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OPTIMAL CONTROL PROBLEM GOVERNED BY WEAKLY COUPLED SYSTEM ON INDUCTION HEATING

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Abstract. In induction heating process, there is a problem of uniform heating. In this paper, an optimal applied electric field control problem for induction heating process of conductive materials is considered. The cost function is defined such that the temperature profile at the final stage has a relative uniform distribution in the field. The controlled system is a coupled by Maxwell's equations with nonlinear heat equation. The existence of solutions for weakly coupled system is proved. We show that there exists an optimal applied electric field which minimizes the cost functional. Moreover, a fist-order necessary condition is derived.

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1. Introduction

Induction heating uses induction eddy current for heating materials. As the characteristics of high heating speed, high heat efficiency and no pollution, induction heating is widely used in metal smelting, casting, welding, heat treatment, hot forging and other modern industrial manufacturing. Meanwhile, the output power of the power source has good controllable conditions. Since 1990s, induction heating and its corresponding control problems have attracted extensive attention(see [1, 17, 18]).

In the process of induction heating, the temperature distribution of the heated object is not uniform. In recent years, many researchers have studied the uniform distribution of heating workpiece temperature and the shortening of heating time through simulation control, which has improved the quality and efficiency of heating (see[14, 21]). However, the mathematical model of induction heating involves the coupling of three-dimensional electromagnetic field and nonlinear temperature field, which is difficult to simulate. Therefore, it is inevitable for the control mode to develop from traditional analog control to digital control. In view of this, we can select suitable applied electric field to improve the uniformity of temperature distribution. In this paper, we formulate this model as an optimal control problem in which the underlying dynamics is governed by Maxwell's equations coupled with nonlinear heat conduction associated with a heat source generated by induction waves. The goal of the control is to reach a relatively uniform heat profile at a specific time by controlling the applied field on the boundary of the domain.

Optimal control associated with partial differential equations or systems as underlying states have been studied by many researchers. There are several classical books, see e.g., [13, 20], and more recent ones,see e.g., [4, 11, 12, 23]. Especially for optimal control problems of microwave heating, Li, Tang and Yin [9] considered a similar problem where the control is chosen from the heat source. Wei and Yin [10, 22] considered a boundary control problem where the external electric field is chosen as the control function. But induction heating is different from microwave heating, induction heating is a parabolic system with the unknown quantity of magnetic field intensity. Therefore, we study induction heating in different ways.

Since the local induction heating model of conductive materials is expressed by Maxwell's equations with the mixed boundary value coupled with nonlinear heat conduction associated with a heat source in this paper, this poses numerous mathematical challenges. Maxwell's equations in a quasi-stationary electromagnetic field with only one boundary value have been studied by Yin in[6]. But we study the corresponding mixed boundary value problem, which needs to deal with some related problems by introducing two trace operators. In addition, the heat equation is strongly nonlinear. Invoking similar ideas in reference [24], we prove the existence of a solution for the coupled system. One of the difficulties in the present paper is the nonlinear heat source often belongs to $L^1(Q_T)$ for this type of coupled systems. This causes a serious problem when one needs to prove the existence of an optimal control. By applying properties of space $H(curl,\Omega)$ and Sobolev's imbedding theorems, we can overcome the difficulties to establish the existence result.

This paper is organized as follows. In Section 2, the mathematical model for induction heating is derived. Moreover,we present the optimal control problem, in which the control function is the applied electric field on a partial boundary. In Section 3, we prove that the controlled system has a unique weak solution for any applied electric field. In Section 4, we prove an existence theorem for the optimal control problem. Finally in Section 5 the state variables are differentiable with respect to the control variable. Based on the previous results the necessary condition is derived.

2. THE FORMULATION OF AN OPTIMAL CONTROL FOR UNIFORMLY TEMPERATURE

Recalling Maxwell's equations, a simplified version has already appeared in [5]. Suppose that a targeted substance is placed in a inductive processor cavity, denoted by $\Omega \subset \mathbf{R}$ with C^1-boundary $S = \partial\Omega$. Let $\mathbf{H}(x,t)$ denote the magnetic fields at $x \in \Omega$ and time t. Hereafter, a bold letter represents a vector function in $\mathbf{R}^3$. Noting that the induction material is highly conductive,therefore from electromagnetic theory [2, 16], Maxwell's equations in Ω can be expressed by

$$\mu\mathbf{H}_t + \nabla \times [\rho(x,t)\nabla \times \mathbf{H}] = 0,$$

where the constant $\mu = \mu_1 - i\mu_2$ ($\mu_1 > 0, \mu_2 > 0$) and $\rho(x,t) = \frac{1}{\sigma(x,t)}$ (σ is the electric conductivity) are a complex value coefficient of permeability and the electric resistivity, respectively.

The heat source produced by Joule's heat is $Q(x,t) = \rho(x,t)|\nabla \times \mathbf{H}|^2$. With this, by using Fourier's law and the conservation of energy, one can easily see that the temperature $u(x,t)$ satisfies a nonlinear heat equation with an internal source generated by induction:

$$\rho_0 c_0 u_t - \nabla \cdot (k(x,u)\nabla u) = \rho(x,t)|\nabla \times \mathbf{H}|^2, (x,t) \in Q_T,$$

where $Q_T = \Omega \times (0,T), \rho_0$ is the density, c_0 the specific heat, and $k(x,u)$ the heat conductivity .

We sum up the above derivation and normalize certain physical constants to obtain the following mathematical model:

$$\mathbf{H}_t + \nabla \times [\rho(x,t)\nabla \times \mathbf{H}] = 0, \quad (x,t) \in Q_T, \tag{2.1}$$

$$u_t - \nabla \cdot (k(x,u)\nabla u) = \rho(x,t)|\nabla \times \mathbf{H}|^2, \quad (x,t) \in Q_T, \tag{2.2}$$

$$\mathbf{n} \times \mathbf{H} = 0, \quad (x,t) \in S_{\Gamma_1} = \Gamma_1 \times (0,T), \tag{2.3}$$

$$\mathbf{n} \times [\rho(x,t)\nabla \times \mathbf{H}(x,t)] = \mathbf{n} \times \mathbf{G}(x,t), (x,t) \in S_{\Gamma_2} = \Gamma_2 \times (0,T), \tag{2.4}$$

$$\mathbf{H}(x,0) = \mathbf{H}_0(x), \quad x \in \Omega, \tag{2.5}$$

$$u_{\mathbf{n}}(x,t) = 0, \quad (x,t) \in S_\Gamma = \partial\Omega \times (0,T), \tag{2.6}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2.7}$$

where the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ is split into two disjoint measurable subsets Γ_1 and Γ_2, both of which are nonempty, $\mathbf{n}$ is the outward unit normal on $S = \partial\Omega$, $u_{\mathbf{n}} = \nabla u \cdot \mathbf{n}$ is the normal derivative on S, and $\mathbf{G}(x,t)$ is the electric field generated by external optoelectronic devices which is considered as a control variable.

Remark 1. The boundary condition (2.4) on Γ_2 is the external electric field applied to the system [8].

The admissible control set is

$$U_{ad} = \{\mathbf{G} \in L^2(0,T;L^2(\Gamma_2)) : \|\mathbf{G}\|_{L^2(0,T;L^2(\Gamma_2))} \leq A_0 < \infty\},$$

where A_0 is a constant.

Optimal control problem (P): Given $T > 0$ and a desired temperature $u_T(\cdot) \in L^2(\Omega)$ at time T, find an optimal control $\mathbf{G}^* \in U_{ad}$ such that the cost functional

$$J(\mathbf{G};\mathbf{H},u) = \frac{1}{2}\int_\Omega |u(x,T) - u_T(x)|^2 dx + \frac{\lambda}{2}\int_0^T \int_{\Gamma_2} |\mathbf{G}(x,t)|^2 dsdt \tag{2.8}$$

reaches its minimum at $(\mathbf{H}^*, u^*)$ for all $\mathbf{G} \in U_{ad}$, where $(\mathbf{H}, u)$ and $(\mathbf{H}^*, u^*)$ are weak solutions of the coupled system (2.1)-(2.7) corresponding to $\mathbf{G}$ and $\mathbf{G}^*$, respectively. Where the number $\lambda > 0$ is a typical regularization parameter.

3. Existence and uniqueness of solution for the underlying system

We recall some standard Banach spaces which we will use in this paper. For convenience, a product space D^n is often simply denoted by D. Let

$$H(curl,\Omega) = \{\mathbf{M} \in L^2(\Omega) : \nabla \times \mathbf{M} \in L^2(\Omega)\},$$
$$X = \{\mathbf{M} \in L^2(\Omega) : \nabla \times \mathbf{M} \in L^2(\Omega), \mathbf{n} \times \mathbf{M} = 0 \text{ on } \Gamma_1\}.$$

Obviously, X is a linear subspace of $H(curl,\Omega)$. $H(curl,\Omega)$ is a Hilbert space equipped with inner product

$$(\mathbf{M},\mathbf{N}) = \int_\Omega [(\nabla \times \mathbf{M}) \times (\nabla \times \mathbf{N}^*) + \mathbf{M} \cdot \mathbf{N}^*] dx$$

where $\mathbf{N}^*$ represents the complex conjugate of $\mathbf{N}$. A norm on $H(curl,\Omega)$ is given by $\|\cdot\|_{H(curl,\Omega)} = \sqrt{(\cdot,\cdot)}$ (see[19]).

Since the boundary consists of two parts, we need to introduce the two trace mapping $\Upsilon_t : H(curl,\Omega) \to Y(\partial\Omega)$ and $\Upsilon_T : H(curl,\Omega) \to Y'(\partial\Omega)$ (the dual space of $Y(\partial\Omega)$) defined by $\Upsilon_t(\mathbf{M}) = \mathbf{n} \times \mathbf{M}|_{\partial\Omega}$ and $\Upsilon_T(\mathbf{M}) = \mathbf{n} \times (\mathbf{n} \times \mathbf{M}|_{\partial\Omega})$ for every $\mathbf{M} \in H(curl,\Omega)$, respectively. Where $\mathbf{n}$ is the above description and $Y(\partial\Omega)$ is a Hilbert space (see[19]) as follow

$$Y(\partial\Omega) = \{\mathbf{f} \in H^{-\frac{1}{2}}(\Omega) : \text{ there exists } \mathbf{M} \in H(curl,\Omega) \text{ with } \Upsilon_t(\mathbf{M}) = \mathbf{f}\}.$$

with norm

$$\|\mathbf{f}\|_{Y(\partial\Omega)} = \inf_{\mathbf{M} \in H(curl,\Omega), \Upsilon_t(\mathbf{M}) = \mathbf{f}} \|\mathbf{M}\|_{H(curl,\Omega)}.$$

We impose some basic assumptions which ensure the well-posedness of the underlying system.

H(1) Functions $u_0(\cdot)$ and $u_T(\cdot)$ are nonnegative with $u_0(\cdot), u_T(\cdot) \in L^2(\Omega)$.

H(2) The function $k(x,u)$ is measurable in x, uniformly Lipschitz continuous with respect to u, and $0 < k_1 \le k(x,u) \le k_2$, for positive constants k_1 and k_2.

H(3) (a) The function $\rho(x,t)$ is real, measurable and bounded. Moreover,

$$0 < \rho_1 \le \rho(x,t) \le \rho_2$$

for some constants $\rho_1 > 0, \rho_2 > 0$.

(b) The vector function $\mathbf{H}_0(\cdot) \in L^2(\Omega)$.

In the following, we will show that the system (2.1) and (2.3)-(2.5) have a solution under the codition **H(3)** for any given $\mathbf{G} \in U_{ad}$.

Definition 1. A vector function $\mathbf{H} \in L^2(0,T;X)$ is said to be a weak solution of problem (2.1) and (2.3)-(2.5) if

$$
\begin{aligned}
&-\int_0^T \int_\Omega \mathbf{H}\cdot\Phi_t dxdt + \int_0^T \int_\Omega \rho(x,t)(\nabla\times\mathbf{H})\cdot(\nabla\times\Phi)dxdt \\
&= \int_\Omega \mathbf{H}_0\cdot\Phi(x,0)dx - \int_0^T \langle \mathbf{n}\times\mathbf{G}, \Upsilon_T(\Phi)\rangle_{\Gamma_2} dt,
\end{aligned}
$$

for any vector function $\Phi \in H^1(0,T;X)$ with $\Phi(x,T) = 0$ $a.e.$ $x \in \Omega$.

Remark 2. Note that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, we need to consider the boundary conditions in Definition 1. On Γ_1 the boundary condition (2.3) gives no information about $\mathbf{n}\times[\rho(x,t)\nabla\times\mathbf{H}]$, so we choose Φ such that $\Upsilon_T(\Phi) = 0$ on Γ_1. Using this fact, the integral of boundary contains only the portion on Γ_2 .

We will use the Galerkin's method to derive the existence of solutions for the system (2.1).

Note that we have a compactness embedding $H(curl,\Omega) \hookrightarrow L^2(\Omega)$ and X is a linear subspace of $H(curl,\Omega)$, we may choose a countable set of linearly independent elements $\{\mathbf{w}_k(x)\}_{k=1}^{\infty}$ in X. After possibly performing an orthogonalization process with respect to the scalar product of $L^2(\Omega)$, we may assume that $\{\mathbf{w}_k(x)\}_{k=1}^{\infty}$ forms an orthonormal system in $L^2(\Omega)$ which is also complete in $L^2(\Omega)$.

Now fix a positive integer m, we will look for a vector-valued function $\mathbf{H}_m : [0,T] \to X$ of the form

$$
\mathbf{H}_m(t) = \sum_{k=1}^{m} d_m^k(t)\mathbf{w}_k. \tag{3.1}
$$

We hope to select the coefficients $d_m^k(t)(0 \le t \le T, k = 1,2,\cdots,m)$ so that

$$
\left(\frac{d\mathbf{H}_m(t)}{dt}, \mathbf{w}_k\right) + a(\mathbf{H}_m, \mathbf{w}_k; t) = -\langle \mathbf{n}\times\mathbf{G}, \Upsilon_T(\mathbf{w}_k)\rangle_{\Gamma_2}, \tag{3.2}
$$

$$
d_m^k(0) = (\mathbf{H}_0, \mathbf{w}_k), \tag{3.3}
$$

where we write $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_{\Gamma_2}$ for two scalar products, respectively, in $L^2(\Omega)$ and $L^2(\Gamma_2)$, $a(\mathbf{H}_\mathrm{m}, \mathbf{w}_\mathrm{k}; \mathrm{t}) = \int_\Omega \rho(\mathrm{x},\mathrm{t})(\nabla\times\mathbf{H})\cdot(\nabla\times\mathbf{w}_\mathrm{k})\mathrm{dxdt}$.

Theorem 1. *(Galerkin approximate). For each integer $m = 1,2,\cdots$, there exists a unique vector function $\mathbf{H}_m$ of the form* (3.1) *satisfying* (3.2) *and* (3.3).

Proof. Assuming $\mathbf{H}_m$ has the structure (3.1), we first note that

$$
\left(\frac{d\mathbf{H}_m(t)}{dt}, \mathbf{w}_k\right) = \left(d_m^k(t)\right)'.
$$

Furthermore

$$
a(\mathbf{H}_m, \mathbf{w}_k; t) = \sum_{l=1}^{m} e^{kl} d_m^l(t)
$$

for $e^{kl} = a(\mathbf{w}_l, \mathbf{w}_k; t), (k, l = 1, 2, \cdots, m)$. Let us further write

$$b^k(t) := -\langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{w}_k) \rangle_{\Gamma_2} (k = 1, 2, \cdots, m).$$

Then (3.2) becomes the linear system of ODE

$$\left(d_m^k(t)\right)' + \sum_{l=1}^{m} e^{kl} d_m^l(t) = b^k(t) \ \ (k = 1, 2, \cdots, m),$$

subject to the initial condition (3.3). Owing to Carathéodory's theorem, this initial value problem for a system of n linear ordinary differential equations on $[0, T]$ for the unknown vector function $d^m = (d_m^1(t), d_m^2(t), \cdots, d_m^m(t))$ has a unique absolutely continuous solution $d^m \in X^m$. Then $\mathbf{H}_m$ defined by (3.1) satisfying (3.2) and (3.3). □

Theorem 2. *(Estimates for $\{\mathbf{H}_m\}$.) There exists constant C_3, depending only on Ω, T and the known data ρ_1, such that*

$$\|\mathbf{H}_m\|_{C([0,T];L^2(\Omega))} + \|\mathbf{H}_m\|_{L^2(0,T;H(curl,\Omega))} \leq C_3(\|\mathbf{H}_0\|_{L^2(\Omega)} + \|\mathbf{G}\|_{L^2(0,T;L^2(\Gamma_2))}). \quad (3.4)$$

Proof. Multiplying (3.2) by $d_m^k(t)$ and adding the resulting equations from $k = 1$ to $k = m$, we obtain, for almost every $t \in (0, T)$,

$$\left(\frac{d\mathbf{H}_m}{dt}, \mathbf{H}_m\right) + a(\mathbf{H}_m, \mathbf{H}_m; t) = -\langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{H}_m) \rangle_{\Gamma_2}. \quad (3.5)$$

For any arbitrary but fixed $\tau \in (0, T]$, we have the identity

$$\int_0^\tau \left(\frac{d\mathbf{H}_m(t)}{dt}, \mathbf{H}_m(t)\right) dt = \frac{1}{2}\|\mathbf{H}_m(\tau)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\mathbf{H}_m(0)\|_{L^2(\Omega)}^2.$$

Integration of (3.5) over $[0, \tau]$ therefore yields that

$$\frac{1}{2}\|\mathbf{H}_m(\tau)\|_{L^2(\Omega)}^2 + \int_0^\tau a[\mathbf{H}_m(t), \mathbf{H}_m(t); t] dt = \frac{1}{2}\|\mathbf{H}_m(0)\|_{L^2(\Omega)}^2 - \int_0^\tau \langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{H}_m) \rangle_{\Gamma_2} dt. \quad (3.6)$$

By virtue of Bessel's inequality, we have

$$\|\mathbf{H}_m(0)\|_{L^2(\Omega)}^2 = \sum_{k=1}^{m} |d_m^k(0)|^2 = \sum_{k=1}^{m} |(\mathbf{H}_0, \mathbf{w}_k)|^2 \leq \|\mathbf{H}_0\|_{L^2(\Omega)}^2. \quad (3.7)$$

Choosing δ sufficiently small and using the assumption **H(3)**, we have

$$|\langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{H}_m) \rangle| \leq \delta \|\mathbf{G}\|_{L^2(\Gamma_2)}^2 + C(\delta)\|\mathbf{H}_m\|_{L^2(\Gamma_2)}^2 \leq \delta \|\mathbf{G}\|_{L^2(\Gamma_2)}^2 + C_0 C(\delta)\|\mathbf{H}_m\|_{H(curl,\Omega)}^2. \quad (3.8)$$

for almost every $t \in (0, T)$. Invoking Gronwall's inequality, we can infer from (3.6)-(3.8) that

$$\max_{t \in [0,T]} \|\mathbf{H}_m\|_{L^2(\Omega)} \leq K\|\mathbf{H}_0\|_{L^2(\Omega} + \|\mathbf{G}\|_{L(0,T;L^2(\Gamma_2))}.$$

It's easy to get inequalities $C\|\mathbf{H}\|^2_{H(curl,\Omega)} \le a(\mathbf{H},\mathbf{H};t)+\|\mathbf{H}\|^2_{L^2(\Omega)}$, where C depends on known data ρ_1. Inserting this estimate for $\mathbf{H}_m$ in $C([0,T],L^2(\Omega))$ into (3.6) with $\tau = T$, we see that there is a constant$C_3 > 0$ such that

$$\|\mathbf{H}_m\|_{C([0,T];L^2(\Omega))} + \|\mathbf{H}_m\|_{L^2(0,T;H(curl,\Omega))} \le C_3\left(\|\mathbf{H}_0\|_{L^2(\Omega)} + \|\mathbf{G}\|_{L^2(0,T;L^2(\Gamma_2))}\right).$$

□

Theorem 3. *(Existence and uniqueness of weak solution). Assume that condition **H(3)** holds, the problem* (2.1) *and* (2.3)-(2.5) *has a unique weak solution* $\mathbf{H} \in L^2(0,T;X)$.

Proof. First, we shall show that the existence of weak solution. From (3.4) and in view of the orthonormality, we have

$$\sum_{k=1}^{m} |d_m^k(t)|^2 = \|\mathbf{H}_m(t)\|^2_{L^2(\Omega)} \le C^2, \forall t \in [0,T], m \in \mathbf{N}, \tag{3.9}$$

where C is a constant. Hence, we have $\left|d_m^k(t)\right| \le C$ for all t,k, and m. Moreover, it follows from (3.2) by integration over time that for any fixed $k \in \mathbf{N}$ the sequence $\left\{d_m^k(t)\right\}_{m=1}^{\infty}$ forms an equicontinuous set in $C[0,T]$. Recalling the Arzelà-Ascoli theorem , there exists a subsequence of $\{d_m^k(t)\}_{m=1}^{\infty}$, denoted by $\{d_m^{k_l}(t)\}_{m=1}^{\infty}$, such that

$$\lim_{l\to\infty} d_{m_l}^k(t) = d^k(t) \text{ strongly in } C[0,T] \quad \forall k \in \mathbf{N}.$$

With the limit functions $d^k(t)$, we define the function

$$\mathbf{H}(x,t) := \sum_{k=1}^{\infty} d^k(t)\mathbf{w}_k(x), (x,t) \in Q_T.$$

It can be shown that $\mathbf{H}_{m_l}(\cdot,t) \to \mathbf{H}(\cdot,t)$ weakly in $L^2(\Omega)$, uniformly with respect to $t \in [0,T]$. Owing to the weak lower sequential semicontinuity of the norm, we infer from the estimate (3.4) that

$$\|\mathbf{H}\|_{L^2(\Omega)} \le C(\|\mathbf{G}\|^2_{L^2(0,T;L^2(\Gamma_2))} + \|\mathbf{H}_0\|^2_{L^2(\Omega)})$$

for almost all t , which means that $\mathbf{H} \in L^\infty(0,T;L^2(\Omega))$.

Since $\sum_{k=1}^{\infty} |d_m^k(0)|^2 < \infty$ by (3.7), we have

$$\|\mathbf{H}_{m_l}(0) - \mathbf{H}_0\|_{L^2(\Omega)} = \|\sum_{k=1}^{m_l} d_{m_l}^k(0)\mathbf{w}_k - \sum_{k=1}^{\infty}(\mathbf{H}_0,\mathbf{w}_k)\mathbf{w}_k\|_{L^2(\Omega)} = \sum_{k=m_l+1}^{\infty} |d_{m_l}^k(0)|^2 \to 0$$

as $l \to \infty$. This means that $\mathbf{H}_{m_l}(0)$ converges strongly in $L^2(\Omega)$ to $\mathbf{H}_0(0)$.

From (3.4), we can assume that $\mathbf{H}_{m_l}(t) \to \mathbf{H}$ weakly in $L^2(0,T;H(curl,\Omega))$. We take as test function in (3.2) any function $\Phi_n(x,t)$ of the form

$$\Phi_n(x,t) = \sum_{j=1}^{n} \alpha_j(t)\mathbf{w}_j(x), n \le m$$

where $\alpha_j \in C^1[0,T]$ satisfies $\alpha_j(T) = 0$ for $1 \le j \le n$. It then follows from (3.5) that

$$\left(\frac{d\mathbf{H}_{m_l}}{dt}, \Phi_n(x,t)\right) + a(\mathbf{H}_{m_l}, \Phi_n; t) = -\langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\Phi_n)\rangle_{\Gamma_2}.$$

Furthermore, we integrate over $[0,T]$ by parts,

$$-\int_0^T \left(\mathbf{H}_{m_l}, \frac{d}{dt}\Phi_n(x,t)\right)dt + \int_0^T a(\mathbf{H}_{m_l}, \Phi_n; t)dt \\ = -\int_0^T \langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\Phi_n)\rangle_{\Gamma_2} dt + (\mathbf{H}_{m_l}(0), \Phi_n(0)).$$

Note that $\mathbf{H}_{m_l} \to \mathbf{H}$ weakly in $L^2(0,T;H(curl,\Omega))$ and $\mathbf{H}_{m_l}(0) \to \mathbf{H}_0$ strongly in $L^2(\Omega)$. Passage to the limit as $l \to \infty$ in the above equation therefore yields

$$-\int_0^T \left(\mathbf{H}, \frac{d}{dt}\Phi_n(x,t)\right) dt + \int_0^T a(\mathbf{H}, \Phi_n; t)dt \\ = -\int_0^T \langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\Phi_n)\rangle_{\Gamma_2} dt + (\mathbf{H}_0, \Phi_n(0)).$$

Since the set of all functions Φ_n is dense in the class of all functions from $H^1(0,T;H(curl,\Omega))$, $\mathbf{H}$ satisfies the variational formulation of Definition 1 and is thus a weak solution.

Finally, we shall show that the uniqueness of weak solution. Note again that $\{\mathbf{H}_{m_l}(t)\}_{l=1}^{\infty}$ converges weakly in $L^2(0,T;H(curl,\Omega))$ to $\mathbf{H}$ and the norm has the weak lower sequential semicontinuity, we infer from the estimate (3.4) that

$$\|\mathbf{H}\|_{L^2(0,T;H(curl,\Omega))} \le C_4(\|\mathbf{G}\|^2_{L^2(0,T;L^2(\Gamma_2))} + \|\mathbf{H}_0\|^2_{L^2(\Omega)}). \tag{3.10}$$

To prove the uniqueness, it suffices to check that the only weak solution of (2.1) and (2.3)-(2.5) with $\mathbf{H}_0 \equiv \mathbf{G} \equiv \mathbf{0}$ is $\mathbf{H} \equiv \mathbf{0}$. However, this is obvious from (3.10). $\square$

Invoking (3.4) and (3.10), we immediately obtain the following theorem.

Theorem 4. *Under the assumption **H(3)**, the problem* (2.1) *and* (2.3)-(2.5) *has a unique weak solution* $\mathbf{H} \in L^2(0,T;X)$ *and the following estimates is hold.*

$$\sup_{t\in[0,T]} \|\mathbf{H}(\cdot,t)\|_{L^2(\Omega)} + \|\mathbf{H}\|_{L^2(0,T;H(curl,\Omega))} \le C_5, \tag{3.11}$$

where C_5 *depend only on known data.*

Next fix the weak $\mathbf{H} \in L^2(0,T;X)$ of problem (2.1) and (2.3)-(2.5), we now turn to the problem (2.2), (2.6) and (2.7). Let $V = H^1(\Omega), H = L^2(\Omega)$, then $V^* = H^{-1}(\Omega)$ and $V \hookrightarrow H \hookrightarrow V^*$ is an evolution triple.

Define space $W(0,T)$ as

$$W(0,T) = \left\{u \in L^2(0,T;V) : u_t \in L^2(0,T;V^*)\right\},$$

with the norm

$$\|u\|^2_{W(0,T)} = \|u\|^2_{L^2(0,T;V)} + \|u_t\|^2_{L^2(0,T;V^*)}.$$

Then $W(0,T)$ is a Banach space, $W(0,T) \hookrightarrow C([0,T];H)$ is continuous and $W(0,T) \hookrightarrow L^2(0,T;H)$ is compact, where $\frac{d}{dt}$ means the generalized derivative of real functions on $[0,T]$.

For almost every $t \in [0,T]$, we can see that $\nabla \times \mathbf{H}(\cdot,t) \in H(curl,\Omega)$ when $\mathbf{H}(\cdot,t)$ lies in $X \subset H(curl,\Omega)$ (see[19]). By using Sobolev's embedding with the dimension N = 3, we know that the $\nabla \times \mathbf{H}(\cdot,t) \in L^6(\Omega)$. Finally, recalling assumption **H(3)**, we have that $F(t) = \rho(\cdot,t)|\nabla \times \mathbf{H}(\cdot,t)|^2 \in L^2(\Omega)$. With this result, the weak solution of the problem (2.2), (2.6) and (2.7) will be well defined.

Theorem 5. *(see[3,24]) Assume that conditions **H(1)–H(3)** hold, the system (2.2), (2.6) and (2.7) has unique weak solution $u \in W(0,T)$ for any given $\mathbf{H} \in L^2(0,T;X)$, which satisfies the following integral identity*

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)dx + \int_\Omega k(x,u)\nabla u(x,t)\cdot\nabla v(x)dx = \int_\Omega \rho(x,t)|\nabla \times \mathbf{H}(x,t)|^2 v(x)dx$$

with the initial condition $u(0) = u_0$, for all $v \in V$ and for almost all $t \in [0,T]$, where $\frac{d}{dt}$ means the generalized derivative.

Moreover,we have estimate

$$\|u\|_{W(0,T)} \le C_6, \tag{3.12}$$

where positive constant C_6 depends on known data.

Summarizing the above considerations, we obtain the existence of solutions of the coupled system (2.1)-(2.7).

Theorem 6. *If the assumptions **H(1)-H(3)** hold, then the coupled system (2.1)-(2.7) has a unique weak solution $(\mathbf{H},u) \in L^2(0,T;X) \times W(0,T)$.*

4. Existence of an optimal control

In this section, we will prove the existence of an optimal control for problem **(P)**. Invoking Theorem 6, for any fixed $\mathbf{G} \in L^2(0,T;L^2(\Gamma_2))$,there is unique weak solution $(\mathbf{H},u) \in L^2(0,T;X) \times W(0,T)$. Hence we can define an control-to-state mapping $P : \mathbf{G} \to (\mathbf{H},u)$ from $L^2(0,T;L^2(\Gamma_2))$ into $L^2(0,T;X) \times W(0,T)$ for the system (2.1)-(2.7).

Theorem 7. *Under the assumptions **H(1)-H(3)**, and for any fixed* $\mathbf{G} \in U_{ad}$*, the control-to-state mapping* P *is weakly sequentially continuous.*

Proof. Let a sequences $\{\mathbf{G}_m\}_{m=1}^{\infty} \subset L^2(0,T;L^2(\Gamma_2))$ such that

$$\mathbf{G}_m \to \mathbf{G}^* \text{ weakly in } L^2(0,T;L^2(\Gamma_2)), \text{ as } m \to +\infty. \tag{4.1}$$

We shall show that

$(\mathbf{H}_m, u_m) = P(\mathbf{G}_m) \to P(\mathbf{G}^*) = (\mathbf{H}^*, u^*)$ weakly in $L^2(0,T;X) \times W(0,T)$, as $m \to \infty$.

According to the definition of mapping P, $(\mathbf{H}_m, u_m)$ is the weak solution of system (2.1)-(2.7) corresponding to the control $\mathbf{G}_m$ for $m = 1, 2, \cdots$. That is,

$$\begin{cases} -\int_0^T \int_\Omega \mathbf{H}_m \cdot \Phi_t dxdt & +\int_0^T \int_\Omega \rho(x,t)\nabla \times \mathbf{H}_m \cdot \nabla \times \Phi dxdt \\ & = \int_\Omega \mathbf{H}_m(x,0) \cdot \Phi(x,0)dx - \int_0^T \langle \mathbf{n} \times \mathbf{G}_m, \Upsilon_T(\Phi)\rangle_{\Gamma_2} dt, \\ \mathbf{n} \times \mathbf{H}_\mathrm{m}(x,t) & = 0, \quad (x,t) \in S_{\Gamma_1}, \\ \mathbf{n} \times [\rho(x,t)\nabla \times \mathbf{H}_m] & = \mathbf{n} \times \mathbf{G}_m(x,t), \quad (x,t) \in S_{\Gamma_2}, \\ \mathbf{H}_m(x,0) & = \mathbf{H}_0(x), \quad x \in \Omega, \\ \frac{d}{dt}\int_\Omega u_m v dx + \int_\Omega k(x,u_m) & \nabla u_m \cdot \nabla v dx = \int_\Omega \rho(x,t)|\nabla \times \mathbf{H}_m|^2 v dx, \quad \text{a.e. } t \in [0,T], \\ (u_m)_\mathbf{n}(x,t) & = 0, \quad (x,t) \in S_\Gamma, \\ u_m(x,0) & = u_0(x), \quad x \in \Omega, \end{cases}$$

where $\Phi \in H^1(0,T;X), v \in V = H^1(\Omega)$.

From Theorems 4 and 5, $(\mathbf{H}_m, u_m)$ is bounded in reflexive space $L^2(0,T;H(curl,\Omega)) \times W(0,T)$. Hence, there exists a subsequence of $(\mathbf{H}_m, u_m)$, again denoted by $(\mathbf{H}_m, u_m)$, such that

$$\mathbf{H}_m \to \mathbf{H}^* \quad \text{weakly in} \quad L^2(0,T;H(curl,\Omega)), \tag{4.2}$$

$$u_m \to u^* \quad \text{weakly in} \quad L^2(Q_T), \tag{4.3}$$

$$\nabla u_m \to \nabla u^* \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)), \tag{4.4}$$

$$\nabla \times \mathbf{H}_m \to \nabla \times \mathbf{H}^* \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)). \tag{4.5}$$

Moreover, the compactness embedding $H(curl,\Omega) \hookrightarrow L^2(\Omega)$ and $W(0,T) \hookrightarrow L^2(0,T;L^2(\Omega))$ imply that

$$\mathbf{H}_m \to \mathbf{H}^* \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)), \tag{4.6}$$

$$u_m \to u^* \quad \text{strongly in} \quad L^2(Q_T). \tag{4.7}$$

The definition of the spaces $L^6(\Omega)$ and $\nabla \times \mathbf{H}_m(\cdot,t) \in L^6(\Omega)$ imply that

$$\nabla \times \mathbf{H}_m(\cdot,t) \to \nabla \times \mathbf{H}^*(\cdot,t) \quad \text{weakly in} \quad L^6(\Omega). \tag{4.8}$$

From (4.6), we know

$$-\lim_{m\to\infty} \int_0^T \int_\Omega \mathbf{H}_m \cdot \Phi_t dxdt = -\int_0^T \int_\Omega \mathbf{H}^* \cdot \Phi_t dxdt,$$

$$\lim_{m\to\infty}\int_\Omega \mathbf{H}_m(x,0)\cdot\Phi(x,0)dx = \int_\Omega \mathbf{H}_0^*(x)\cdot\Phi(x,0)dx.$$

Computing

$$\Big|\int_0^T\int_\Omega \rho(x,t)(\nabla\times\mathbf{H}_m)\cdot(\nabla\times\Phi)dxdt - \int_0^T\int_\Omega \rho(x,t)(\nabla\times\mathbf{H}^*)\cdot(\nabla\times\Phi)dxdt\Big|$$
$$\le \rho_2\int_0^T\int_\Omega \Big|(\nabla\times\mathbf{H}_m - \nabla\times\mathbf{H}^*)\cdot(\nabla\times\Phi)\Big|dxdt.$$

and using (4.5), we have

$$\int_0^T\int_\Omega (\nabla\times\mathbf{H}_m)\cdot(\nabla\times\Phi)dxdt \to \int_0^T\int_\Omega (\nabla\times\mathbf{H}^*)\cdot(\nabla\times\Phi)dxdt$$

as $m\to\infty$.

We can infer from (4.1) that

$$\int_0^T \langle \mathbf{n}\times\mathbf{G}_m, \Upsilon_T(\Phi)_T\rangle_{\Gamma_2}dt \to \int_0^T \langle \mathbf{n}\times\mathbf{G}^*, \Upsilon_T(\Phi)_T\rangle_{\Gamma_2}dt$$

as $m\to\infty$.

By performing integration by parts, for all $\eta\in C^\infty(0,T)$ and $\Psi\in H^1(\Omega)$, we have

$$\int_0^T\int_\Omega [(\mathbf{H}_m)_t\cdot\eta_t\Psi(x) + \mathbf{H}_m\cdot\eta_t(t)\Psi(x)]dxdt$$
$$= \int_\Omega \mathbf{H}_m(x,T)\cdot\eta(T)\Psi(x)dx - \int_\Omega \mathbf{H}_m(x,0)\cdot\eta(0)\Psi(x)dx. \tag{4.9}$$

Passage to the limit as $m\to\infty$ in (4.9) therefore yields

$$\int_\Omega \mathbf{H}^*(x,T)\cdot\eta_t(T)\Psi(x)dx - \int_\Omega \mathbf{H}_0(x)\cdot\eta_t(0)\Psi(x)dx$$
$$= \int_\Omega \mathbf{H}^*(x,T)\cdot\eta(T)\Psi(x)dx - \int_\Omega \mathbf{H}^*(x,0)\cdot\eta(0)\Psi(x)dx. \tag{4.10}$$

Taking $\eta(T)=0$ and $\eta(0)=1$ in (4.10), we have

$$\int_\Omega [\mathbf{H}_0(x) - \mathbf{H}^*(x,0)]\cdot\eta(0)\Psi(x)dx = 0.$$

Since $H^1(\Omega)$ is dense in $L^2(\Omega)$, this implies

$$\mathbf{H}^*(\cdot,0) = \mathbf{H}_0(\cdot) \text{ in } L^2(\Omega).$$

Owing to the continuity of embedding $L^6(\Omega)\hookrightarrow L^4(\Omega)$ and (4.8), for any $t\in[0,T]$, we have

$$\int_\Omega \Big||\nabla\times\mathbf{H}_m(x,t)|^2 - |\nabla\times\mathbf{H}^*(x,t)|^2\Big|v(x)dx \to 0 \ \text{ for all } \ v\in H^1(\Omega).$$

Now recall that $0<\rho_1\le\rho(x,t)\le\rho_2$. Then, for any $t\in[0,T]$, we obtain

$$\Big|\int_\Omega \rho(x,t)|\nabla\times\mathbf{H}_m(x,t)|^2 v(x)dx - \int_\Omega \rho(x,t)|\nabla\times\mathbf{H}^*(x,t)|^2 v(x)dx\Big|$$

$$\leq \rho_2 \int_\Omega \left| \left(|\nabla \times \mathbf{H}_m(x,t)|^2 - |\nabla \times \mathbf{H}^*(x,t)|^2 \right) v(x) \right| dx \to 0, \tag{4.11}$$

as $m \to \infty$. From Majorized convergence theorem and (4.11), for all $\varphi \in C_0^\infty(0,T)$, we have

$$\lim_{m\to\infty} \int_0^T \int_\Omega \rho(x,t)|\nabla \times \mathbf{H}_m(x,t)|^2 v(x)\varphi(t)dxdt$$
$$= \int_0^T \int_\Omega \rho(x,t)|\nabla \times \mathbf{H}^*(x,t)|^2 v(x)\varphi(t)dxdt.$$

We can obtain from (4.7) that

$$-\lim_{m\to\infty} \int_0^T \int_\Omega u_m(x,t)v(x)\varphi_t(t)dxdt = -\int_0^T \int_\Omega u^*(x,t)v(x)\varphi_t(t)dxdt$$

for all $\varphi \in C_0^\infty(0,T)$ and $v \in H^1(\Omega)$(see [24]).

By virtue of **H(2)**, (4.4) and (4.7), one can easy to see that

$$\lim_{m\to\infty} \int_0^T \int_\Omega k(x,u_m)\nabla u_m(x,t)\cdot\nabla v(x)\varphi(t)dxdt$$
$$= \int_0^T \int_\Omega k(x,u^*)\nabla u^*(x,t)\cdot\nabla v(x)\varphi(t)dxdt$$

for all $\varphi \in C_0^\infty(0,T)$ and $v \in H^1(\Omega)$(see [24]).

Since the condition $u^*(\cdot,0) = u_0(\cdot) \in L^2(\Omega)$ has the same structure as condition $\mathbf{H}^*(\cdot,0) = \mathbf{H}_0(\cdot) \in L^2(\Omega)$, it may be shown by the same way.

Therefore, $(\mathbf{H}^*, u^*)$ is a weak solution of (2.1)-(2.7) corresponding to $\mathbf{G}^*$. That is, $P(\mathbf{G}^*) = (\mathbf{H}^*, u^*)$. In the step above we have shown that $(\mathbf{H}_m, u_m) \to (\mathbf{H}^*, u^*)$ for a subsequence. With the arguments above , we can prove that every subsequence has a subsequence converging to the same $(\mathbf{H}^*, u^*)$. Therefore, the entire sequence $(\mathbf{H}_m, u_m)$ converges to $(\mathbf{H}^*, u^*)$ weakly. □

Theorem 8. *(Existence of an Optimal Control). Under the assumptions **H(1)-H(3)**, there exists at least one global minimizer* $(\mathbf{G}^*; \mathbf{H}^*, u^*)$ *of the problem **(P)** such that*

$$\mathbf{G}^* \in U_{ad}, \mathbf{H}^* \in L^2(0,T;X), u^* \in W(0,T).$$

Proof. Using the control-to-state map $P : \mathbf{G} \to (\mathbf{H}, u)$, we reduce the functional $J(\mathbf{G})$ in (2.8) such that $J(\mathbf{G}) := J(\mathbf{G}; \mathbf{H}, u)$. Obviously, $J(\mathbf{G})$ is bounded from (2.8). Hence, we get the existence of an infimum J, that is,

$$J^* := \inf_{\mathbf{G}\in U_{ad}} J(\mathbf{G}) \in \mathbf{R}.$$

Put $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$ be a minimizing sequence such that $\lim_{m\to\infty} J(\mathbf{G}_m) = J^*$.

Since $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$ and $\|\mathbf{G}_m\|_{L^2(0,T;L^2(\Gamma_2))} < \infty$, there exists a subsequence of $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$, again denoted by $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$, such that

$$\mathbf{G}_m \to \mathbf{G}^* \text{ weakly in } L^2(0,T;L^2(\Gamma_2)).$$

where $\mathbf{G}^* \in U_{ad}$ from the Mazur's lemma.

Observe that J is weakly sequentially lower semi-continuous in (2.8) and P is weakly sequentially continuous from Theorem 7. Therefore, we find

$$J^* = \lim_{m\to\infty} J(\mathbf{G}_m) \geq J(\mathbf{G}^*) \geq J^*.$$

This means that $\mathbf{G}^*$, also $(\mathbf{G}^*, \mathbf{H}^*, u^*)$ is a global minimizer. □

5. Necessary optimality condition

Next, we will show the necessary optimality condition of (2.1)–(2.7).First, a variational inequality will be derived that still involves the states $\mathbf{H}$ and u; then, $\mathbf{H}$ and u will be eliminated by means of the adjoint states to deduce a variational inequality for the control.

Theorem 9. *Suppose that the assumption **H(3)** holds. Then the mapping $\Lambda_1 : U_{ad} \to L^2(0,T;X)$ is differentiable and the following limit*

$$\hat{\mathbf{H}}(\mathbf{G};\mathbf{K}) = \lim_{\varepsilon\to 0} \frac{\mathbf{H}(\mathbf{G}+\varepsilon\mathbf{K}) - \mathbf{H}(\mathbf{G})}{\varepsilon} \ \textit{in}\ L^2(0,T;X),$$

exists for any $\mathbf{G},\mathbf{K} \in U_{ad}$ such that $\mathbf{G}+\varepsilon\mathbf{K} \in U_{ad}$ for small ε. Moreover, in the weak sense, $\hat{\mathbf{H}}$ satisfies

$$\begin{cases} \hat{\mathbf{H}}_t + \nabla\times[\rho(x,t)\nabla\times\hat{\mathbf{H}}] = 0, (x,t)\in Q_T, \\ \mathbf{n}\times\hat{\mathbf{H}} = 0, (x,t)\in S_{\Gamma_1}, \\ \mathbf{n}\times[\rho(x,t)\nabla\times\hat{\mathbf{H}}] = \mathbf{n}\times\mathbf{K}, (x,t)\in S_{\Gamma_2}, \\ \hat{\mathbf{H}}(x,0) = 0, x\in\Omega, \end{cases} \tag{5.1}$$

Proof. Let $\mathbf{H} := \mathbf{H}(\mathbf{G}); \mathbf{H}_\varepsilon := \mathbf{H}(\mathbf{G}+\varepsilon\mathbf{K})$ and put $\hat{\mathbf{H}}_\varepsilon = (\mathbf{H}_\varepsilon - \mathbf{H})/\varepsilon,\ (x,t)\in Q_T$. where $\mathbf{H}_\varepsilon$ and $\mathbf{H}$ is a solutions of (2.1) and (2.3)–(2.7) corresponding to $\mathbf{G}+\varepsilon\mathbf{K}$ and $\mathbf{G}$, respectively. By computing, in the weak sense, we easily obtain that $\hat{H}_\varepsilon$ satisfies

$$\begin{cases} (\hat{\mathbf{H}}_\varepsilon)_t + \nabla\times[\rho(x,t)\nabla\times\hat{\mathbf{H}}_\varepsilon] = 0, (x,t)\in Q_T, \\ \mathbf{n}\times\hat{\mathbf{H}}_\varepsilon = 0, (x,t)\in S_{\Gamma_1}, \\ \mathbf{n}\times[\rho(x,t)\nabla\times\hat{\mathbf{H}}_\varepsilon] = \mathbf{n}\times\mathbf{K}, (x,t)\in S_{\Gamma_2}, \\ \hat{\mathbf{H}}_\varepsilon(x,0) = 0, x\in\Omega, \end{cases} \tag{5.2}$$

As several estimates were shown in Theorem 3.7, therefore under the condition **H(3)**, $\hat{\mathbf{H}}_\varepsilon$ can be estimated similarly:

$$\sup_{t\in[0,T]} \|\hat{\mathbf{H}}_\varepsilon\|_{L^2(\Omega)} + \|\hat{\mathbf{H}}_\varepsilon\|_{L^2(0,T;H(curl,\Omega))} \leq C_7, \tag{5.3}$$

where C_7 depend only on known data. From estimate (5.3), there exists a subsequence of $\hat{\mathbf{H}}_\varepsilon$, again denoted by $\hat{\mathbf{H}}_\varepsilon$, and there exist $\hat{\mathbf{H}} \in L^2(0,T;H(curl,\Omega))$ such that

$$\hat{\mathbf{H}}_\varepsilon \to \hat{\mathbf{H}} \ \text{ weakly in }\ L^2(0,T;H(curl,\Omega)), \tag{5.4}$$

$$\nabla\times\hat{\mathbf{H}}_\varepsilon \to \nabla\times\hat{\mathbf{H}} \ \text{ weakly in }\ L^2(0,T;L^2(\Omega)) \tag{5.5}$$

as $\varepsilon \to 0$. Again remember that two embeddings $H(curl,\Omega) \hookrightarrow L^2(\Omega)$ is compact. Thus we have

$$\hat{\mathbf{H}}_\varepsilon \to \hat{\mathbf{H}} \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{5.6}$$

In view of the system (5.2) and limits (5.4)–(5.6), we can obtain the system (5.1) by taking limit as $\varepsilon \to 0$ and using a similar technique as in Theorem 7.

Note that the system (5.1) and the boundary with respect to $\mathbf{K}$ are linear. Hence, the solution $\hat{\mathbf{H}}$ is unique and the mapping

$$\wedge_1 : U_{ad} \to L^2(0,T;H(curl,\Omega))$$

is Fréchet differentiable ,which Fr*é*chet derivative is $\hat{\mathbf{H}}$. □

Secondly, in analogy to Theorem 9, we have the following theorem:

Theorem 10. *Suppose that the assumptions **H(1)–H(3)** hold. Then the mapping $\Lambda_2 : U_{ad} \to W(0,T)$ is differentiable and the following limit*

$$\hat{u}(\mathbf{G};\mathbf{K}) = \lim_{\varepsilon \to 0} \frac{u(\mathbf{G}+\varepsilon\mathbf{K}) - u(\mathbf{G})}{\varepsilon} \text{ in } W(0,T),$$

exists for any $\mathbf{G},\mathbf{K} \in U_{ad}$ such that $\mathbf{G}+\varepsilon\mathbf{K} \in U_{ad}$ for small ε. Moreover, in the weak sense, $\hat{u}$ satisfies

$$\begin{cases} \hat{u}_t - \nabla\cdot[k(x,u)\nabla\hat{u}] - \nabla\cdot[k_u(x,u)\hat{u}\nabla u] \\ \qquad = 2\rho(x,t)\nabla\times\hat{\mathbf{H}}\cdot\nabla\times\mathbf{H}, (x,t) \in Q_T, \\ \hat{u}_{\mathbf{n}}(x,t) = 0, (x,t) \in S_\Gamma, \\ \hat{u}(x,0) = 0, x \in \Omega, \end{cases} \tag{5.7}$$

where$(\mathbf{H},u)$ is a weak solution of (2.1)–(2.7) corresponding to $\mathbf{G}$.

Theorem 11. *Suppose that the assumptions **H(1)–H(3)** hold and $(\mathbf{H}^0,u^0)$ is the optimal solution of the system (2.1)–(2.7) corresponding to the optimal control $\mathbf{G}^0 \in U_{ad}$. Then there exists a pair of functions $(\mathbf{N},p) \in L^2(0,T;X)\times W(0,T)$,which satisfy the adjoint system*

$$\begin{cases} \mathbf{N}_t - \nabla\times[\rho(x,t)\nabla\times\mathbf{N}] = -\nabla\times[2\rho(x,t)p\nabla\times\mathbf{H}^0], & (x,t)\in Q_T, \\ p_t + \nabla\cdot[k(x,u^0)\nabla p] - k_u(x,u^0)\nabla u^0\cdot\nabla p = 0, & (x,t)\in Q_T, \\ \mathbf{n}\times\mathbf{N} = 0, & (x,t)\in S_{\Gamma_1}, \\ \mathbf{n}\times[\rho(x,t)\nabla\times\mathbf{N}] = \mathbf{n}\times[2\rho(x,t)p\nabla\times\mathbf{H}^0], & (x,t)\in S_{\Gamma_2}, \\ \mathbf{N}(x,T) = 0, & x\in\Omega, \\ p_{\mathbf{n}}(x,t) = 0, & (x,t)\in S_\Gamma, \\ p(x,T) = u^0(T) - u_T, & x\in\Omega. \end{cases} \tag{5.8}$$

Moreover, the following inequality is satisfied:

$$\int_0^T \int_{\Gamma_2} [-\mathbf{n}\times(\mathbf{G}-\mathbf{G}^0)\cdot\Upsilon_T(\mathbf{N}) + \lambda\mathbf{G}^0\cdot(\mathbf{G}-\mathbf{G}^0)]dsdt \geq 0, \forall\mathbf{G}\in U_{ad}. \tag{5.9}$$

Proof. Using the Lagrange technique, the adjoint system (5.8) is derived. According to similar ideas in some literatures [4, 7], one can show that there exists a unique solution for the adjoint system. It remains to derive that the inequality (5.9) holds. Since a similar derivation process of the inequality can be seen [15], we skip the details here. □

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