Miskolc Mathematical Notes
Vol. 23 (2022), No. 1, pp. 131–154
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3656

MEASURE THEORETIC GENERALIZATIONS OF JENSEN'S INEQUALITY BY FINK'S IDENTITY

S. I. BUTT, T. RASHEED, Đ. PEČARIĆ, AND J. PEČARIĆ

Received 17 January, 2021

Abstract. We generalize integral Jensen's inequality and its converse for real Stieltjes measure utilizing the theory of n−convex function by employing Fink's identity. We also give several versions of discrete Jensen's inequality along with its reverses and its converse for real weights. As an application we give generalized variants of Hermite-Hadamard's inequality. Also we give applications in information theory by giving new estimations of generalized divergence functional, Shannon and relative entropies. Finally we give connections to Zipf-Mandelbrot and hybrid Zipf-Mandelbrot entropies.

2010 *Mathematics Subject Classification:* 26D15; 26D10; 94A17

Keywords: Jensen's inequality, n-convex function, Fink's identity, divergence functional, entropy, hybrid Zipf-Mandelbrot entropy

1. Introduction

Jensen's inequality for differentiable convex functions plays a significant role in the field of inequalities as several other inequalities can be seen as special cases of it. It is used in order to make claims regarding the function while just a little is known or is needed to be known about the distribution. It is also used in defining the lower bound on the probability of a random variable. Some key applications covers derivation of AM-GM inequality, estimation of Shannon and non-Shannon entropies, convergence of maximization algorithm and non-negativity of divergence functionals. Taking into consideration the very numerous applications of Jensen's inequality in various fields of mathematics and other applied sciences, the generalizations and improvements of Jensen's inequality has been a topic of supreme interest for the researchers during the last few decades as evident from a large number of publications on the topic see [6, 14, 15, 21, 22] and the references therein.

The first author was supported by H.E.C. Pakistan under NRPU project 7906. The fourth author has been supported by the RUDN University Strategic Academic Leadership Program.

The well-known Jensen's inequality asserts that for the function Γ holds

$$\Gamma\Big(\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}} p_{\jmath} x_{\jmath}\Big) \leq \frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}} p_{\jmath}\Gamma(x_{\jmath}), \tag{1.1}$$

if Γ is a convex function on interval $I \subset \mathbb{R}$, where $p_{\jmath}$ be positive real numbers and $x_{\jmath} \in I$ ($\jmath = 1,\ldots,\mathfrak{r}$), while $P_{\mathfrak{r}} = \sum_{\jmath=1}^{\mathfrak{r}} p_{\jmath}$.

However the well known integral analogue of Jensen's inequality is:

Theorem 1. *Let $\hbar : [\nu_1,\nu_2] \to [\varrho_1,\varrho_2]$ be a continuous function and $\lambda : [\nu_1,\nu_2] \to \mathbb{R}$ is an increasing and bounded function with $\lambda(\nu_1) \neq \lambda(\nu_2)$. Then for every continuous convex function $\Gamma : [\varrho_1,\varrho_2] \to \mathbb{R}$, the following inequality holds*

$$\Gamma(\widetilde{\hbar}) \leq \frac{\int_{\nu_1}^{\nu_2} \Gamma(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)}, \tag{1.2}$$

where

$$\widetilde{\hbar} = \frac{\int_{\nu_1}^{\nu_2} \hbar(\zeta)\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)} \in [\varrho_1,\varrho_2]. \tag{1.3}$$

There are several inequalities coming from Jensen's inequality both in integral and discrete cases which can be obtained by varying conditions on the function $\hbar$ and measure λ defined in Theorem 1.

Jensen Steffensen's Conditions. If $\hbar$ is continuous monotonic and λ is a continuous or a function of bounded variation such that

$$\lambda(\nu_1) \leq \lambda(x) \leq \lambda(\nu_2), \ \ \forall\, x \in [\nu_1,\nu_2], \ \ \lambda(\nu_2) > \lambda(\nu_1), \tag{1.4}$$

then (1.2) holds and is called Jensen Steffensen's inequality given by Boas [4]. Boas [4] also gave a generalization of the above inequality [19, p. 59].

Jensen Boas Conditions. Let λ be a continuous or a function of bounded variation such that

$$\lambda(\nu_1) \leq \lambda(x_1) \leq \lambda(y_1) \leq \lambda(x_2) \leq \lambda(y_2) \leq \ldots \leq \lambda(y_{\mathfrak{r}-1}) \leq \lambda(x_{\mathfrak{r}}) \leq \lambda(\nu_2) \tag{1.5}$$

$\forall\, x_k \in (y_{k-1},y_k)$ $(y_0 = \nu_1, y_{\mathfrak{r}} = \nu_2)$ and $\lambda(\nu_2) > \lambda(\nu_1)$. If $\hbar$ is continuous and monotonic on the $\mathfrak{r}$ intervals (y_{k-1},y_k) then again (1.2) holds and called Jensen-Boas inequality.

In 1992, Fink introduced a novel representation of real $\mathfrak{n}-$times differentiable function whose $\mathfrak{n}-$th derivative $(\mathfrak{n} \geq 1)$ is absolutely continuous by connecting Taylor series and Peano kernel approach together in an identity given as:

Theorem 2 ([10]). *Let $\varrho_1, \varrho_2 \in \mathbb{R}$, $\Gamma : [\varrho_1, \varrho_2] \to \mathbb{R}$, $\mathfrak{n} \geq 1$ and $\Gamma^{(\mathfrak{n}-1)}$ is absolutely continuous on $[\varrho_1, \varrho_2]$. Then*

$$\Gamma(x) = \frac{\mathfrak{n}}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \Gamma(s)\,ds - \sum_{z=1}^{\mathfrak{n}-1} \left(\frac{\mathfrak{n} - z}{z!}\right) \left(\frac{\Gamma^{(z-1)}(\varrho_1)(x - \varrho_1)^z - \Gamma^{(z-1)}(\varrho_2)(x - \varrho_2)^z}{\varrho_2 - \varrho_1}\right)$$

$$+ \frac{1}{(\mathfrak{n}-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} (x - s)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}(s, x) \Gamma^{(\mathfrak{n})}(s)\,ds \tag{1.6}$$

where

$$F_{\varrho_1}^{\varrho_2}(s, x) = \begin{cases} s - \varrho_1, & \varrho_1 \leq s \leq x \leq \varrho_2, \\ s - \varrho_2, & \varrho_1 \leq x < s \leq \varrho_2. \end{cases} \tag{1.7}$$

Fink's identity helps to give more insight of bounds on the deviation of a function from its averages [10]. One can get Ostrowski's inequality and formulate its new variants with improve and optimal quadrature formulas using Fink's identity [9]. It is used to generalize renowned Guessab-Schmeisser and Popoviciu's inequalities for $\mathfrak{n}-$times differentiable function, for instance see [1, 7].

2. Generalization of integral Jensen's inequality by Fink's identity

Before giving our main results, we consider the following assumptions that we use throughout our paper:

- M_1**:** $\hbar : [\nu_1, \nu_2] \to \mathbb{R}$ be continuous function such that $\hbar([\nu_1, \nu_2]) \subset [\varrho_1, \varrho_2]$.
- M_2**:** $\lambda : [\nu_1, \nu_2] \to \mathbb{R}$ be a continuous function or the functions of bounded variation such that $\lambda(\nu_1) \neq \lambda(\nu_2)$.

In our first main result we employ Fink's identity to obtain the following real Stieltjes measure theoretic representations of Jensen's inequality.

Theorem 3. *Let $\hbar, \lambda$ be as defined in M_1, M_2 and $\Gamma : [\varrho_1, \varrho_2] \to \mathbb{R}$ be such that $\Gamma^{(\mathfrak{n}-1)}$ is absolutely continuous for $\mathfrak{n} \geq 1$. Suppose, that Γ is $\mathfrak{n}-$convex such that*

$$\left(\widetilde{\hbar} - s\right)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}\left(s, \widetilde{\hbar}\right) \leq \frac{\int_{\nu_1}^{\nu_2} (\hbar(\zeta) - s)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}(s, \hbar(\zeta))\,\mathrm{d}\lambda(\zeta)}{\int_a^b \mathrm{d}\lambda(\zeta)}, \quad s \in [\varrho_1, \varrho_2] \tag{2.1}$$

with $\widetilde{\hbar}$ and $F_{\varrho_1}^{\varrho_2}(s, \cdot)$ be as defined in (1.3) and (1.7), respectively. Then we have

$$\Gamma(\widetilde{\hbar}) - \frac{\int_{\nu_1}^{\nu_2} \Gamma(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)} \leq \sum_{z=2}^{\mathfrak{n}-1} \left(\frac{\mathfrak{n} - z}{z!(\varrho_2 - \varrho_1)}\right) \tag{2.2}$$

$$\times\left\{\Gamma^{(z-1)}(\varrho_2)\left(\left(\widetilde{\hbar}-\varrho_2\right)^z-\frac{\int_{\nu_1}^{\nu_2}(\hbar(\zeta)-\varrho_2)^z\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\right)-\Gamma^{(z-1)}(\varrho_1)\left(\left(\widetilde{\hbar}-\varrho_1\right)^z-\frac{\int_{\nu_1}^{\nu_2}(\hbar(\zeta)-\varrho_1)^z\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\right)\right\}.$$

Proof. As $\Gamma^{(\mathfrak{n}-1)}$ is absolutely continuous for $(\mathfrak{n}\geq 1)$, we can use the representation of Γ using Fink's identity (1.6) and can calculate

$$\Gamma(\widetilde{\hbar})=\frac{\mathfrak{n}}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Gamma(s)\,ds$$
$$+\sum_{z=1}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right)\left\{\Gamma^{(z-1)}(\varrho_2)\left(\left(\widetilde{\hbar}-\varrho_2\right)^z\right)-\Gamma^{(z-1)}(\varrho_1)\left(\left(\widetilde{\hbar}-\varrho_1\right)^z\right)\right\}$$
$$+\frac{1}{(\mathfrak{n}-1)!(\varrho_2-\varrho_1)}\int_{\varrho_1}^{\varrho_2}\left(\left(\widetilde{\hbar}-s\right)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}\left(s,\widetilde{\hbar}\right)\right)\Gamma^{(\mathfrak{n})}(s)\,ds.$$

The integration of the composition of functions $\Gamma\circ\hbar$ for the real measure λ on $[\nu_1,\nu_2]$ gives

$$\frac{\int_{\nu_1}^{\nu_2}\Gamma(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}=\frac{\mathfrak{n}}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Gamma(s)\,ds+\sum_{z=1}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right)$$
$$\times\left\{\Gamma^{(z-1)}(\varrho_2)\left(\frac{\int_{\nu_1}^{\nu_2}(\hbar(\zeta)-\varrho_2)^z\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\right)-\Gamma^{(z-1)}(\varrho_1)\left(\frac{\int_{\nu_1}^{\nu_2}(\hbar(\zeta)-\varrho_1)^z\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\right)\right\}$$
$$+\frac{1}{(\mathfrak{n}-1)!(\varrho_2-\varrho_1)}\int_{\varrho_1}^{\varrho_2}\left(\frac{\int_{\nu_1}^{\nu_2}(\hbar(\zeta)-s)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}(s,\hbar(\zeta))\,\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\right)\Gamma^{(\mathfrak{n})}(s)\,ds.$$

Now computing the difference $\Gamma(\widetilde{\hbar})-\frac{\int_{\nu_1}^{\nu_2}\Gamma(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}$, we get the following generalized identity involving real Stieltjes measure:

$$\Gamma(\widetilde{\hbar})-\frac{\int_{\nu_1}^{\nu_2}\Gamma(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}=\sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right) \tag{GI.1}$$

$$\times\left\{\Gamma^{(z-1)}(\varrho_2)\left(\left(\widetilde{\hbar}-\varrho_2\right)^z-\frac{\int_{\nu_1}^{\nu_2}(\hbar(\zeta)-\varrho_2)^z\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\right)-\Gamma^{(z-1)}(\varrho_1)\left(\left(\widetilde{\hbar}-\varrho_1\right)^z-\frac{\int_{\nu_1}^{\nu_2}(\hbar(\zeta)-\varrho_1)^z\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\right)\right\}$$

$$+\frac{1}{(\mathfrak{n}-1)!(\varrho_2-\varrho_1)}\int_{\varrho_1}^{\varrho_2}\left(\left(\widetilde{\hbar}-s\right)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}\left(s,\widetilde{\hbar}\right)-\frac{\int_{\nu_1}^{\nu_2}(\hbar(\zeta)-s)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}(s,\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\right)\Gamma^{(\mathfrak{n})}(s)\,ds.$$

Finally by our assumption $\Gamma^{(\mathfrak{n}-1)}$ is absolutely-continuous on $[\varrho_1,\varrho_2]$, as a result $\Gamma^{(\mathfrak{n})}$ exists almost everywhere. Moreover, Γ is supposed to be $\mathfrak{n}-$convex, so we have $\Gamma^{(\mathfrak{n})}(x)\geq 0$ almost everywhere on $[\varrho_1,\varrho_2]$. Therefore by taking into account the last term in generalized identity (GI.1) and inequality (2.1), we get (2.2). □

In the later part of this section, we will employ convexity properties and Theorem 3 by alternating conditions on functions $\hbar$ and Stieltjes measure $\mathrm{d}\lambda$ to obtain generalized variants of Jensen–Steffensen's, Jensen–Boas, Jensen–Brunk and Jensen type inequalities. We start with the following generalization of Jensen–Steffensen's inequality for $\mathfrak{n}-$convex functions.

Theorem 4. *Let Γ be as defined in Theorem 3 is $\mathfrak{n}-$convex and $\hbar$ be as defined in M_1 is monotonic. Then the following results hold.*

(i) *Let λ be as defined in M_2 satisfying*

$$\lambda(\nu_1)\leq\lambda(x)\leq\lambda(\nu_2),\ \ \forall\ x\in[\nu_1,\nu_2],\ \ \lambda(\nu_2)>\lambda(\nu_1).$$

Then for even $n\geq 3$, (2.1) is valid.

(ii) *Moreover if (2.1) is valid and the function*

$$H(x):=\sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!\varrho_2-\varrho_1}\right)\left(\Gamma^{(z-1)}(\varrho_1)(x-\varrho_1)^z-\Gamma^{(z-1)}(\varrho_2)(x-\varrho_2)^z\right) \tag{2.3}$$

is convex, then we get inequality (1.2) and it is called the generalized Jensen–Steffensen's inequality for $\mathfrak{n}-$convex functions.

Proof. (i) We consider the following function in the remainder term of Fink's identity as:

$$\wp(x) := (x-\xi)^{n-1} F_{\varrho_1}^{\varrho_2}(\xi,x) = \begin{cases} (x-\xi)^{n-1}(\xi-\varrho_1), & \xi \leq x \leq \varrho_2, \\ (x-\xi)^{n-1}(\xi-\varrho_2), & x < \xi \leq \varrho_2. \end{cases}$$

Now taking derivative twice, we have

$$\wp''(x) := \begin{cases} (n-1)(n-2)(x-\xi)^{n-3}(\xi-\varrho_1), & \xi \leq x \leq \varrho_2, \\ (n-1)(n-2)(x-\xi)^{n-3}(\xi-\varrho_2), & x < \xi \leq \varrho_2. \end{cases}$$

By applying the second derivative test, the function $\wp$ is convex for even $\mathfrak{n} > 3$. Now using the assumed conditions one can employ Jensen Steffensen's inequality given by Boas (see [4] or [19, p. 59]) for the convex function $\wp(x)$ to obtain (2.1).

(ii) We can rewrite the r.h.s. of (2.2) in the difference

$$H(\widetilde{\hbar}) - \frac{\int_{\nu_1}^{\nu_2} H(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)}.$$

For the convex function H and by our assumed conditions on functions $\hbar$ and λ, this difference is non-positive by using Jensen–Steffensen's inequality difference [4]. As a result, the r.h.s. of inequality (2.2) is non-positive and we get the generalized Jensen–Steffensen's inequality (1.2) for $\mathfrak{n}-$convex functions. □

Now we give a similar result related to Jensen-Boas inequality [19, p. 59], that is a generalization of the Jensen–Steffensen's inequality:

Corollary 1. *Let Γ be as defined in Theorem 3 is $\mathfrak{n}-$convex function. Also let $\hbar$ be as defined in M_1 with $\nu_1 = y_0 < y_1 < \ldots < y_k < \ldots < y_{\mathfrak{r}-1} < y_{\mathfrak{r}} = \nu_2$ and $\hbar$ is monotonic in each of the $\mathfrak{r}$ intervals $((y_{k-1}, y_k))$. Then the following results hold.*

(i) *Let λ be as defined in M_2 satisfying*

$$\lambda(\nu_1) \leq \lambda(x_1) \leq \lambda(y_1) \leq \lambda(x_2) \leq \lambda(y_2) \leq \ldots \leq \lambda(y_{\mathfrak{r}-1}) \leq \lambda(x_{\mathfrak{r}}) \leq \lambda(\nu_2)$$

$\forall\ x_k \in (y_{k-1}, y_k)$ *and $\lambda(\nu_2) > \lambda(\nu_1)$. Then for even $\mathfrak{n} \geq 3$, (2.1) is valid.*

(ii) *Moreover if (2.1) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then again inequality (1.2) holds and is called Jensen–Boas inequality for $\mathfrak{n}-$convex functions.*

Proof. We follow the similar idea as in the proof of Theorem 4 but under the conditions of this corollary, we utilize Jensen–Boas inequality (see [4] or [19, p. 59]) instead of Jensen-Steffensen's inequality. □

Next we give results for Jensen–Brunk inequality.

Corollary 2. *Let Γ be as defined in Theorem 3 is $\mathfrak{n}-$convex and $\hbar$ be as defined in M_1 is an increasing function. Then the following results hold.*

(i) *Let λ be as defined in M_2 with $\lambda(\nu_2) > \lambda(\nu_1)$,*

$$\int_{\nu_1}^{x} \Big(\hbar(x) - \hbar(\zeta)\Big) \mathrm{d}\lambda(\zeta) \geq 0 \ \text{ and } \int_{x}^{\nu_2} \Big(\hbar(x) - \hbar(\zeta)\Big) \mathrm{d}\lambda(\zeta) \leq 0$$

$\forall\ x \in [\nu_1, \nu_2]$ *hold. Then for even $\mathfrak{n} \geq 3$, (2.1) is valid.*

(ii) *Moreover if (2.1) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then again inequality (1.2) holds and is called Jensen–Brunk inequality for $\mathfrak{n}-$convex functions.*

Proof. We follow the similar idea as in the proof of Theorem 4 but under the conditions of this corollary, we employ Jensen–Brunk inequality (see [5] or [19, p. 59]) instead of Jensen-Steffensen's inequality. □

Remark 1. The similar result in Corollary 2 is also valid provided that the function $\hbar$ is decreasing. Also assuming that the function $\hbar$ is monotonic one can replace the conditions in Corollary 2(i) by

$$0 \leq \int_{\nu_1}^{x} \Big|\hbar(x) - \hbar(\zeta)\Big| \mathrm{d}\lambda(\zeta) \leq \int_{x}^{\nu_2} \Big|\hbar(x) - \hbar(\zeta)\Big| \mathrm{d}\lambda(\zeta).$$

Remark 2. It is interesting to see that by employing a similar method as in Theorem 4, we can also get the generalization of classical Jensen's inequality (1.2) for $\mathfrak{n}-$convex functions by assuming the functions $\hbar$ and λ along with there respective conditions in Theorem 1.

Another important application of Theorem 3 is by setting the function $\hbar$ as $\hbar(\zeta) = \zeta$ gives the generalized version of the r. h. s. inequality of the Hermite-Hadamard inequality:

Corollary 3. *Let $\lambda : [\nu_1, \nu_2] \to \mathbb{R}$ be a function of bounded variation such that $\lambda(\nu_1) \neq \lambda(\nu_2)$ with $[\nu_1, \nu_2] \subset [\varrho_1, \varrho_2]$ and $\widetilde{\zeta} = \frac{\int_{\nu_1}^{\nu_2} \zeta \, \mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)} \in [\varrho_1, \varrho_2]$. Under the assumptions of Theorem 3, if Γ is $\mathfrak{n}-$convex such that*

$$\Big(\widetilde{\zeta} - s\Big)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}\Big(s, \widetilde{\zeta}\Big) \leq \frac{\int_{\nu_1}^{\nu_2} (\zeta - s)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}(s, \zeta) \, \mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)}, \quad s \in [\varrho_1, \varrho_2], \tag{2.4}$$

then we have

$$\Gamma(\widetilde{\zeta}) \leq \frac{\int_{\nu_1}^{\nu_2} \Gamma(\zeta) \mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)} + \sum_{z=2}^{\mathfrak{n}-1} \left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)} \right) \tag{2.5}$$

$$\times \left\{ \Gamma^{(z-1)}(\varrho_2) \left(\left(\widetilde{\zeta}-\varrho_2\right)^z - \frac{\int_{\nu_1}^{\nu_2} (\zeta-\varrho_2)^z \mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)} \right) - \Gamma^{(z-1)}(\varrho_1) \left(\left(\widetilde{\zeta}-\varrho_1\right)^z - \frac{\int_{\nu_1}^{\nu_2} (t-\varrho_1)^z \mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)} \right) \right\}.$$

If inequality (2.4) *holds in the reverse direction, then* (2.5) *also holds reversely.*

The special case of the above corollary can be given in the form of the following remark:

Remark 3. It is interesting to see that substituting $\lambda(\zeta) = \zeta$ gives $\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta) = \nu_2 - \nu_1$ and $\widetilde{\zeta} = \frac{\nu_1+\nu_2}{2}$. Using these substitutions in (1.2) and by following Remark 2 we get the l.h.s. inequality of the renowned Hermite Hadamard inequality for $\mathfrak{n}-$convex functions.

3. Generalization of discrete Jensen's inequality by Fink's identity

In this section, we give generalizations for the discrete case by using Fink's identity. The proofs are similar to that of continuous case were given in the previous section, therefore we give results directly. In discrete case we have that $p_\jmath > 0$ for all $\jmath = 1,2,\ldots,\mathfrak{r}$. Here we give generalizations of results allowing $p_\jmath$ to be negative real numbers. Also with usual notations for $p_\jmath x_\jmath$ $(\jmath = 1,2,\ldots,\mathfrak{r})$, we notate the $\mathfrak{r}-$tuples

$$\mathbf{x} = (x_1, x_2, \ldots, x_\mathfrak{r}) \text{ and } \mathbf{p} = (p_1, p_2, \ldots, p_\mathfrak{r}),$$

$$P_l = \sum_{\jmath=1}^{l} p_\jmath, \ \overline{P}_l = P_\mathfrak{r} - P_{l-1} \ (l = 1,2,\ldots,\mathfrak{r})$$

and

$$\overline{x} = \frac{1}{P_\mathfrak{r}} \sum_{\jmath=1}^{\mathfrak{r}} p_\jmath x_\jmath .$$

Using Fink's identity (1.6), we obtain the following representations of discrete Jensen's inequality.

Theorem 5. *Let* $\Gamma : [\varrho_1, \varrho_2] \to \mathbb{R}$ *be such that for* $\mathfrak{n} \geq 1$, $\Gamma^{(\mathfrak{n}-1)}$ *is absolutely continuous. Also let* $x_\jmath \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$, $p_\jmath \in \mathbb{R}(\jmath = 1,\ldots,\mathfrak{r})$, *be such that* $P_\mathfrak{r} \neq 0$ *and* $\overline{x} \in [\varrho_1, \varrho_2]$.

(i) *Then the following generalized identity holds*

$$\Gamma(\overline{x})-\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}}p_{\jmath}\Gamma(x_{\jmath})=\sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right) \tag{DGI.1}$$

$$\times\left\{\Gamma^{(z-1)}(\varrho_2)\left((\overline{x}-\varrho_2)^z-\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}}p_{\jmath}(x_{\jmath}-\varrho_2)^z\right)-\Gamma^{(z-1)}(\varrho_1)\left((\overline{x}-\varrho_1)^z-\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}}p_{\jmath}(x_{\jmath}-\varrho_1)^z\right)\right\}$$

$$+\frac{1}{(\mathfrak{n}-1)!(\varrho_2-\varrho_1)}\int_{\varrho_1}^{\varrho_2}\left[\left((\overline{x}-s)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}(s,\overline{x})\right)-\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}}p_{\jmath}\left(\left(x_{\jmath}-s\right)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}\left(s,x_{\jmath}\right)\right)\right]\Gamma^{(\mathfrak{n})}(s)\,ds,$$

where $F_{\varrho_1}^{\varrho_2}(s,\cdot)$ *is defined in* (1.7).

(ii) *Moreover, if* Γ *is* $\mathfrak{n}-$*convex and the inequality*

$$(\overline{x}-s)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}(s,\overline{x})\leq\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}}p_{\jmath}\left(\left(x_{\jmath}-s\right)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}\left(s,x_{\jmath}\right)\right) \tag{3.1}$$

holds, then we have the following generalized inequality

$$\Gamma(\overline{x})-\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}}p_{\jmath}\Gamma(x_{\jmath})\leq\sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right) \tag{3.2}$$

$$\times\left\{\Gamma^{(z-1)}(\varrho_2)\left((\overline{x}-\varrho_2)^z-\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}}p_{\jmath}(x_{\jmath}-\varrho_2)^z\right)-\Gamma^{(z-1)}(\varrho_1)\left((\overline{x}-\varrho_1)^z-\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}}p_{\jmath}(x_{\jmath}-\varrho_1)^z\right)\right\}.$$

If inequality (3.1) *holds in the reverse direction, then* (3.2) *also holds reversely.*

Proof. Similar to that of Theorem 3. □

In the later part of this section, we will vary our conditions on $p_{\jmath}x_{\jmath}$ ($\jmath=1,2,\ldots,\mathfrak{r}$) to obtain generalized discrete variants of Jensen–Steffensen's, Jensen's and Jensen-Petrović type inequalities. We start with the following generalization of discrete Jensen–Steffensen's inequality for $\mathfrak{n}-$convex functions:

Theorem 6. *Let* Γ *be as defined in Theorem* 5*. Also let* $\mathbf{x}$ *be monotonic* $\mathfrak{r}-$*tuple,* $x_{\jmath}\in[\nu_1,\nu_2]\subseteq[\varrho_1,\varrho_2]$ *and* $\mathbf{p}$ *be a real* $\mathfrak{r}-$*tuple such that*

$$0\leq P_l\leq P_{\mathfrak{r}},\ (l=1,2,\ldots,\mathfrak{r}-1),\quad P_{\mathfrak{r}}>0$$

is satisfied.

(i) *If* Γ *is* $\mathfrak{n}-$*convex, then for even* $\mathfrak{n}\geq 3$, (3.1) *is valid.*

(ii) *Moreover if* (3.1) *is valid and the function* $H(\cdot)$ *defined in* (2.3) *is convex, then we get the following generalized discrete Jensen–Steffensen's inequality*

$$\Gamma(\overline{x})\leq\frac{1}{P_{\mathfrak{r}}}\sum_{\jmath=1}^{\mathfrak{r}}p_{\jmath}\Gamma(x_{\jmath}). \tag{3.3}$$

Proof. It is interesting to see that under the assumed conditions on tuples $\mathbf{x}$ and $\mathbf{p}$, we have that $\overline{x} \in [\nu_1, \nu_2]$. Since for $x_1 \geq x_2 \geq \ldots \geq x_{\mathfrak{r}}$,

$$P_{\mathfrak{r}}(x_1 - \overline{x}) = \sum_{\jmath=2}^{\mathfrak{r}} p_\jmath (x_1 - x_\jmath) = \sum_{l=2}^{\mathfrak{r}} (x_{l-1} - x_l)(P_{\mathfrak{r}} - P_{l-1}) \geq 0.$$

This shows that $x_1 \geq \overline{x}$. Also $\overline{x} \geq x_{\mathfrak{r}}$, since we have

$$P_{\mathfrak{r}}(\overline{x} - x_{\mathfrak{r}}) = \sum_{\jmath=1}^{\mathfrak{r}-1} p_\jmath (x_\jmath - x_{\mathfrak{r}}) = \sum_{l=1}^{\mathfrak{r}-1} (x_l - x_{l-1}) P_l \geq 0.$$

For further details see the proof of discrete Jensen–Steffensen's inequality [19, p. 57]. The idea of the rest of the proof is similar to that of Theorem 3, but here we employ Theorem 5 and discrete Jensen–Steffensen's inequality. □

Corollary 4. *Let Γ be as defined in Theorem 5 and let $x_\jmath \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$ with a positive $\mathfrak{r}-$tuple* $\mathbf{p}$.

(i) *If Γ is $\mathfrak{n}-$convex, then for even $\mathfrak{n} \geq 3$, (3.1) is valid.*

(ii) *Moreover if (3.1) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then again we get (3.3) and is called Jensen's inequality for $\mathfrak{n}-$convex functions.*

Proof. For $p_\jmath > 0$, $x_\jmath \in [\nu_1, \nu_2]$ ($\jmath = 1, 2, 3, \ldots, \mathfrak{r}$) ensures that $\overline{x} \in [\nu_1, \nu_2]$. So by applying classical discrete Jensen inequality (1.1) and the idea of Theorem 6 we will get the required results. □

Remark 4. Under the assumptions of Corollary 4, if we choose $P_{\mathfrak{r}} = 1$, then Corollary 4(ii) gives the following inequality for $\mathfrak{n}-$convex functions:

$$\Gamma\left(\sum_{\jmath=1}^{\mathfrak{r}} p_\jmath x_\jmath\right) \leq \sum_{\jmath=1}^{\mathfrak{r}} p_\jmath \Gamma(x_\jmath). \tag{3.4}$$

Now we give the following reverses of Jensen–Steffensen's and Jensen type inequalities:

Corollary 5. *Let Γ be as defined in Theorem 5. Also let* $\mathbf{x}$ *be a monotonic $\mathfrak{r}-$tuple, $x_\jmath \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$ and* $\mathbf{p}$ *be a real $\mathfrak{r}-$tuple such that there exist $m \in \{1, 2, \ldots, \mathfrak{r}\}$ with*

$$0 \geq P_l, \text{ for } l < m \text{ and } 0 \geq \overline{P}_l, \text{ for } l > m,$$

where $P_{\mathfrak{r}} > 0$ and $\overline{x} \in [\varrho_1, \varrho_2]$.

(i) *If Γ is $\mathfrak{n}-$convex, then for even $\mathfrak{n} \geq 3$, the reverse of inequality (3.1) holds.*

(ii) *Moreover if (3.1) holds reversely and the function $H(\cdot)$ defined in (2.3) is convex, then we get the reverse of generalized Jensen–Steffensen inequality (3.3) for $\mathfrak{n}-$convex functions.*

Proof. We follow the idea of Theorem 6 but according to our assumed conditions we employ the reverse of Jensen-Steffensen inequality to obtain results. □

In the next corollary we give explicit conditions on real tuple $\mathbf{p}$ such that we get the reverse of classical Jensen inequality:

Corollary 6. *Let Γ be as defined in Theorem 5 and let $x_\jmath \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$ such that $\overline{x} \in [\varrho_1, \varrho_2]$. Let $\mathbf{p}$ be a real $\mathfrak{r}-$tuple such that*

$$0 < p_1, \ 0 \geq p_2, p_3, \ldots, p_{\mathfrak{r}}, \ 0 < P_{\mathfrak{r}}$$

is satisfied.

(i) *If Γ is $\mathfrak{n}-$convex, then for even $n \geq 3$, the reverse of inequality (3.1) is valid.*
(ii) *Also if reverse of (3.1) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the reverse of (3.3).*

Proof. We follow the idea of Theorem 6 but according to our assumed conditions we employ the reverse of Jensen inequality to obtain results. □

In [2] (see also [19]) one can find the result which is equivalent to the Jensen–Steffensen and the reverse Jensen–Steffensen inequality together. It is the so-called Jensen-Petrović inequality. Here, without the proof, we give the adequate corollary which uses that result. The proof goes the same way as in the previous corollaries.

Corollary 7. *Let Γ be as defined in Theorem 5 and let $x_i \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$ be such that $x_{\mathfrak{r}} \geq x_{\mathfrak{r}-1}, \ldots, x_2 \geq x_1$. Let $\mathbf{p}$ be a real $\mathfrak{r}-$tuple with $P_{\mathfrak{r}} = 1$ such that*

$$0 \leq P_l, \text{ for } 1 \leq l < \mathfrak{r} - 1 \text{ and } 0 \leq \overline{P}_l, \text{ for } 2 \leq l < \mathfrak{r}$$

is satisfied. Then we get the equivalent results given in Theorem 6 (i) and (ii) respectively.

Remark 5. Under the assumptions of Corollary 7, if there exist $m \in \{1, 2, \ldots, \mathfrak{r}\}$ such that

$$0 \geq P_l, \text{ for } l < m \text{ and } 0 \geq \overline{P}_l, \text{ for } l > m$$

and $\overline{x} \in [\varrho_1, \varrho_2]$, then we get equivalent results for the reverse Jensen–Steffensen's inequality given in Corollary 5(i) and (ii) respectively.

Remark 6. It is interesting to see that the conditions on $p_\jmath, \jmath = 1, 2, \ldots, \mathfrak{r}$ given in Corollary 7 and Remark 5 are coming from Jensen-Petrović inequality which becomes equivalent to conditions for $p_\jmath, \jmath = 1, 2, \ldots, \mathfrak{r}$ for Jensen–Steffensen's results given in Theorem 6 and Corollary 5 respectively when $P_{\mathfrak{r}} = 1$.

Now we give results for Jensen and its reverses for $\mathfrak{r}-$tuples $\mathbf{x}$ and $\mathbf{p}$ when $\mathfrak{r}$ is an odd number.

Corollary 8. *Let Γ be as defined in Theorem 5 and let $x_\jmath \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$ for $\jmath = 1, 2, \ldots, \mathfrak{r}$ be such that $\mathbf{x}$, $\mathbf{p}$ be real $\mathfrak{r}-$tuples, $\mathfrak{r} = 2m + 1$, $m \in \mathbb{N}$ and $\hat{x} = \frac{1}{\sum_{\jmath=1}^{2k+1} p_\jmath} \sum_{\jmath=1}^{2k+1} p_\jmath x_\jmath \in [\varrho_1, \varrho_2]$ for all $k = 1, 2, \ldots, m$. If for every $k = 1, 2, \ldots, m$, we have* (i*) $p_1 > 0,\ p_{2k} \le 0,\ p_{2k} + p_{2k+1} \le 0,\ \sum_{\jmath=1}^{2k} p_\jmath \ge 0,\ \sum_{\jmath=1}^{2k+1} p_\jmath > 0$

and

(ii*) $x_{2k} \le x_{2k+1},\ \sum_{\jmath=1}^{2k} p_\jmath (x_\jmath - x_{2k+1}) \ge 0,$

then we have the following statements to be valid:

(i) *If* Γ *is* $\mathfrak{n}-$*convex, then for even* $\mathfrak{n} \ge 3$,

$$(\hat{x} - s)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}(s, \hat{x}) \ge \frac{1}{P_{2m+1}} \sum_{\jmath=1}^{2m+1} p_\jmath \left(\left(x_\jmath - s\right)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}\left(s, x_\jmath\right)\right). \tag{3.5}$$

(ii) *Also if* (3.5) *is valid and the function* $H(\cdot)$ *defined in* (2.3) *is convex, then we get the following generalized inequality*

$$\Gamma(\hat{x}) \ge \frac{1}{P_{2m+1}} \sum_{\jmath=1}^{2m+1} p_\jmath \Gamma(x_\jmath). \tag{3.6}$$

Proof. We employ the idea of the proofs of Theorem 5 and Theorem 6 for $\mathfrak{r} = odd$ along with inequality of P. M. Vasić and J. R. Janic [18]. □

Remark 7. We can also obtain inequalities (3.5) and (3.6) along with their reverses respectively given in Corollary 8 by employing the important cases and corresponding explicit conditions given in [18].

4. Generalization of converse of integral Jensen's inequality by Fink's identity

In this section, we give results for the converse of the Jensen's inequality to hold, giving the conditions on the real Stieltjes measure $d\lambda$, such that $\lambda(\nu_1) \ne \lambda(\nu_2)$, allowing that the measure can also be negative, but employing Fink's identity.

To start we need the following assumption for the results of this section:

M_3**:** Let $m, M \in [\varrho_1, \varrho_2]$ $(m \ne M)$ be such that $m \le \hbar(\zeta) \le M$ for all $\zeta \in [\nu_1, \nu_2]$ where $\hbar$ is defined in M_1.

For a given function $\Gamma : [\varrho_1, \varrho_2] \to R$, we consider the difference

$$CJ(\Gamma, \hbar_{\{m,M\}}; \lambda) = \frac{\int_{\nu_1}^{\nu_2} \Gamma(\hbar(\zeta)) d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} - \frac{M - \widetilde{\hbar}}{M - m}\Gamma(m) - \frac{\widetilde{\hbar} - m}{M - m}\Gamma(M), \tag{4.1}$$

where $\widetilde{\hbar}$ is defined in (1.3).

Using Fink's identity, we obtain the following representation of the converse of Jensen's inequality.

Theorem 7. *Let $\hbar, \lambda$ be as defined in M_1, M_2 and let $\Gamma : [\varrho_1, \varrho_2] \to \mathbb{R}$ be such that $\Gamma^{(\mathfrak{n}-1)}$ is absolutely continuous for $\mathfrak{n} \geq 1$. If Γ is $\mathfrak{n}-$convex such that*

$$CJ\Big((x-s)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}(s,x), \hbar_{\{m,M\}}; \lambda\Big) \leq 0, \quad s \in [\varrho_1, \varrho_2], \tag{4.2}$$

or

$$\frac{\int_{\nu_1}^{\nu_2} (\hbar(\zeta)-s)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}(s,\hbar(\zeta))\, \mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)} \leq \frac{M-\widetilde{\hbar}}{M-m}\Big(F_{\varrho_1}^{\varrho_2}(s,m)(m-s)^{\mathfrak{n}-1}\Big) \tag{4.3}$$

$$+\frac{\widetilde{\hbar}-m}{M-m}\Big(F_{\varrho_1}^{\varrho_2}(s,M)(M-s)^{\mathfrak{n}-1}\Big), \quad s \in [\varrho_1, \varrho_2],$$

then we get the following extension of the converse of the Jensen's difference

$$\frac{\int_{\nu_1}^{\nu_2} \Gamma(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} \mathrm{d}\lambda(\zeta)} \leq \frac{M-\widetilde{\hbar}}{M-m}\Gamma(m) + \frac{\widetilde{\hbar}-m}{M-m}\Gamma(M) + \sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right) \tag{4.4}$$

$$\times\Big(\Gamma^{(z-1)}(\varrho_2)CJ((x-\varrho_2)^z, \hbar_{\{m,M\}}; \lambda) - \Gamma^{(z-1)}(\varrho_1)CJ((x-\varrho_1)^z, \hbar_{\{m,M\}}; \lambda)\Big).$$

where $F_{\varrho_1}^{\varrho_2}(s,\cdot)$ is defined in (1.7).

Proof. As $\Gamma^{(\mathfrak{n}-1)}$ is absolutely continuous for $\mathfrak{n} \geq 1$ we can use the representation of Γ using Fink's identity (1.6) in the difference $CJ(\Gamma, \hbar_{\{m,M\}}; \lambda)$:

$$\begin{aligned} CJ(\Gamma, \hbar_{\{m,M\}}; \lambda) = {} & CJ\left(\frac{\mathfrak{n}}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} \Gamma(\zeta)d\zeta, \hbar_{\{m,M\}}; \lambda\right) \\ & + \sum_{z=1}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right)\Gamma^{(z-1)}(\varrho_2)CJ((x-\varrho_2)^z, \hbar_{\{m,M\}}; \lambda) \\ & - \sum_{z=1}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right)\Gamma^{(z-1)}(\varrho_1)CJ((x-\varrho_1)^z, \hbar_{\{m,M\}}; \lambda) \\ & + \frac{1}{(\mathfrak{n}-1)!(\varrho_2-\varrho_1)}\int_{\varrho_1}^{\varrho_2} CJ\Big((x-s)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}(s,x), \hbar_{\{m,M\}}; \lambda\Big)\Gamma^{(\mathfrak{n})}(s)ds. \end{aligned} \tag{4.5}$$

After simplification and following the fact that $CJ(\Gamma, \hbar_{\{m,M\}}; \lambda)$ is zero for Γ to be constant or linear we get the following generalized identity

$$CJ(\Gamma, \hbar_{\{m,M\}}; \lambda) = \sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right) \tag{CGI.1}$$

$$\times\Big(\Gamma^{(z-1)}(\varrho_2)CJ((x-\varrho_2)^z,\hbar_{\{m,M\}};\lambda)-\Gamma^{(z-1)}(\varrho_1)CJ((x-\varrho_1)^z,\hbar_{\{m,M\}};\lambda)\Big)$$
$$+\frac{1}{(\mathfrak{n}-1)!(\varrho_2-\varrho_1)}\int_{\varrho_1}^{\varrho_2} CJ\Big((x-s)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}(s,x),\hbar_{\{m,M\}};\lambda\Big)\Gamma^{(\mathfrak{n})}(s)ds.$$

Now using characterizations of $\mathfrak{n}-$convex functions like in the proof of Theorem 3, we get (4.4). □

The next result gives the converse of Jensen's inequality for higher order convex functions.

Theorem 8. *Let Γ be as defined in Theorem 7 is $\mathfrak{n}-$convex and $\hbar$ be as defined in M_3. Then the following results hold.*

(i) *If λ is a non-negative measure on $[\nu_1,\nu_2]$, then for even $\mathfrak{n}\geq 3$, (4.3) is valid.*
(ii) *Moreover if (4.3) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the following inequality for $\mathfrak{n}-$convex functions to be valid*

$$\frac{\int_{\nu_1}^{\nu_2}\Gamma(\hbar(\zeta))\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\leq\frac{M-\widetilde{\hbar}}{M-m}\Gamma(m)-\frac{\widetilde{\hbar}-m}{M-m}\Gamma(M). \tag{4.6}$$

Proof. The idea of the proof is similar to that of Theorem (3), but we use the converse of Jensen's inequality (see [3] or [19, p. 98]). □

Another important consequence of Theorem 7 is by setting the function $\hbar$ as $\hbar(\zeta)=\zeta$ gives the generalized version of the l. h. s. inequality of the Hermite-Hadamard inequality:

Corollary 9. *Let $\lambda:[\nu_1,\nu_2]\to\mathbb{R}$ be a function of bounded variation such that $\lambda(\nu_1)\neq\lambda(\nu_2)$ with $[\nu_1,\nu_2]\subset[\varrho_1,\varrho_2]$ and $\widetilde{\zeta}=\frac{\int_{\nu_1}^{\nu_2}\zeta\,\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\in[\varrho_1,\varrho_2]$. Under the assumptions of Theorem 7, if Γ is $\mathfrak{n}-$convex such that*

$$\frac{\int_{\nu_1}^{\nu_2}(\zeta-s)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}(s,\zeta)\mathrm{d}\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}\mathrm{d}\lambda(\zeta)}\leq\frac{\nu_2-\widetilde{\zeta}}{\nu_2-\nu_1}\Big(F_{\varrho_1}^{\varrho_2}(s,\nu_1)(\nu_1-s)^{\mathfrak{n}-1}\Big) \tag{4.7}$$
$$+\frac{\widetilde{\zeta}-\nu_1}{\nu_2-\nu_1}\Big(F_{\varrho_1}^{\varrho_2}(s,\nu_2)(\nu_2-s)^{\mathfrak{n}-1}\Big),\quad s\in[\varrho_1,\varrho_2],$$

then we have

$$\frac{\int_{\nu_1}^{\nu_2}\Gamma(\zeta)d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2}d\lambda(\zeta)} \leq \frac{\nu_2-\widetilde{\zeta}}{\nu_2-\nu_1}\Gamma(\nu_1)+\frac{\widetilde{\zeta}-\nu_1}{\nu_2-\nu_1}\Gamma(\nu_2)+\sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right) \tag{4.8}$$
$$\times\Big(\Gamma^{(z-1)}(\varrho_2)CJ((x-\varrho_2)^z,id_{\{\nu_1,\nu_2\}};\lambda)-\Gamma^{(z-1)}(\varrho_1)CJ((x-\varrho_1)^z,id_{\{\nu_1,\nu_2\}};\lambda)\Big).$$

If the inequality (4.7) *holds in the reverse direction, then* (4.8) *also holds reversely.*

The special case of the above corollary can be given in the form of the following remark:

Remark 8. It is interesting to see that substituting $\lambda(\zeta)=\zeta$ and by following Theorem 8 we get the r.h.s. inequality of the renowned Hermite-Hadamard inequality for $\mathfrak{n}-$convex functions.

5. Generalization of converse of discrete Jensen's inequality by Fink's identity

In this section, we give the results for the converse of Jensen's inequality in discrete case by using the Fink's identity.

Let $x_\jmath \in [\nu_1,\nu_2] \subseteq [\varrho_1,\varrho_2]$, $\nu_1 \neq \nu_2$, $p_\jmath \in \mathbb{R}(\jmath = 1,...,\mathfrak{r})$ be such that $P_\mathfrak{r} \neq 0$. Then we have the following difference of the converse of Jensen's inequality for $\Gamma : [\varrho_1,\varrho_2] \to \mathbb{R}$:

$$CJ_{dis}(\Gamma)=\frac{1}{P_\mathfrak{r}}\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath\Gamma(x_\jmath)-\frac{\nu_2-\overline{x}}{\nu_2-\nu_1}\Gamma(\nu_1)-\frac{\overline{x}-\nu_1}{\nu_2-\nu_1}\Gamma(\nu_2) \tag{5.1}$$

Similarly, we assume the Giaccardi difference given as:

$$G_{cardi}(\Gamma)=\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath\Gamma(x_\jmath)-A\Gamma\left(\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath x_\jmath\right)-B\left(\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath-1\right)\Gamma(x_0), \tag{5.2}$$

where

$$A=\frac{\left(\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath(x_\jmath-x_0)\right)}{\left(\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath x_\jmath-x_0\right)},\quad B=\frac{\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath x_\jmath}{\left(\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath x_\jmath-x_0\right)}\quad\text{and}\quad\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath x_\jmath\neq x_0.$$

Theorem 9. *Let $\Gamma : [\varrho_1,\varrho_2] \to \mathbb{R}$ be such that for $\mathfrak{n} \geq 1$, $\Gamma^{(\mathfrak{n}-1)}$ is absolutely continuous. Also let $x_0, x_\jmath \in [\nu_1,\nu_2] \subseteq [\varrho_1,\varrho_2]$, $p_\jmath \in \mathbb{R}(\jmath = 1,...,\mathfrak{r})$, be such that $\sum_{\jmath=1}^{\mathfrak{r}} p_\jmath x_\jmath \neq x_0$.*

(i) *Then the following generalized identity hold*

$$CJ_{dis}(\Gamma) = \sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right)\left(\Gamma^{(z-1)}(\varrho_2)CJ_{dis}(\left(x_\jmath-\varrho_2\right)^z)-\Gamma^{(z-1)}(\varrho_1)CJ_{dis}(\left(x_\jmath-\varrho_1\right)^z)\right)$$
$$+\frac{1}{(\mathfrak{n}-1)!(\varrho_2-\varrho_1)}\int_{\varrho_1}^{\varrho_2} CJ_{dis}(\left(x_\jmath-s\right)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}\left(s,x_\jmath\right))\Gamma^{(\mathfrak{n})}(s)ds, \tag{DC.GI}$$

where $F_{\varrho_1}^{\varrho_2}(s,\cdot)$ *is defined in* (1.7).

(ii) *Moreover, if* Γ *is* $\mathfrak{n}-$*convex and the inequality*

$$CJ_{dis}(\left(x_\jmath-s\right)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}\left(s,x_\jmath\right))\le 0 \tag{5.3}$$

holds, then we have the following generalized inequality

$$CJ_{dis}(\Gamma) \le \sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right)\left(\Gamma^{(z-1)}(\varrho_2)CJ_{dis}(\left(x_\jmath-\varrho_2\right)^z)-\Gamma^{(z-1)}(\varrho_1)CJ_{dis}(\left(x_\jmath-\varrho_1\right)^z)\right). \tag{5.4}$$

If inequality (5.3) *holds in the reverse direction, then* (5.4) *also holds reversely.*

Proof. Similar to that of Theorem 5. □

Theorem 10. *Let* $\Gamma:[\varrho_1,\varrho_2]\to\mathbb{R}$ *be such that for* $\mathfrak{n}\ge 1$, $\Gamma^{(\mathfrak{n}-1)}$ *is absolutely continuous. Also let* $x_\jmath\in[\nu_1,\nu_2]\subseteq[\varrho_1,\varrho_2]$, $p_\jmath\in\mathbb{R}(\jmath=1,\dots,\mathfrak{r})$, *be such that* $P_\mathfrak{r}\ne 0$ *and* $\overline{x}\in[\varrho_1,\varrho_2]$.

(i) *Then the following generalized Giaccardi identity hold*

$$G_{cardi}(\Gamma) = \sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right) \tag{GIA.GI}$$
$$\cdot\left(\Gamma^{(z-1)}(\varrho_2)G_{cardi}(\left(x_\jmath-\varrho_2\right)^z)-\Gamma^{(z-1)}(\varrho_1)G_{cardi}(\left(x_\jmath-\varrho_1\right)^z)\right)$$
$$+\frac{1}{(\mathfrak{n}-1)!(\varrho_2-\varrho_1)}\int_{\varrho_1}^{\varrho_2} G_{cardi}(\left(x_\jmath-s\right)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}\left(s,x_\jmath\right))\Gamma^{(\mathfrak{n})}(s)ds$$

where $F_{\varrho_1}^{\varrho_2}(s,\cdot)$ *is defined in* (1.7).

(ii) *Moreover, if* Γ *is* $\mathfrak{n}-$*convex and the inequality*

$$G_{cardi}(\left(x_\jmath-s\right)^{\mathfrak{n}-1}F_{\varrho_1}^{\varrho_2}\left(s,x_\jmath\right))\le 0 \tag{5.5}$$

holds, then we have the following generalized Giaccardi inequality

$$G_{cardi}(\Gamma) \le \sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right)\left(\Gamma^{(z-1)}(\varrho_2)G_{cardi}(\left(x_\jmath-\varrho_2\right)^z)-\Gamma^{(z-1)}(\varrho_1)G_{cardi}(\left(x_\jmath-\varrho_1\right)^z)\right). \tag{5.6}$$

If inequality (5.5) *holds in the reverse direction, then* (5.6) *also holds reversely.*

Proof. Similar to that of Theorem 5. □

In the later part of this section, we will vary our conditions on $p_\jmath x_\jmath$ $(\jmath = 1,2,\ldots,\mathfrak{r})$ to obtain generalized converse discrete variants of Jensen's inequality and Giaccardi inequality for $\mathfrak{n}-$convex functions.

Theorem 11. *Let Γ be as defined in Theorem 9. Also let $x_\jmath \in [\nu_1,\nu_2] \subseteq [\varrho_1,\varrho_2]$ and* $\mathbf{p}$ *be a positive $\mathfrak{r}-$tuple.*

(i) *If Γ is $\mathfrak{n}-$convex, then for even $\mathfrak{n} \geq 3$, (5.3) is valid.*
(ii) *Moreover if (5.3) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the following generalized converse of Jensen's inequality*

$$\frac{1}{P_\mathfrak{r}}\sum_{\jmath=1}^{\mathfrak{r}} p_\jmath \Gamma(x_\jmath) \leq \frac{\nu_2-\overline{x}}{\nu_2-\nu_1}\Gamma(\nu_1)+\frac{\overline{x}-\nu_1}{\nu_2-\nu_1}\Gamma(\nu_2). \tag{5.7}$$

Proof. We follow the idea of Theorem 6 but according to our assumed conditions we employ the converse of Jensen's inequality (see [3] or [19, p. 98]) to obtain results. □

Finally in this section we give Giaccardi inequality for higher order convex functions.

Theorem 12. *Let Γ be as defined in Theorem 9. Also let $x_0, x_\jmath \in [\nu_1,\nu_2] \subseteq [\varrho_1,\varrho_2]$ and* $\mathbf{p}$ *be a positive $\mathfrak{r}-$tuple such that*

$$\sum_{\jmath=1}^{\mathfrak{r}} p_\jmath x_\jmath \neq x_0 \text{ and } (x_l - x_0)\left(\sum_{\jmath=1}^{\mathfrak{r}} p_\jmath x_\jmath - x_l\right) \geq 0, \ (l = 1,\ldots,\mathfrak{r}).$$

.

(i) *If Γ is $\mathfrak{n}-$convex, then for even $\mathfrak{n} \geq 3$, (5.5) is valid.*
(ii) *Moreover if (5.5) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the following generalized Giaccardi inequality*

$$\sum_{\jmath=1}^{\mathfrak{r}} p_\jmath \Gamma(x_\jmath) \leq A\Gamma\left(\sum_{\jmath=1}^{\mathfrak{r}} p_\jmath x_\jmath\right) + B\left(\sum_{\jmath=1}^{\mathfrak{r}} p_\jmath - 1\right)\Gamma(x_0), \tag{5.8}$$

where A and B are defined in (5.2).

Proof. We follow the idea of Theorem 6 but according to our assumed conditions we employ Giaccardi inequality (see [17, p. 11]) to obtain results. □

6. Application in information theory for discrete Jensen's inequality

Jensen's inequality plays a key role in information theory to construct lower bounds for some notable inequalities, but here we will use it to make connections between inequalities in information theory.

Consider a convex function $\Gamma:\mathbb{R}^+\to\mathbb{R}^+$, let $\mathbf{p}:=(p_1,...,p_\mathfrak{r})$ and $\mathbf{q}:=(q_1,...,q_\mathfrak{r})$ be positive probability distributions and Γ-divergence functional is defined in [8] as follows:

$$I_\Gamma(\mathbf{p},\mathbf{q})=\sum_{\jmath=1}^{\mathfrak{r}} q_\jmath \Gamma\left(\frac{p_\jmath}{q_\jmath}\right).$$

L. Horváth et al. in [11] defined the generalized Csiszár divergence functional in the following way:

Definition 1. Consider I is an interval in $\mathbb{R}$ and let $\Gamma:I\to\mathbb{R}$ be a function. Also let $\mathbf{p}:=(p_1,\ldots,p_\mathfrak{r})\in\mathbb{R}^\mathfrak{r}$ and $\mathbf{q}:=(q_1,\ldots,q_\mathfrak{r})\in(0,\infty)^\mathfrak{r}$ such that

$$\frac{p_\jmath}{q_\jmath}\in I,\quad \jmath=1,\ldots,\mathfrak{r}.$$

Then let

$$\widetilde{I}_\Gamma(\mathbf{p},\mathbf{q})=\sum_{\jmath=1}^{\mathfrak{r}} q_\jmath \Gamma\left(\frac{p_\jmath}{q_\jmath}\right). \tag{6.1}$$

We need the following representation for the results of this section:

$$F\left(\mathbf{p},x_\jmath,\Gamma\right)=\sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right)\left\{\Gamma^{(z-1)}(\varrho_1)\left((\bar{x}-\varrho_1)^z-\frac{1}{P_\mathfrak{r}}\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath\left(x_\jmath-\varrho_1\right)^z\right)\right.$$
$$\left.-\Gamma^{(z-1)}(\varrho_2)\left((\bar{x}-\varrho_2)^z-\frac{1}{P_\mathfrak{r}}\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath\left(x_\jmath-\varrho_2\right)^z\right)\right\}. \tag{6.2}$$

Theorem 13. *Under the assumptions of Theorem 5(ii), let (3.1) hold and Γ be $\mathfrak{n}-$convex. Also let $\mathbf{p}:=(p_1,\ldots,p_\mathfrak{r})\in\mathbb{R}^\mathfrak{r}$ and $\mathbf{q}:=(q_1,\ldots,q_\mathfrak{r})\in(0,\infty)^\mathfrak{r}$, then we have the following result:*

$$\widetilde{I}_\Gamma(\mathbf{p},\mathbf{q})\geq P_\mathfrak{r}\Gamma(1)+P_\mathfrak{r}F\left(\mathbf{q},\frac{p_\jmath}{q_\jmath},\Gamma\right). \tag{6.3}$$

Proof. From Theorem 5 we can rearrange (3.2) as

$$\frac{1}{P_\mathfrak{r}}\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath\Gamma(x_\jmath)\geq\Gamma(\bar{x})-\sum_{z=2}^{\mathfrak{n}-1}\left(\frac{\mathfrak{n}-z}{z!(\varrho_2-\varrho_1)}\right) \tag{6.4}$$
$$\times\left\{\Gamma^{(z-1)}(\varrho_2)\left((\bar{x}-\varrho_2)^z-\frac{1}{P_\mathfrak{r}}\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath(x_\jmath-\varrho_2)^z\right)-\Gamma^{(z-1)}(\varrho_1)\left((\bar{x}-\varrho_1)^z-\frac{1}{P_\mathfrak{r}}\sum_{\jmath=1}^{\mathfrak{r}}p_\jmath(x_\jmath-\varrho_1)^z\right)\right\}.$$

Now by replacing $p_\jmath$ with $q_\jmath$ and $x_\jmath$ with $\frac{p_\jmath}{q_\jmath}$, and following (6.2) we get (6.3). □

For a positive $\mathfrak{r}$-tuple $\mathbf{q}=(q_1,...,q_\mathfrak{r})$ such that $\sum_{\jmath=1}^{\mathfrak{r}}q_\jmath=1$, the **Shannon entropy** [23] is defined by

$$S(\mathbf{q})=-\sum_{\jmath=1}^{\mathfrak{r}}q_\jmath\ln q_\jmath. \tag{6.5}$$

Corollary 10. *Under the assumptions of Theorem 13, let $\mathfrak{n}$ be even. Then we have the following results:*

(i) *If $\mathbf{q} := (q_1, \ldots, q_\mathfrak{r}) \in (0,\infty)^\mathfrak{r}$, then*

$$\sum_{\mathfrak{j}=1}^{\mathfrak{r}} q_\mathfrak{j} \ln q_\mathfrak{j} \geq \mathfrak{r} F\left(\mathbf{q}, \frac{1}{q_\mathfrak{j}}, -\ln(\cdot)\right). \tag{6.6}$$

(ii) *We can get bounds for the Shannon entropy of $\mathbf{q}$, if we choose $\mathbf{q} := (q_1, \ldots, q_\mathfrak{r})$ a positive probability distribution:*

$$S(\mathbf{q}) \leq -\mathfrak{r} F\left(\mathbf{q}, \frac{1}{q_\mathfrak{j}}, -\ln(\cdot)\right). \tag{6.7}$$

Proof. (i) For even $\mathfrak{n}$, $\Gamma(x) := -\ln x$ is $\mathfrak{n}-$convex and using $\mathbf{p} := (1,1,\ldots,1)$ in Theorem 13 we get (6.6) by following (6.2).

(ii) Since we have $\sum_{\mathfrak{j}=1}^{\mathfrak{r}} q_\mathfrak{j} = 1$, therefore by multiplying -1 on both side to (6.6) and taking into account expression (6.5), we get (6.7). □

The **Kullback-Leibler** distance [16] between the positive probability distributions $\mathbf{p} = (p_1, ..., p_\mathfrak{r})$ and $\mathbf{q} = (q_1, ..., q_\mathfrak{r})$ is defined by

$$D(\mathbf{q} \,\|\, \mathbf{p}) = \sum_{\mathfrak{j}=1}^{\mathfrak{r}} q_\mathfrak{j} \ln\left(\frac{q_\mathfrak{j}}{p_\mathfrak{j}}\right). \tag{6.8}$$

Corollary 11. *Under the assumption of Corollary 10, we have the following results:*

(i) *If $\mathbf{q} := (q_1, \ldots, q_\mathfrak{r}), \mathbf{p} := (p_1, \ldots, p_\mathfrak{r}) \in (0,\infty)^\mathfrak{r}$, then*

$$\sum_{\mathfrak{j}=1}^{\mathfrak{r}} q_\mathfrak{j} \ln\left(\frac{q_\mathfrak{j}}{p_\mathfrak{j}}\right) \geq \mathfrak{r} F\left(\mathbf{q}, \frac{p_\mathfrak{j}}{q_\mathfrak{j}}, -\ln(\cdot)\right). \tag{6.9}$$

(ii) *Now if we take positive probability distributions $\mathbf{q} := (q_1, \ldots, q_\mathfrak{r})$ and $\mathbf{p} := (p_1, \ldots, p_\mathfrak{r})$, then we have*

$$D(\mathbf{q} \,\|\, \mathbf{p}) \geq \mathfrak{r} F\left(\mathbf{q}, \frac{p_\mathfrak{j}}{q_\mathfrak{j}}, -\ln(\cdot)\right). \tag{6.10}$$

Proof. (i) Using $\Gamma(x) := -\ln x$ (which is $\mathfrak{n}-$convex for even $\mathfrak{n}$) in Theorem 13 and following (6.2), after simplification we get (6.9).

(ii) It is a special case of (i). □

7. Results for Zipf and hybrid Zipf-Mandelbrot entropy

One of the basic laws in information science is Zipf's law [20, 24] and is highly applied in linguistics. Let $c \geq 0$, $d > 0$ and $N \in \{1,2,3,\ldots\}$. The **Zipf-Mandelbrot**

entropy can be given as

$$Z_M(H,c,d) = \frac{d}{H^N_{c,d}} \sum_{\jmath=1}^{N} \frac{\ln(\jmath + c)}{(\jmath + c)^d} + \ln(H^N_{c,d}), \tag{7.1}$$

where

$$H^N_{c,d} = \sum_{\sigma=1}^{N} \frac{1}{(\sigma + c)^d}.$$

Consider

$$q_\jmath = \Gamma(\jmath; N, c, d) = \frac{1}{((\jmath + c)^d H^N_{c,d})} \tag{7.2}$$

where $\Gamma(\jmath; N, c, d)$ is a discrete probability distribution known as **Zipf-Mandelbrot** law. The Zipf-Mandelbrot law has many applications in linguistics and information science. Some of the recent studies about Zipf-Mandelbrot law can be seen in the listed references (see [11, 13]). Now we state our results involving entropy introduced by Mandelbrot law by establishing the relationship with Shannon and relative entropies.

Theorem 14. *Let* $\mathbf{q}$ *be a* ***Zipf-Mandelbrot*** *law as defined in* (7.2) *with parameters* $c \geq 0$, $d > 0$ *and* $N \in \{1,2,\ldots\}$. *Then we have*

$$Z_M(H,c,d) = S(\mathbf{q}) \leq -N \times F\Big(\mathbf{q}, ((\jmath + c)^d H^N_{c,d}), -\ln(\cdot)\Big). \tag{7.3}$$

Proof. It is interesting to see that for $q_\jmath$, defined as in (7.2), $\sum_{\jmath=1}^{N} q_\jmath = 1$. Therefore, using the above $q_\jmath$ in Shannon entropy (6.5), we get Mandelbrot entropy (7.1):

$$S(\mathbf{q}) = -\sum_{\jmath=1}^{N} q_\jmath \ln q_\jmath = -\sum_{\jmath=1}^{N} \frac{1}{((\jmath + c)^d H^N_{c,d})} \ln \frac{1}{((\jmath + c)^d H^N_{c,d})}$$

$$= \frac{d}{(H^N_{c,d})} \sum_{\jmath=1}^{N} \frac{\ln(\jmath + c)}{(\jmath + c)^d} + \ln(H^N_{c,d}). \tag{7.4}$$

Finally, substituting $q_\jmath = \frac{1}{((\jmath+c)^d H^N_{c,d})}$ in Corollary 10(ii) and following (6.2) we get the desired result. □

Corollary 12. *Let* $\mathbf{q}$ *and* $\mathbf{p}$ *be* ***Zipf-Mandelbrot*** *laws with parameters* $c_1, c_2 \in [0,\infty)$, $d_1, d_2 > 0$, *let* $H^N_{c_1,d_1} = \sum_{\sigma=1}^{N} \frac{1}{(\sigma+c_1)^{d_1}}$ *and* $H^N_{c_2,d_2} = \sum_{\sigma=1}^{N} \frac{1}{(\sigma+c_2)^{d_2}}$. *Now using*

$$q_\jmath = \frac{1}{(\jmath + c_1)^{d_1} H^N_{c_1,d_1}} \text{ and } p_\jmath = \frac{1}{(\jmath + c_2)^{d_2} H^N_{c_2,d_2}}$$

in Corollary 11(ii) and following (6.2), *we have*

$$D(\mathbf{q}\,\|\,\mathbf{p}) = \sum_{J=1}^{N} \frac{1}{(J+c_1)^{d_1} H^N_{c_1,d_1}} \ln\left(\frac{(J+c_2)^{d_2} H^N_{c_2,d_2}}{(J+c_1)^{d_1} H^N_{c_1,d_1}}\right)$$

$$= -Z_M(H,c_1,d_1) + \frac{d_2}{H^N_{c_1,d_1}} \sum_{J=1}^{N} \frac{\ln(J+c_2)}{(J+c_1)^{d_1}} + \ln\left(H^N_{c_2,d_2}\right) \tag{7.5}$$

$$\geq N \times F\left(\mathbf{q}, \frac{(J+c_1)^{d_1} H^N_{c_1,d_1}}{(J+c_2)^{d_2} H^N_{c_2,d_2}}, -\ln(\cdot)\right).$$

The final results are about **hybrid Zipf-Mandelbrot entropy**, which is a further generalization of Zipf-Mandelbrot entropy. Let $N \in \{1,2,3,\ldots\}$, $c \geq 0$, $d,\omega > 0$. Then hybrid Zipf-Mandelbrot entropy can be given as follows:

$$\widehat{Z}_M(H^*,c,d,\omega) = \frac{1}{H^*_{c,d,\omega}} \sum_{J=1}^{N} \frac{\omega^J}{(J+c)^d} \ln\left(\frac{(J+c)^d}{\omega^J}\right) + \ln(H^*_{c,d,\omega}), \tag{7.6}$$

where

$$H^*_{c,d,\omega} = \sum_{J=1}^{N} \frac{\omega^J}{(J+c)^d} \tag{7.7}$$

Consider

$$q_J = \Gamma(J;N,c,d,\omega) = \frac{\omega^J}{(J+c)^d H^*_{c,d,\omega}}, \tag{7.8}$$

which is called **hybrid Zipf-Mandelbrot** law. There is a unified approach, maximization of Shannon entropy [25], that naturally follows the path of generalization from Zipf's to hybrid Zipf's law. Extending this idea Jakšetic et al. in [12] presented a transition from Zipf-Mandelbrot to hybrid Zipf-Mandelbrot law by employing maximum entropy technique with one additional constraint. It is interesting that the examination of its densities provides some new insights of Lerch's transcendent.

Theorem 15. *Let* $\mathbf{q}$ *be a* ***hybrid Zipf-Mandelbrot*** *law as defined in* (7.8) *with parameters* $c \geq 0$, $d,\omega > 0$ *and* $N \in \{1,2,\ldots\}$, *then we have*

$$\widehat{Z}_M(H^*,c,d,\omega) = S(\mathbf{q}) \leq -N \times F\left(\mathbf{q}, \frac{(J+c)^d H^*_{c,d,\omega}}{\omega^J}, -\ln(\cdot)\right). \tag{7.9}$$

Proof. It is interesting to see that for q_J, defined as in (7.8), $\sum_{J=1}^{\mathrm{r}} q_J = 1$. Therefore, using the above q_J in Shannon entropy (6.5), we get hybrid Zipf-Mandelbrot entropy (7.6):

$$S(\mathbf{q}) = -\sum_{J=1}^{N} q_J \ln q_J = -\sum_{J=1}^{N} \frac{\omega^J}{(J+c)^d H^*_{c,d,\omega}} \ln \frac{\omega^J}{(J+c)^d H^*_{c,d,\omega}}$$

$$= \frac{-1}{H^*_{c,d,\omega}} \sum_{J=1}^{N} \frac{\omega^J}{(J+c)^d} \left[\ln\left(\frac{\omega^J}{(J+c)^d}\right) + \ln\left(\frac{1}{H^*_{c,d,\omega}}\right) \right]$$

$$= \frac{1}{H^*_{c,d,\omega}} \sum_{J=1}^{N} \frac{\omega^J}{(J+c)^d} \left[\ln\left(\frac{(J+c)^d}{\omega^J}\right) + \ln\left(H^*_{c,d,\omega}\right) \right]$$

$$= \frac{1}{H^*_{c,d,\omega}} \sum_{J=1}^{N} \frac{\omega^J}{(J+c)^d} \ln\left(\frac{(J+c)^d}{\omega^J}\right) + \ln\left(H^*_{c,d,\omega}\right). \tag{7.10}$$

Finally, substituting this $q_J = \frac{\omega^J}{(J+c)^d H^*_{c,d,\omega}}$ in Corollary 10(ii) and following (6.2) we get the desired result. □

Corollary 13. *Let* **q** *and* **p** *be* ***hybrid Zipf-Mandelbrot*** *laws with parameters* $c_1, c_2 \in [0,\infty)$, $\omega_1, \omega_2, d_1, d_2 > 0$. *Now using*

$$q_J = \frac{\omega_1^J}{(J+c_1)^{d_1} H^*_{c_1,d_1,\omega_1}} \text{ and } p_J = \frac{\omega_2^J}{(J_2+c_2)^{d_2} H^*_{c_2,d_2,\omega_2}}$$

in Corollary 11(ii) and following (6.2)*, then we have*

$$D(\mathbf{q} \,\|\, \mathbf{p}) = \sum_{J=1}^{N} \frac{\omega_1^J}{(J+c_1)^{d_1} H^*_{c_1,d_1,\omega_1}} \ln\left(\frac{\omega_1^J}{\omega_2^J} \frac{(J+c_2)^{d_2} H^*_{c_2,d_2,\omega_2}}{(J+c_1)^{d_1} H^*_{c_1,d_1,\omega_1}}\right)$$

$$= -\widehat{Z}_M(H^*, c_1, d_1, \omega_1) + \frac{1}{H^*_{c_1,d_1,\omega_1}} \sum_{J=1}^{N} \frac{\omega_1^J}{(J+c_1)^{d_1}} \ln\left(\frac{(J+c_2)_2^d}{\omega_2^J}\right) + \ln\left(H^*_{c_2,d_2,\omega_2}\right).$$

$$\geq N \times F\left(\mathbf{q}, \frac{\omega_2^J (J_1+c_1)^{d_1} H^*_{c_1,d_1,\omega_1}}{\omega_1^J (J_2+c_2)^{d_2} H^*_{c_2,d_2,\omega_2}}, -\ln(\cdot)\right). \tag{7.11}$$

Remark 9. Similarly, we can give results for Shannon entropy, Kullback-Leibler distance, Zipf-Mandelbrot entropy and hybrid Zipf-Mandelbrot entropy by using results for converse discrete Jensen's inequality and therefore also for generalized Giaccardi inequality defined in (5.6) on the same approach.

8. Concluding remarks

Jensen's inequality is considered to be much fruitful for the characterization of convex functions. First we gave real Stieltjes measure theoretic representations of integral Jensen's inequality by using Fink's identity. Then we formulate results for other inequalities like Jensen Steffensen's inequality, Jensen-Boas inequality and Jensen–Brunk inequality. We can obtain Jensen Steffensen's inequality, Jensen-Boas inequality and Jensen–Brunk inequality by changing the assumptions in Jensen's inequality. We also gave generalized Jensen's inequality in discrete case having real weights. As a result we give its reverses and converses by studying conditions on tuples. At the end we gave applications in information theory for our obtained results, specially we gave results for hybrid Zipf-Mandelbrot entropy.

References

[1] M. W. Alomari, "Generalizations of Guessab-Schmeisser formula via Fink type identity with applications to Quadrature rules," *arXiv preprint arXiv:1703.01535*, 2017.

[2] R. Barlow, A. Marshall, and F. Proschan, "Some inequalities for starshaped and convex functions," *Pacific Journal of Mathematics*, vol. 29, no. 1, pp. 19–42, 1969, doi: 10.2140/pjm.1969.29.19.

[3] P. R. Beesack and J. E. Pečarić, "On Jessen's inequality for convex functions," *Journal of Mathematical Analysis and Applications*, vol. 110, no. 2, pp. 536–552, 1985, doi: 10.1016/0022-247X(85)90315-4.

[4] R. P. Boas, "The Jensen-Steffensen inequality," *Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika*, no. 302/319, pp. 1–8, 1970.

[5] H. Brunk, "On an inequality for convex functions," *Proceedings of the American Mathematical Society*, vol. 7, no. 5, pp. 817–824, 1956, doi: 10.2307/2033544.

[6] S. I. Butt, M. Klaričić Bakula, Đ. Pečarić, and J. Pečarić, "Jensen-Grüss inequality and its applications for the Zipf-Mandelbrot law," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 2, pp. 1664–1673, 2021, doi: 10.1002/mma.6869.

[7] S. I. Butt, J. Pečarić, and A. Vukelić, "Generalization of Popoviciu-type inequalities Via Fink's identity," *Mediterranean Journal of Mathematics*, vol. 13, no. 4, pp. 1495–1511, 2016, doi: 10.1007/s00009-015-0573-8.

[8] I. Csiszár, "Information-type measures of difference of probability distributions and indirect observation," *Studia Scientiarum Mathematicarum Hungarica*, vol. 2, pp. 229–318, 1967.

[9] S. S. Dragomir, T. M. Rassias, and I. Aleksanova, *Ostrowski type inequalities and applications in numerical integration*. Springer, 2002.

[10] A. Fink, "Bounds on the deviation of a function from its averages," *Czechoslovak Mathematical Journal*, vol. 42, no. 2, pp. 289–310, 1992, doi: 10.21136/cmj.1992.128336.

[11] L. Horváth, Đ. Pečarić, and J. Pečarić, "Estimations of f-and Rényi divergences by using a cyclic refinement of the Jensen's inequality," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 3, pp. 933–946, 2019, doi: 10.1007/S40840-017-0526-4.

[12] J. Jaksetic, D. Pecaric, and J. Pecaric, "Hybrid Zipf–Mandelbrot law," *J. Math. Inequal*, vol. 9, no. 1, pp. 275–286, 2019, doi: 10.7153/jmi-2019-13-20.

[13] J. Jakšetić, Đ. Pečarić, and J. Pečarić, "Some properties of Zipf-Mandelbrot law and Hurwitz ζ-function," *Mathematical Inequalities & Applications*, vol. 21, no. 2, pp. 575–584, 2018, doi: 10.7153/mia-2018-21-42.

[14] M. A. Khan, Đ. Pečarić, and J. Pečarić, "New refinement of the Jensen inequality associated to certain functions with applications," *Journal of Inequalities and Applications*, vol. 2020, no. 1, pp. 1–11, 2020, doi: 10.1186/s13660-020-02343-7.

[15] S. Khan, M. A. Khan, S. I. Butt, and Y.-M. Chu, "A new bound for the Jensen gap pertaining twice differentiable functions with applications," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–11, 2020, doi: 10.1186/s13662-020-02794-8.

[16] S. Kullback, *Statistics and Information theory*. Wiley, New York, 1959.

[17] D. Mitrinović, J. Pečarić, and A. Fink, *Classical and New Inequalities in Analysis, Mathematics and Its Applications (East European Series)*. Kluwer Academic Publishers Dordrecht, The Netherlands, 1993.

[18] J. E. Pečarić, "On an inequality of P.M. Vasić and R.R. Janić," *Publ. Inst. Math.(Beograd)(NS)*, vol. 28, no. 42, pp. 145–149, 1980.

[19] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*. Academic Press, 1992.

[20] S. T. Piantadosi, "Zipf's word frequency law in natural language: A critical review and future directions," *Psychonomic Bulletin & Review*, vol. 21, no. 5, pp. 1112–1130, 2014, doi: 10.3758/s13423-014-0585-6.

[21] M. A. Ragusa and A. Tachikawa, "Partial regularity of the minimizers of quadratic functionals with VMO coefficients," *Journal of the London Mathematical Society*, vol. 72, no. 3, pp. 609–620, 2005, doi: 10.1112/S002461070500699X.

[22] M. A. Ragusa and A. Tachikawa, "On minimizers for functionals under the non-standard growth conditions," in *AIP Conference Proceedings*, vol. 1738, no. 1, 2016, p. 480112.

[23] C. E. Shannon, "A mathematical theory of communication," *The Bell System Technical Journal*, vol. 27, no. 3, pp. 379–423, 1948, doi: 10.1002/j.1538-7305.1948.tb01338.x.

[24] Z. Silagadze, "Citations and the Zipf-Mandelbrot's law," *Complex Systems*, vol. 11, pp. 487–499, 1997.

[25] M. Visser, "Zipf's law, power laws and maximum entropy," *New Journal of Physics*, vol. 15, no. 4, p. 043021, 2013, doi: 10.1088/1367-2630/15/4/043021.

Authors' addresses

S. I. Butt
(**Corresponding author**) COMSATS University Islamabad, Lahore Campus, Pakistan
E-mail address: saadihsanbutt@gmail.com

T. Rasheed
COMSATS University Islamabad, Lahore Campus, Pakistan
E-mail address: tahirtishna24@gmail.com

Đ. Pečarić
Catholic University of Croatia, Ilica 242, 10000 Zagreb, Croatia
E-mail address: gildapeca@gmail.com

J. Pečarić
Peoples Friendship University of Russia (RUDN University), Miklukho-Maklaya str. 6, 117198, Moscow, Russia
E-mail address: pecaric@element.hr