pmatrix},$$

evaluated at P_0, the eigenvalues are

$$\lambda_1 = 1-p-\beta, \quad \lambda_2 = \frac{\alpha\beta}{(\beta+p)} + (1-\beta-\gamma).$$

The conditions $0<\beta+p<1$, $0<\beta+\gamma<1$ and $R_0<1$ implies that $|\lambda_1|<1$ and $|\lambda_2|=|1-(\beta+\gamma)(1-R_0)|<1$. It follows from Lemma 1 that the DFE point P_0 is locally asymptotically stable if $R_0<1$. When $R_0>1$, we have $\lambda_2=1-(\beta+\gamma)(1-R_0)>1$, then the DFE point P_0 is unstable. From the first equation of system (2.1), it is easy to see that S_t satisfies the following inequality:

$$S_t \le \beta N+(1-p-\beta)S_{t-1}.$$

The equation $u_t=\beta N+(1-p-\beta)u_{t-1}$ has a unique equilibrium point $u^*=\frac{\beta N}{p+\beta}$, which is globally asymptotically stable. From inequality (2.2) and the comparison principle, we know that for any small ε_1, there exists a positive integer T_1 such that $S_t\le\frac{\beta N}{p+\beta}+\varepsilon_1$ for all $t>T_1$. From the second equation of system (2.1) and $S_t\le\frac{\beta N}{p+\beta}+\varepsilon_1$, $t>T_1$, we have

$$I_t \le \left(1-(\beta+\gamma)(1-R_0)+\varepsilon_1\frac{\alpha}{N}\right)I_{t-1}, \text{ for } t>T_1.$$

The conditions $0<\beta+\gamma<1$, $R_0<1$ and the arbitrarily of ε_1 imply $0<1-(\beta+\gamma)(1-R_0)+\varepsilon_1\frac{\alpha}{N}<1$. Then we can rewrite (2.2) as

$$I_t \le \left(1-(\beta+\gamma)(1-R_0)+\varepsilon_1\frac{\alpha}{N}\right)^t I_0, \text{ for } t>T_1,$$

from which we can derive that

$$\lim_{t\to\infty} I_t=0.$$

Therefore, for any small $\varepsilon_2>0$, there exists a larger positive integer $T_2\ge T_1$ such that $I_t<\varepsilon_2$ for $t>T_2$. Then from the first equation of system (2.1), we have

$$S_t \ge \beta N+\left(1-p-\beta-\varepsilon_2\frac{\alpha}{N}\right)S_{t-1}, \text{ for } t>T_2.$$

Then from the comparison principle, we know that for any small $\varepsilon_3>0$, there exists an integer $T_3>T_2$ such that $S_t\ge\frac{\beta N}{\beta+p+\varepsilon_2\alpha/N}-\varepsilon_3$ for all $t>T_3$. Let $T_4=T_1+T_3$, then the inequalities

$$\frac{\beta N}{\beta+p+\varepsilon_2\alpha/N}-\varepsilon_3 \le S_t \le \frac{\beta N}{\beta+p}+\varepsilon_1, \text{ for } t>T_4,$$

and the arbitrary of $\varepsilon_1,\varepsilon_2$ and ε_3 imply that

$$\lim_{t\to\infty} S_t=\frac{\beta N}{\beta+p},$$

Thus the limits

$$\lim_{t\to\infty} S_t=\frac{\beta N}{\beta+p} \text{ and } \lim_{t\to\infty} I_t=0.$$

imply that the DFE point P_0 is a global attractor if $R_0<1$. Consequently; since P_0 is locally asymptotically stable if $R_0<1$ and P_0 is global attractor, we have the DFE point P_0 of system (2.1) is globally asymptotically stable if $R_0<1$. □

Remark 1. [3, 14] The basic reproductive ratio R_0 is referred as $\frac{\alpha\beta}{(\beta+p)(\beta+\gamma)}$. This ratio is a threshold parameter for the SIR epidemic model. If $R_0 < 1$, then there exists the DFE point which is locally asymptotically stable .

Theorem 3. *The endemic equilibrium (EE) point* $P_1 = \left(\frac{(\beta+\gamma)N}{\alpha}, \frac{N(\alpha\beta-(\beta+\gamma)(p+\beta))}{\alpha(\beta+\gamma)}\right)$ *of the system* (2.1) *is locally asymptotically stable if* $R_0 > 1$.

Proof. By considering (2.1), we can write the Jacobian matrix evaluated at P_1 as

$$J_{P_1} = \begin{pmatrix} 1-p-\beta-\frac{\alpha I^*}{N} & -\beta-\gamma \\ \frac{\alpha I^*}{N} & 1 \end{pmatrix}. \tag{2.3}$$

The characteristic polynomial of the Jacobian matrix J_{P_1} is as follows:

$$G(\lambda) = \lambda^2 - \left(2-p-\beta-\frac{\alpha I^*}{N}\right)\lambda + 1 - p - \beta - (1-\beta-\gamma)\frac{\alpha I^*}{N}. \tag{2.4}$$

It is easy to verify that $G(1) = (\beta+\gamma)\frac{\alpha I^*}{N}$, $G(0) = 1-p-\beta-(1-\beta-\gamma)\frac{\alpha I^*}{N}$ and

$$\begin{aligned} G(-1) &= 2(1-p-\beta) + 2\left(1-\frac{\alpha I^*}{N}\right) + (\beta+\gamma)\frac{\alpha I^*}{N} \\ &> 2(1-p-\beta) + 2(1-\alpha) + (\beta+\gamma)\frac{\alpha}{N}I^*. \end{aligned}$$

Since $R_0 = \frac{\alpha\beta}{(\beta+p)(\beta+\gamma)} > 1$ implies $\alpha > \beta + p$, then using the assumptions $0 < p+\alpha < 1,\ 0 < \beta+\gamma < 1$, we have $G(1) > 0,\ G(-1) > 0$ and $C = G(0) < 1$ when $R_0 > 1$. It follows from Theorem 1 that the two roots, λ_1 and λ_2, of the equation $G(\lambda) = 0$ satisfies $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Therefore, from Lemma 2, we conclude that EE point P_1 is locally asymptotically stable when $R_0 > 1$. □

Let $L_t = S_t + I_t$, then we can rewrite system (2.1) as follows

$$\begin{aligned} L_{t+1} &= \beta N + (1-\beta-p)L_t + (p-\gamma)I_t, \\ I_{t+1} &= \frac{\alpha}{N}I_t(L_t - I_t) + (1-\beta-\gamma)I_t. \end{aligned} \tag{2.5}$$

The global stability of the positive equilibrium point of system (2.5) is equivalent to that of system (2.1).

Theorem 4. *The endemic equilibrium(EE) point of the system (2.5) is globally asymptotically stable if*

(i) $1 < R_0 < \frac{\beta+p+\gamma}{\beta+p} < \frac{2+\beta+\min\{p,\gamma\}}{\beta+p}$ *and* $p \le \gamma$,

(ii) $1 < \frac{\beta+p+\gamma}{\beta+p} < R_0 < \frac{2+\beta+\min\{p,\gamma\}}{\beta+p}$ *and* $p > \gamma$.

Proof. The global stability of the positive equilibrium point of (2.5) is quite difficult, so we consider two cases to prove it.

Case 1. $p \le \gamma$.

From the inequality $0 < I_t < L_t$ and the first equation of system (2.5), we obtain

$$\begin{aligned} L_{t+1} &\leq \beta N + (1-\beta-p)L_t, \\ L_{t+1} &\geq \beta N + (1-\beta-p-\gamma)L_t. \end{aligned} \tag{2.6}$$

From these two inequalities in (2.6) and the comparison theorem [15], we know that for any small $\varepsilon > 0$, there exists a positive integer T_1 such that $L_1^l \leq L_t \leq L_1^r$ for $t > T_1$, where $L_1^l = \frac{\beta N}{\beta+p+\gamma} - \varepsilon$ and $L_1^r = \frac{\beta N}{\beta+p} + \varepsilon$.

When $t > T_1$, we substitute $L_1^l \leq L_t \leq L_1^r$ into the second equation of (2.5), then we obtain

$$\begin{aligned} I_{t+1} &\geq \frac{\alpha}{N} I_t \left(L_1^l - I_t\right) + (1-\beta-\gamma)I_t, \\ I_{t+1} &\leq \frac{\alpha}{N} I_t \left(L_1^r - I_t\right) + (1-\beta-\gamma)I_t. \end{aligned} \tag{2.7}$$

Let us consider the following auxiliary equations corresponding to the inequalities in (2.7)

$$\begin{aligned} I_{t+1}^l &= \frac{\alpha}{N} I_t^l \left(L_1^l - I_t^l\right) + (1-\beta-\gamma)I_t^l, \\ I_{t+1}^r &= \frac{\alpha}{N} I_t^r \left(L_1^r - I_t^r\right) + (1-\beta-\gamma)I_t^r. \end{aligned} \tag{2.8}$$

Define $I_t^l = \frac{(1-\beta-\gamma)N+\alpha L_1^l}{\alpha} U_t^l$ and $I_t^r = \frac{(1-\beta-\gamma)N+\alpha L_1^r}{\alpha} U_t^r$. Then we can rewrite (2.8) as follows

$$\begin{aligned} U_{t+1}^l &= r^l U_t^l \left(1 - U_t^l\right), \text{ with } r^l = 1-\beta-\gamma+\frac{\alpha}{N}L_1^l, \\ U_{t+1}^r &= r^r U_t^r \left(1 - U_t^r\right), \text{ with } r^r = 1-\beta-\gamma+\frac{\alpha}{N}L_1^r. \end{aligned} \tag{2.9}$$

It follows from [10] that the first equation of (2.8) has a positive equilibrium $I_{1*}^l = L_1^l - \frac{(\beta+\gamma)N}{\alpha}$, which is global asymptotically stable if $1 < R_0 < \frac{\beta+p+\gamma}{\beta+p} < \frac{2+\beta+p+\gamma}{\beta+p}$. A similar argument implies that the second equation of (2.8) has a positive equilibrium $I_{1*}^r = L_1^r - \frac{(\beta+\gamma)N}{\alpha}$, which is globally asymptotically stable if $1 < R_0 < \frac{2+\beta+p}{\beta+p}$.

The global asymptotic stability and the comparison theory imply that there exists a $T_2 \geq T_1$ such that $I_1^l < I_t < I_1^r$ for all $t > T_2$, where $I_1^l = I_{1*}^l - \varepsilon$ and $I_1^r = I_{1*}^r + \varepsilon$. When $t > T_2$, by substituting the inequality $I_1^l < I_t < I_1^r$ into the first equation of (2.5), we have

$$\begin{aligned} L_{t+1} &\geq \beta N + (1-\beta-p)L_t + (p-\gamma)I_1^r, \\ L_{t+1} &\leq \beta N + (1-\beta-p)L_t + (p-\gamma)I_1^l. \end{aligned} \tag{2.10}$$

From (2.10) and a similar argument, we can derive that there exists a positive integer $T_3 \geq T_2$ such that $L_2^l \leq L_t \leq L_2^r$ for all $t > T_3$, where $L_2^l = \frac{\beta N + (p-\gamma)I_1^r}{\beta+p} - \varepsilon$ and $L_2^r = \frac{\beta N + (p-\gamma)I_1^l}{\beta+p} + \varepsilon$. Obviously, $L_1^l < L_2^l < L_2^r < L_1^r$.

When $t > T_3$, the inequality $L_2^l \leq L_t \leq L_2^r$ substitute into the second equation of (2.5) implies

$$I_{t+1} \geq \frac{\alpha}{N} I_t \left(L_2^l - I_t\right) + (1-\beta-\gamma) I_t,$$
$$I_{t+1} \leq \frac{\alpha}{N} I_t \left(L_2^r - I_t\right) + (1-\beta-\gamma) I_t. \tag{2.11}$$

A similar argument implies that there exists a $T_4 \geq T_3$ such that $I_2^l \leq I_t \leq I_2^r$ for $t > T_4$, where $I_2^l = L_2^l - \frac{(\beta+\gamma)N}{\alpha} - \varepsilon$ and $I_2^r = L_2^r - \frac{(\beta+\gamma)N}{\alpha} + \varepsilon$. By substituting L_2^r, L_2^l into I_2^l, I_2^r, we derive

$$I_2^l = \frac{p-\gamma}{\beta+p} I_1^r + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) - 2\varepsilon,$$
$$I_2^r = \frac{p-\gamma}{\beta+p} I_1^l + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) + 2\varepsilon. \tag{2.12}$$

When $t > T_4$ and $I_2^l \leq I_t \leq I_2^r$, equations in (2.12) hold. Then from the induction, we know that there exist sequences $T_{2k-1}, T_{2k}, L_k^l, L_k^r, I_k^l, I_k^r$ such that $I_k^l \leq I_t \leq I_k^l$ for $t > T_{2k}$, and I_k^l, I_k^r satisfy that

$$I_{k+1}^l = \frac{p-\gamma}{\beta+p} I_k^r + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) - 2\varepsilon,$$
$$I_{k+1}^r = \frac{p-\gamma}{\beta+p} I_k^l + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) + 2\varepsilon. \tag{2.13}$$

One can easily check that linear system (2.13) has a unique positive equilibrium $P_*(I_*^l(\varepsilon), I_*^r(\varepsilon))$ with

$$I_*^l(\varepsilon) = \frac{\beta+p}{\alpha} N(R_0 - 1) - \frac{2(\beta+p)}{\beta+2p-\gamma}\varepsilon \; I_*^l(\varepsilon) = \frac{\beta+p}{\alpha} N(R_0 - 1) + \frac{2(\beta+p)}{\beta+2p-\gamma}\varepsilon,$$

which is globally asymptotically stable. That is,

$$\lim_{k\to\infty} I_k^l = I_*^l(\varepsilon) \;\text{ and }\; \lim_{k\to\infty} I_k^r = I_*^r(\varepsilon).$$

The arbitrary small of ε means that

$$\lim_{\varepsilon\to 0} I_*^l(\varepsilon) = \lim_{\varepsilon\to 0} I_*^r(\varepsilon) = \frac{\beta+p}{\alpha} N(R_0 - 1).$$

Then from the inequality $I_k^l < I_t < I_k^r$ and those limits, we derive that

$$\lim_{t\to\infty} I_t = \frac{\beta+p}{\alpha} N(R_0 - 1).$$

Similarly, we can prove that the sequences $\{L_k^l\}$ and $\{L_k^r\}$ is a linear system of difference equations and satisfy

$$\lim_{k\to\infty} L_k^l = \lim_{k\to\infty} L_k^r = \frac{\beta N}{\beta+p} + \frac{p-\gamma}{\alpha} N(R_0 - 1), \text{ as } \varepsilon \to 0.$$

The inequality $L_k^l < L_t < L_k^r$ and the limits implies that

$$\lim_{t\to\infty} L_t = \frac{\beta N}{\beta+p} + \frac{p-\gamma}{\alpha} N(R_0-1).$$

Finally, from $L_t = S_t + I_t$, then we have

$$\lim_{t\to\infty} S_t = \lim_{t\to\infty} L_t - \lim_{t\to\infty} I_t = \frac{(\beta+\gamma)N}{\alpha}.$$

Therefore, the endemic equilibrium P_1 of system (2.5) is globally asymptotically stable when $1 < R_0 < \frac{\beta+p+\gamma}{\beta+p} < \frac{2+\beta+p}{\beta+p}$ and $p \leq \gamma$.

Case 2. $p > \gamma$.

For this case, from the inequality $0 < I_t < L_t$ and the first equation of system (2.5), we obtain

$$\begin{aligned} L_{t+1} &\leq \beta N + (1-\beta-\gamma)L_t, \\ L_{t+1} &\geq \beta N + (1-\beta-p-\gamma)L_t. \end{aligned} \tag{2.14}$$

Similarly, using the comparison theory, we have that for any small $\delta > 0$, there exists a positive integer $\tilde{T}_1$ such that $L_1^s \leq L_t \leq L_1^b$ for $t > \tilde{T}_1$, where $L_1^s = \frac{\beta N}{\beta+p+\gamma} - \delta$ and $L_1^b = \frac{\beta N}{\beta+\gamma} + \delta$.

When $t > \tilde{T}_1$, by substituting inequality $L_1^s \leq L_t \leq L_1^b$ into the second equation of (2.5), we have

$$\begin{aligned} I_{t+1} &\geq \frac{\alpha}{N} I_t \left(L_1^s - I_t\right) + (1-\beta-\gamma)I_t, \\ I_{t+1} &\leq \frac{\alpha}{N} I_t \left(L_1^b - I_t\right) + (1-\beta-\gamma)I_t. \end{aligned} \tag{2.15}$$

Define

$$\begin{aligned} I_{t+1} &= \frac{\alpha}{N} I_t \left(L_1^s - I_t\right) + (1-\beta-\gamma)I_t, \\ I_{t+1} &= \frac{\alpha}{N} I_t \left(L_1^b - I_t\right) + (1-\beta-\gamma)I_t. \end{aligned} \tag{2.16}$$

Using transform $I_t^s = \frac{(1-\beta-\gamma)N+\alpha L_1^s}{\alpha} V_t^s$ and $I_t^b = \frac{(1-\beta-\gamma)N+\alpha L_1^b}{\alpha} V_t^b$, then we can rewrite (2.16) as follows

$$\begin{aligned} V_{t+1}^s &= r^s V_t^s \left(1-V_t^s\right), \text{ with } r^s = 1-\beta-\gamma+\frac{\alpha}{N} L_1^s, \\ V_{t+1}^b &= r^b V_t^b \left(1-V_t^b\right), \text{ with } r^b = 1-\beta-\gamma+\frac{\alpha}{N} L_1^b. \end{aligned} \tag{2.17}$$

It follows from [10], we know that the first equation of (2.16) has a positive equilibrium $I_{1*}^s = L_1^s - \frac{(\beta+\gamma)N}{\alpha}$, which is global asymptotically stable if $\frac{\beta+p+\gamma}{\beta+p} < R_0 < \frac{2+\beta+p+\gamma}{\beta+p}$, and that the second equation of (2.16) has a positive equilibrium $I_{1*}^b = L_1^b - \frac{(\beta+\gamma)N}{\alpha}$, which is globally asymptotically stable if $1 < R_0 < \frac{2+\beta+\gamma}{\beta+p}$.

It follows from the asymptotic stability and the comparison theory that there exists a $\tilde{T}_2 \geq \tilde{T}_1$ such that $I_1^s < I_t < I_1^b$ for all $t > \tilde{T}_2$, where $I_1^s = I_{1*}^s - \delta$ and $I_1^b = I_{1*}^b + \delta$. When $t > T_2$, by substituting the inequality $I_1^s < I_t < I_1^b$ into the first equation of (2.5), we have

$$L_{t+1} \geq \beta N + (1-\beta-p)L_t + (p-\gamma)I_1^s,$$
$$L_{t+1} \leq \beta N + (1-\beta-p)L_t + (p-\gamma)I_1^b. \tag{2.18}$$

Similarly, from (2.18), we derive that there exists a positive integer $\tilde{T}_3$ such that $L_2^s \leq L_t \leq L_2^b$ for all $t > \tilde{T}_3$, where $L_2^s = \frac{\beta N+(p-\gamma)I_1^s}{\beta+p} - \delta$ and $L_2^b = \frac{\beta N+(p-\gamma)I_1^b}{\beta+p} + \delta$. Obviously, $L_1^s < L_2^s < L_2^b < L_1^b$.

When $t > \tilde{T}_3$, the inequality $L_2^s \leq L_t \leq L_2^b$ substitute into the second equation of (2.5) implies

$$I_{t+1} \geq \frac{\alpha}{N} I_t\,(L_2^s - I_t) + (1-\beta-\gamma)I_t,$$
$$I_{t+1} \leq \frac{\alpha}{N} I_t\left(L_2^b - I_t\right) + (1-\beta-\gamma)I_t. \tag{2.19}$$

Similar argument implies that there exists a $\tilde{T}_4 \geq \tilde{T}_3$ such that $I_2^s \leq I_t \leq I_2^b$ for $t > \tilde{T}_4$, where

$$I_2^s = L_2^s - \frac{(\beta+\gamma)N}{\alpha} - \delta = \frac{p-\gamma}{\beta+p} I_1^s + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) - 2\delta,$$
$$I_2^b = L_2^b - \frac{(\beta+\gamma)N}{\alpha} + \delta = \frac{p-\gamma}{\beta+p} I_1^b + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) + 2\delta. \tag{2.20}$$

Equations in (2.20) hold if $t > \tilde{T}_4$ and $I_2^s \leq I_t \leq I_2^b$. Then by the induction, there exists sequences $\{I_k^s\}, \{I_k^b\}$ such that $I_k^s < I_t < I_k^b$ for all $t > \tilde{T}_{2k}$, where I_k^s, I_k^b satisfy that

$$I_{k+1}^s = \frac{p-\gamma}{\beta+p} I_k^s + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) - 2\delta,$$
$$I_{k+1}^b = \frac{p-\gamma}{\beta+p} I_k^b + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) + 2\delta. \tag{2.21}$$

Note that the first and the second equation of system (2.21) are both linear difference equations and $0 < \frac{p-\gamma}{\beta+p} < 1$, then it is easy to verify that

$$\lim_{k\to\infty} I_k^s = \frac{\beta+p}{\alpha} N(R_0 - 1) - \frac{2(\beta+\gamma)}{\beta+p}\delta, \lim_{k\to\infty} I_k^b = \frac{\beta+p}{\alpha} N(R_0 - 1) + \frac{2(\beta+\gamma)}{\beta+p}\delta.$$

The inequality $I_k^s < I_t < I_k^b$ for $t > \tilde{T}_{2k}$ and the arbitrary of δ suggest that

$$\lim_{t\to\infty} I_k = \frac{\beta+p}{\alpha} N(R_0 - 1).$$

Similar to the proof of case 1, we have

$$\lim_{t\to\infty} S_t = \frac{(\beta+\gamma)N}{\alpha}.$$

Therefore, the endemic equilibrium point P_1 of system (2.5) is globally asymptotically stable when $1 < \frac{\beta+p+\gamma}{\beta+p} < R_0 < \frac{2+\beta+\gamma}{\beta+p}$ and $p > \gamma$.

Therefore, we conclude that the endemic equilibrium point P_1 of system (2.5) is globally asymptotically stable when $1 < \frac{\beta+p+\gamma}{\beta+p} < R_0 < \frac{2+\beta+\min\{p,\gamma\}}{\beta+p}$. □

3. Numerical Simulations

In this section, our aim is to present numerical simulations to illustrate the key results of theoretical analysis and graphical representations in the form of time plots, phase portrait diagrams of system (2.5).

Example 1. For DFE point, we take the parameter values as $N = 100$, $p = 0.5$, $\beta = 0.95$, $\alpha = 0.8$, $\gamma = 0.0005$ and initial values as $(S_0, I_0) = (100, 30)$. The eigen values are $|\lambda_1| = 0.9795 < 1$, $|\lambda_2| = 0.7780 < 1$ and basic reproductive ratio $R_0 = 0.5514 < 1$ then DFE point of the system (2.5) is globally asymptotically stable (see Figure 1).

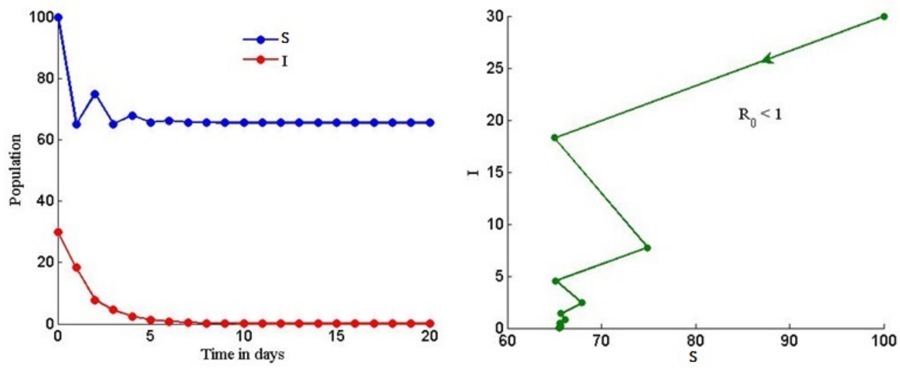

FIGURE 1. Time plots and phase portraits of DFE point of the system (2.5) with stability $R_0 < 1$.

Example 2. For EE point, we take the parameter values as $N = 100$, $p = 0.0005$, $\beta = 0.025$, $\alpha = 0.6$, $\gamma = 0.3$ and initial values as $(S_0, I_0) = (100, 30)$. We apply the conditions $1 < R_0(= 1.81) < \frac{\beta+p+\gamma}{\beta+p}(= 12.7647) < \frac{2+\beta+p}{\beta+p}(= 79.4314)$ and $p(= 0.0005) \leq \gamma(= 0.3)$ then the endemic equilibrium point of the system (2.5) is globally asymptotically stable (see Figure 2).

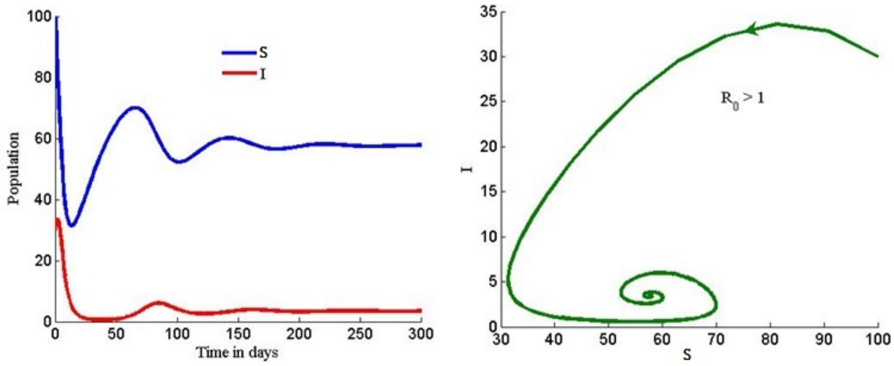

FIGURE 2. Time plots and phase portraits of EE point of the system (2.5) with stability $R_0 > 1$

Example 3. For EE point, we take the parameter values as $N = 100$, $p = 0.5$, $\beta = 0.2$, $\alpha = 0.8$, $\gamma = 0.0005$ and initial values as $(S_0, I_0) = (100, 30)$. We apply the conditions $\frac{\beta+p+\gamma}{\beta+p}(= 1.0007) < R_0(= 1.14) < \frac{2+\beta+\gamma}{\beta+p}(= 3.1436)$ and $p > \gamma$ then the endemic equilibrium point of the system (2.5) is globally asymptotically stable (see Figure 3).

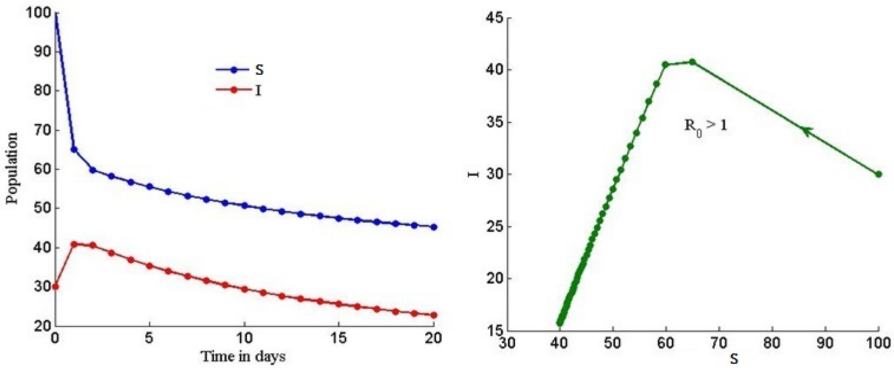

FIGURE 3. Time plots and phase portraits of EE point of the system (2.5) with stability $R_0 > 1$.

4. BIFURCATION AND CHAOTIC BEHAVIOR

In this section, we present the bifurcation diagrams and maximum Lyapunov exponent and chaotic attractors of the system (2.5). It is known that Maximum Lyapunov exponent qualifies the exponential divergence of initially close state-space trajectories and frequency employ to identify a chaotic behavior.

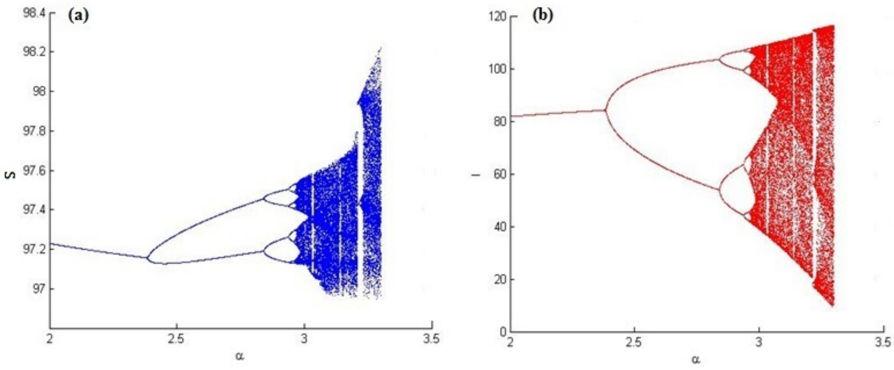

FIGURE 4. The bifurcation diagrams of the system (2.5) for $\alpha \in (2, 3.4)$.

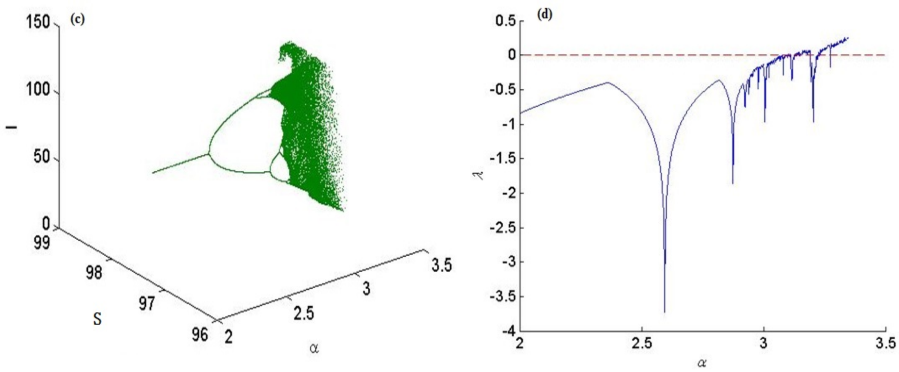

FIGURE 5. The bifurcation diagram in $(\alpha - S - I)$ space and the Maximum Lyapunov exponent corresponding to Figure 4(a-b).

The bifurcation diagrams are considered for three cases:

Case 1. Fixing parameters $N = 100$, $\beta = 0.3$, $p = 0.0009$, $\gamma = 0.01$ and varying $\alpha \in (2, 3.4)$.
The bifurcation diagrams of the system (2.5) plotted in particular range of $\alpha \in (2, 3.4)$ with the contact rate α as the bifurcation parameter are given in Figures 4(a-b). The bifurcation diagram of the system (2.5) in $(\alpha - S - I)$ space is given in Figure 5(c). The Maximum Lyapunov exponent corresponding to Figures 4(a-b) are computed and plotted in Figure 5(d) confirming the existence of the chaotic regions and period orbits in the parametric space. Figures 6(a-b) is the local amplification corresponding to Figures 4(a-b) for $\alpha \in [3, 3.3]$. The phase portraits for various α-values corresponding to Figure.4(a-b) are plotted in Figures 7(a)-(f) to illustrate the observations.

Furthermore, the stable equilibria at $\alpha = 2.2$, the emergence of periodic-2,4,8 orbits are observed when $\alpha = 2.6, 2.9, 2.95$ in Figures 7(b)-(d). For instance, when $\alpha = 2.99, 3.0$ chaotic attractors appears in Figures 7(e)-(f). Some interesting phenomena are also seen in different chaotic attractors in the range $\alpha = 3.1$ to $\alpha = 3.3$. The occurrence of chaotic regions is observed in Figures 6(a)-(b), these phenomena are illustrated by the phase portraits in Figures 8(g)-(l).

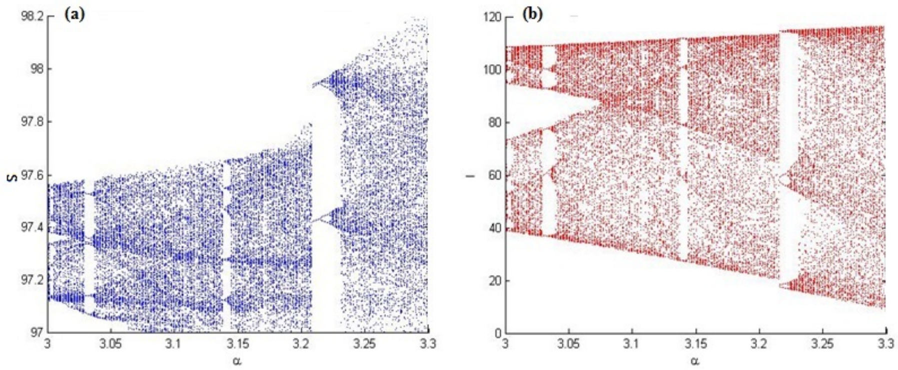

FIGURE 6. Local amplification corresponding to Figure $4(a-b)$.

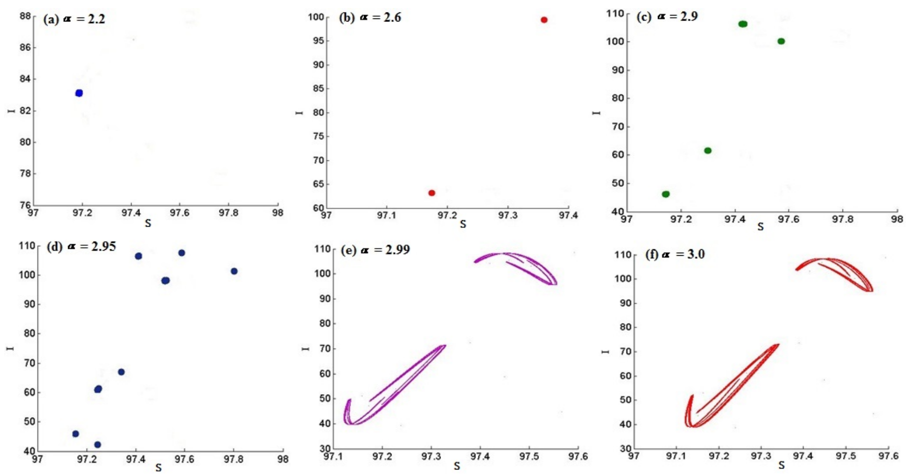

FIGURE 7. Phase portrait diagrams of the system (2.5) for various α corresponding to Figure 4.

Case 2. Fixing parameters $N = 100$, $p = 0.0005$, $\gamma = 0.1$, $\alpha = 4.16$ and varying $\beta \in (1, 2.5)$. The bifurcation diagrams of the system (2.5) plotted in particular range

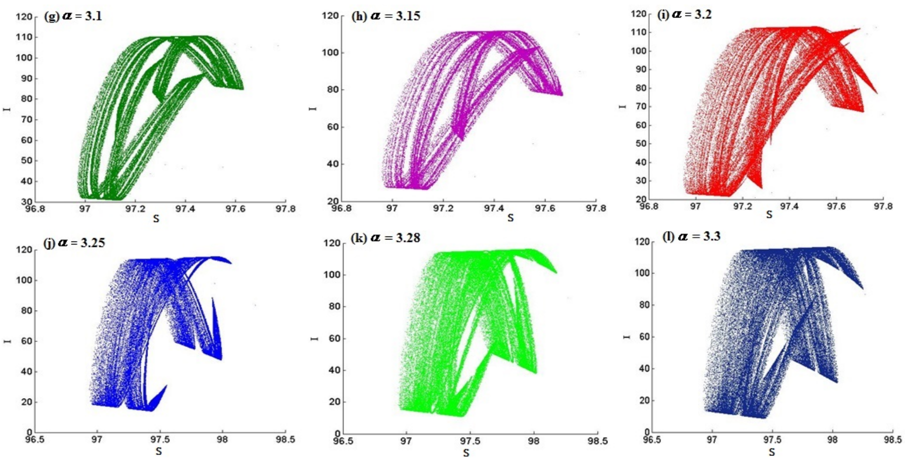

FIGURE 8. Phase portrait diagrams of the system (2.5) for various α corresponding to Figure 4.

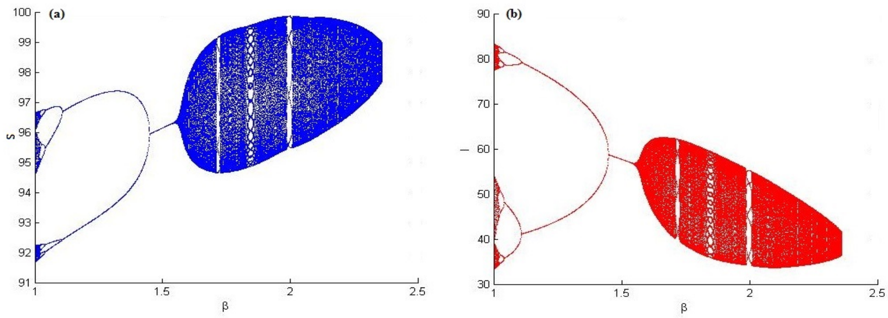

FIGURE 9. Bifurcation diagrams of the system (2.5) with varying $\beta \in (1, 2.5)$.

of $\beta \in (1, 2.5)$ with birth rate as the bifurcation parameter are given in Figures 9(a-b). The bifurcation diagram of the system (2.5) in $(\beta - S - I)$ space are given in Figure 10(c). The Maximum Lyapunov exponent corresponding to Figures 9(a-b) are computed and plotted in Figure 10(d) confirming the existence of the chaotic regions and periodic orbits in the parametric space. The chaotic regions are also observed in Figures 9(a)-(b): these phenomena are illustrated by the phase portrait in Figure 11(a). The phase portraits for various β-values corresponding to Figures 9(a-b) are plotted in Figures 11(b)-(f) to illustrate the observations. Furthermore, the emergence of periodic-8,4,2 orbits are observed when $\beta = 1.03, 1.05, 1.25$ in Figure.11(b)-(d), the stable equilibria at $\beta = 1.56$ in Figure 11(e). Attracting invariant circle appears

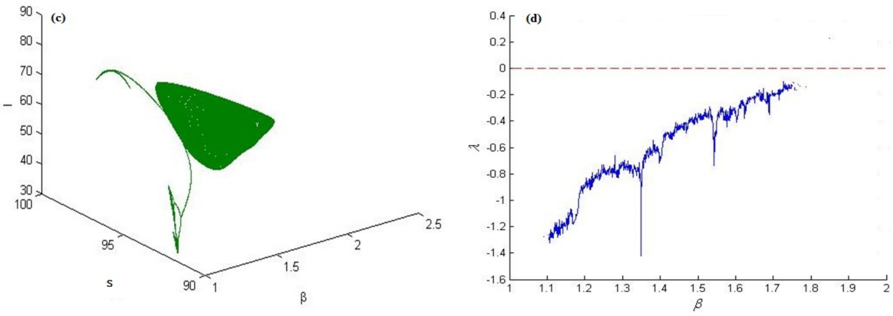

FIGURE 10. Bifurcation diagram in $(\beta - S - I)$ space and Maximum Lyapunov exponents of the system (2.5) with varying $\beta \in (1, 2.5)$.

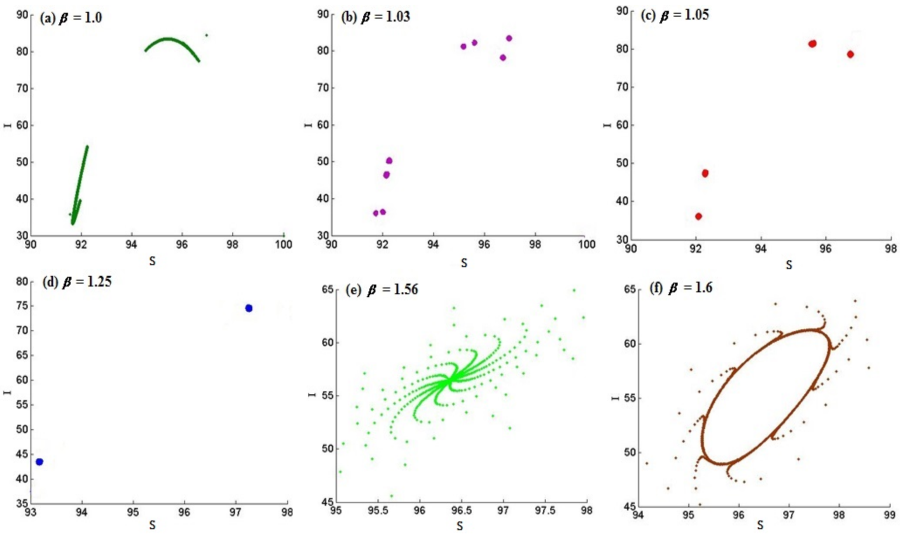

FIGURE 11. Phase portrait diagrams of the system (2.5) for various β corresponding to figure 9.

when $\beta = 1.6$ in Figure 11(f) and when $\beta = 1.7, 1.71, 1.85, 1.95, 2.3$ in Figures 12(g)-(j) and Figure 12(l). Furthermore, quasi-periodic orbits appears when $\beta = 2.0$ in Figure 12(k).

Case 3. Fixing parameters $N = 100$, $p = 0.0009$, $\beta = 0.3$, $\alpha = 3.3$ and varying $\gamma \in (0, 1)$. The bifurcation diagrams of the system (2.5) plotted in particular range of $\gamma \in (0, 1)$ recovery rate as the bifurcation parameter are given in Figure 13(a). The Maximum Lyapunov exponent corresponding to Figure 13(a) are computed and plotted in Figure 13(b) confirming the existence of the chaotic regions and periodic orbits in the parametric space.

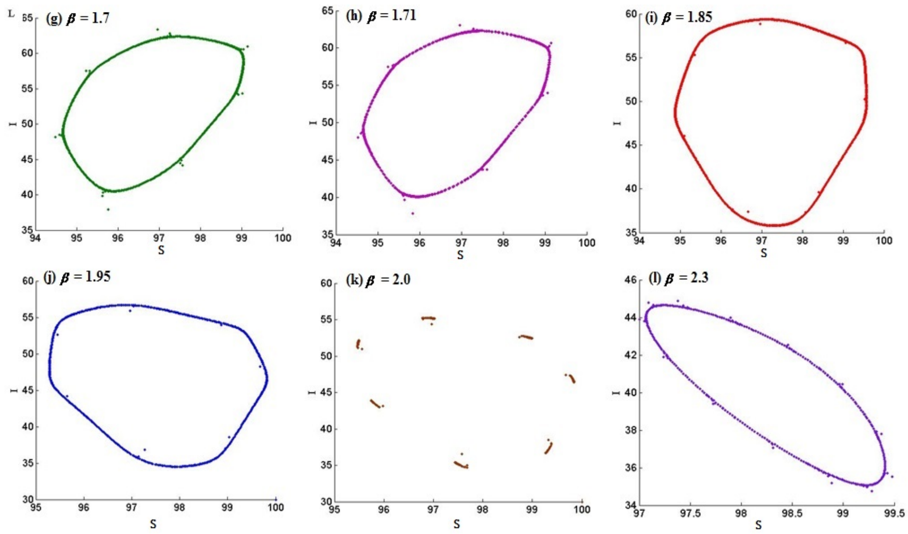

FIGURE 12. Phase portrait diagrams of the system (2.5) for various β corresponding to figure 9.

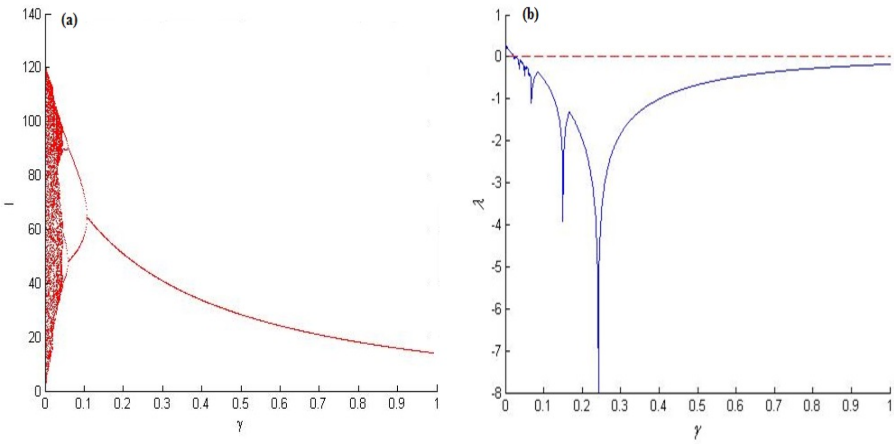

FIGURE 13. Bifurcation diagram and Maximum Lyapunov exponent of the system (2.5) with varying $\gamma \in (0, 1)$.

5. CONCLUSION

In this paper, we considered the dynamical behaviors of a discrete-time SIR epidemic model (1.1). The basic reproduction number completely determine the stability of discrete model (1.1). If the basic reproduction number is less than one, model

(1.1) only has a DFE and it is globally asymptotically stable. If the basic reproduction number is bigger than one, model has a EE except DFE, and EE is globally asymptotically stable. Numerical simulations are provided to justify our analytical findings. Furthermore, let α, β, γ be the bifurcation parameters, numerical simulations show that model (1.1) has periodic orbits, period-2,4,8 orbits, and chaotic sets, which implies that the susceptible and infective can coexists in the stable period-n orbits and cycle. We also present the maximum Lyapunov exponent, when the maximum Lyapunov exponent is positive which is an evidence for chaos. These results reveal far richer dynamics of the discrete model compared to the continuous model. The theoretical analysis of bifurcation and chaos will be studied in the further.

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Authors' addresses

Özlem Ak Gümüş
Adiyaman University, Faculty of Arts and Sciences, Department of Mathematics, Adiyaman University, 02040 Adiyaman, Turkey
E-mail address: akgumus@adiyaman.edu.tr

Qianqian Cui
(**Corresponding author**) School of Mathematics and Statistics, Ningxia University, 750021, Yinchuan, P.R. China
E-mail address: cmengqian@nxu.edu.cn

George Maria Selvam
Department of Mathematics, Sacred Heart College (Autonomous), 635 601, Tamil Nadu, India
E-mail address: agmshc@gmail.com

Abraham Vianny
Department of Mathematics, Sacred Heart College (Autonomous), 635 601, Tamil Nadu, India
E-mail address: abrahamvianny1988@gmail.com