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SOME COMBINATORIAL IDENTITIES VIA k-ORDER FIBONACCI MATRICES

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Abstract. Matrix factorizations have been widely used in recent years, especially in engineering problems, to facilitate performance-requiring computations. In this paper, we investigate some interesting relationships between some combinatorial matrices such as Pascal matrix, Stirling matrices and k-order Fibonacci matrices. We give factorizations and inverse factorizations of the Pascal and Stirling matrices via k-order Fibonacci matrices. Moreover, we derive various combinatorial properties by using relationships between these matrices. Finally, compared to previous studies, we present more general results for specific values of k.

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1. INTRODUCTION

The Pascal matrix arises in linear algebra, probability and combinatorial analysis. Until now, several researchers have studied the factorizations, generalizations and eigenvalues of some combinatorial matrices [1, 2, 4, 6, 11, 13–15]. In [2], the authors produced some combinatorial identities and an existence theorem for diophantine equation systems by applying linear algebra. Lee et al. gave a factorization of the Pascal matrix via Fibonacci matrix and they obtained some combinatorial identities by using Pascal matrix, the Stirling matrices of the first kind and the second kind and the Fibonacci matrix [11]. In another way, Zhang and Wang presented a factorization of the Pascal matrix and they found various combinatorial properties with the help of the Fibonacci matrix [15]. For more information about Pascal matrices, Stirling matrices and properties of the binomial coefficients, the reader is referred to book such as [5].

For any positive integer n, the $n\times n$ lower triangular Pascal matrix $\mathscr{P}_n=[p_{ij}]$ and its inverse $\mathscr{P}_n^{-1}=[p'_{ij}]$ are defined by

$$p_{ij}=\begin{cases}\binom{i-1}{j-1} & \text{if } i\geqslant j,\\ 0 & \text{otherwise,}\end{cases}\qquad p'_{ij}=\begin{cases}(-1)^{i-j}\binom{i-1}{j-1} & \text{if } i\geqslant j,\\ 0 & \text{otherwise,}\end{cases}\tag{1.1}$$

where the binomial coefficient $\binom{i}{j}$ counts the number of lattice paths from $(0,0)$ to $(i-j,j)$ with steps $(1,0)$ and $(0,1)$.

For example

$$\mathscr{P}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \quad \text{and} \quad \mathscr{P}_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}.$$

For nonnegative integers n,k and $n \geqslant k \geqslant 0$, the Stirling numbers of the first kind $s(n,k)$ and of the second kind $S(n,k)$, which are the coefficients of polynomials that commonly arise in combinatorial problems, are defined as the coefficients in the following expansions (see [3]):

$$[x]_n = \sum_{k=0}^{n} s(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S(n,k)[x]_k$$

where

$$[x]_n = \begin{cases} x(x-1)\cdots(x-n+1) & \text{if } n \geqslant 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Moreover, the $S(n,k)$ satisfy the following formula

$$S(n,k) = \sum_{l=k-1}^{n-1} \binom{n-1}{l} S(l,k-1). \tag{1.2}$$

For the Stirling numbers $s(i,j)$ and $S(i,j)$ of the first kind and of the second kind, respectively, define $\mathscr{S}_n^{(1)} = [s_{ij}]$ and $\mathscr{S}_n^{(2)} = [S_{ij}]$ to be the $n \times n$ matrices by

$$s_{ij} = \begin{cases} s(i,j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad S_{ij} = \begin{cases} S(i,j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

We call the matrices $\mathscr{S}_n^{(1)}$ and $\mathscr{S}_n^{(2)}$ Stirling matrix of the first kind and of the second kind, respectively (see [5]). For example,

$$\mathscr{S}_5^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 \\ 24 & -50 & 35 & -10 & 1 \end{bmatrix} \quad \text{and} \quad \mathscr{S}_5^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}.$$

The Fibonacci numbers are one of the most well-known numbers, and have many important applications to a wide variety of research areas such as mathematics, computer science, physics, biology, and statistics [8]. The Fibonacci numbers, $\{F_n\}_{n=0}^{\infty}$, are the terms of the sequence $0,1,1,2,3,5,\ldots$ where each term is the sum of the two previous terms, and starting with the initial values $F_0 = 0$ and $F_1 = 1$. Up to this time, some researchers have studied the generalizations and applications of the k-generalized Fibonacci numbers [7,9,12].

For $n > k \geqslant 2$, an interesting generalization of the Fibonacci sequence, k-order Fibonacci sequence, k-order Fibonacci sequence $\{g(k)_n\}_{n=0}^{\infty}$, were defined in [9] by

$$g(k)_n = g(k)_{n-1} + g(k)_{n-2} + \cdots + g(k)_{n-k},$$

with the initial conditions $g(k)_0 = \cdots = g(k)_{k-2} = 0, g(k)_{k-1} = g(k)_k = 1$. For example, if $k = 2$, then $\{g(2)_n\}_{n=0}^{\infty}$ is the Fibonacci sequence, $\{F_n\}_{n=0}^{\infty}$ and if $k = 3$, then $\{g(3)_n\}_{n=0}^{\infty}$ is the 3-Fibonacci sequence, namely the Tribonacci sequence $\{T_n\}_{n=0}^{\infty}$.

For any positive integer n, the $n \times n$ k-order Fibonacci matrix, $\mathscr{F}_n^{(k)} = [f_{ij}]$, is defined by [10]

$$f_{ij} = \begin{cases} g_{i-j+1} & \text{if } i-j \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

where $g_t = g(k)_{t+k-2}$, $1 \leq t \leq n$. Here we note that $f_{n1} = g_n$ and each column of $\mathscr{F}_n^{(k)}$ is the vector of k-order Fibonacci numbers. For example, the Fibonacci and 4-Fibonacci matrices are as follows:

$$\mathscr{F}_6^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \\ 5 & 3 & 2 & 1 & 1 & 0 \\ 8 & 5 & 3 & 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathscr{F}_6^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 \\ 15 & 8 & 4 & 2 & 1 & 1 \end{bmatrix}.$$

From (1.3), the matrix $\mathscr{F}_n^{(k)}$ is lower triangular and $\det\left(\mathscr{F}_n^{(k)}\right) = 1 \neq 0$. Thus, the matrix $\mathscr{F}_n^{(k)}$ has an inverse. Therefore, for any positive integer n, the $n \times n$ inverse k-order Fibonacci matrix, $\left(\mathscr{F}_n^{(k)}\right)^{-1} = [f'_{ij}]$, is defined by [10]

$$f'_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i-k \leqslant j \leqslant i-1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

The relations between the Pascal matrix, Stirling matrices and Fibonacci matrix motivate us to study a more generalized situation. Thus, motivated by the above cited works, we give the factorizations of the Pascal and Stirling matrices via k-order Fibonacci matrices. We derive various generalized combinatorial identities by using these matrices. As compared to earlier works [11],[15] and [14], we give k-extended results.

2. First factorization of the Pascal matrix

In this section, we consider the first factorization of the Pascal matrix by means of the k-order Fibonacci matrices.

Definition 1. For $n \in \mathbb{N}$, the $n \times n$ matrix $\mathscr{L}_n^{(k)} = [l_{ij}]$ is defined by

$$l_{ij} = \binom{i-1}{j-1} - \sum_{t=1}^{k} \binom{i-t-1}{j-1}. \tag{2.1}$$

By Definition 1, we observe that $l_{11} = 1$, $l_{1j} = 0$, for $j \geqslant 2$, $l_{21} = 0$, $l_{22} = 1$, $l_{2j} = 0$, for $j \geqslant 3$, $l_{31} = -1$, $l_{32} = 1$, $l_{33} = 1$, $l_{3j} = 0$, for $j \geqslant 4$ and

$$l_{i1} = \begin{cases} 2-i, & 1 \leqslant i \leqslant k, \\ 1-k, & i > k. \end{cases} \tag{2.2}$$

Moreover, for $i, j \geqslant 2$ we have the recursion $l_{ij} = l_{i-1,j-1} + l_{i-1,j}$. In particular, if we take $n = 8$ and $k = 5$, we get

$$\mathscr{L}_8^{(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -3 & -2 & 2 & 3 & 1 & 0 & 0 & 0 \\ -4 & -5 & 0 & 5 & 4 & 1 & 0 & 0 \\ -4 & -9 & -5 & 5 & 9 & 5 & 1 & 0 \\ -4 & -13 & -14 & 0 & 14 & 14 & 6 & 1 \end{bmatrix}.$$

By virtue of the equations (1.1), (1.3) and (2.1), we can derive the following theorem.

Theorem 1. *Let $\mathscr{L}_n^{(k)}$ be the matrix as in (2.1). For the Pascal matrix $\mathscr{P}_n$ and the k-order Fibonacci matrix $\mathscr{F}_n^{(k)}$, we have*

$$\mathscr{P}_n = \mathscr{F}_n^{(k)} \mathscr{L}_n^{(k)}.$$

Proof. Since the matrix $\mathscr{F}_n^{(k)}$ is invertible, we will prove that $\left(\mathscr{F}_n^{(k)}\right)^{-1} \mathscr{P}_n = \mathscr{L}_n^{(k)}$.

Let $\left(\mathscr{F}_n^{(k)}\right)^{-1} = [f'_{ij}]$ be the inverse of $\mathscr{F}_n^{(k)}$. Since $f'_{1j} = 0$ for $j \geqslant 2, f'_{11} p_{11} = 1$ and $l_{11} = 1 = \sum_{t=1}^{n} f'_{1t} p_{t1}$. Since $p_{1j} = 0$ and $f'_{1j} = 0$ for $j \geqslant 2, \sum_{t=1}^{n} f'_{1t} p_{tj} = 0 = l_{1j}$ for $j \geqslant 2$. Since $f'_{2j} = 0$ for $j \geqslant 3, f'_{21} = -1$ and $f'_{22} = 1$, we have $\sum_{t=1}^{n} f'_{2t} p_{t1} = l_{21} = 0$. By virtue of (1.4), we have, for $i = 3, 4, \ldots, n, \sum_{t=1}^{n} f'_{it} p_{t1} = l_{i1}$.

Now, we consider $i \geqslant 3$ and $j \geqslant 2$. From (1.4) and the recurrence relation of l_{ij}, we have $\sum_{t=1}^{n} f'_{it} p_{tj} = l_{ij}$. Therefore, we have $\left(\mathscr{F}_n^{(k)}\right)^{-1} \mathscr{P}_n = \mathscr{L}_n^{(k)}$, and the proof is completed. □

By virtue of Theorem 1, we have the following Corollary.

Corollary 1. *For $1 \leqslant r \leqslant n$, we have*

$$\binom{n-1}{r-1} = \sum_{t=r}^{n} g_{n-t+1}\left[\binom{t-1}{r-1} - \sum_{s=1}^{k}\binom{t-s-1}{r-1}\right]. \tag{2.3}$$

In particular, if $r = 1$ then

$$g_n = 1 + \sum_{i=2}^{k}(i-2)g_{n-i+1} + (k-1)\sum_{i=k+1}^{n} g_{n-i+1}.$$

Proof. Since $\mathscr{P}_n = \mathscr{F}_n^{(k)}\mathscr{L}_n^{(k)}$, $g_1 = 1, g_2 = 1$ and $l_{ij} = 0$ for $i \leqslant j-1$, we have

$$\binom{n-1}{r-1} = p_{nr} = g_n l_{1r} + g_{n-1}l_{2r} + g_{n-2}l_{3r} + \cdots + g_3 l_{n-2,r} + g_2 l_{n-1,r} + g_1 l_{nr}.$$

From (2.2) and Theorem 1, we know $l_{11}, l_{21}, l_{22}, l_{i1}$ and the recurrence $l_{ij} = l_{i-1,j-1} + l_{i-1,j}$. Thus we obtain

$$p_{nr} = \sum_{t=r}^{n} g_{n-t+1} l_{t,r},$$

which is equal to (2.3). In particular, if we take $r = 1$, we have

$$p_{n1} = g_n l_{11} + g_{n-1}l_{21} + g_{n-2}l_{31} + \cdots + g_{n-k}l_{k+1,1} + \cdots + g_1 l_{n1}.$$

By virtue of (2.2), we get

$$1 = g_n + (0)g_{n-1} + (-1)g_{n-2} + (-2)g_{n-3} + (-3)g_{n-4} + \cdots + (2-k)g_{n-k+1} + (1-k)\left(g_{n-k} + g_{n-k+1} + \cdots + g_2 + g_1\right).$$

By arranging the above expressions, we find that

$$g_n = 1 + \sum_{i=2}^{k}(i-2)g_{n-i+1} + (k-1)\sum_{i=k+1}^{n} g_{n-i+1}.$$

Hence the proof is completed. □

From (2.1), we can find the inverse of the matrix $\left(\mathscr{L}_n^{(k)}\right)^{-1} = [l'_{ij}]$ as

$$l'_{ij} = \sum_{t=j}^{i}(-1)^{i+t}\binom{i-1}{t-1} g_{t-j+1}. \tag{2.4}$$

We observe that $l'_{11} = 1$, $l'_{1j} = 0$, for $j \geqslant 2$, $l'_{21} = 0$, $l'_{22} = 1$, $l'_{2j} = 0$, for $j \geqslant 3$ and the recursion $l'_{ij} = l'_{i-1,j-1} - l'_{i-1,j}$ holds for $i, j \geqslant 2$. Since $\mathscr{F}_n^{(k)} = \mathscr{P}_n\left(\mathscr{L}_n^{(k)}\right)^{-1}$, we have

$$f_{nr} = p_{n1}l'_{1r} + p_{n2}l'_{2r} + p_{n3}l'_{3r} + \cdots + p_{n,n-1}l'_{n-1,r} + p_{n,n}l'_{nr}.$$

From (1.1) and (2.4), we get

$$g_{n-r+1} = \sum_{t=r}^{n}\left[\binom{n-1}{t-1}\left[\sum_{s=r}^{t}(-1)^{t+s}\binom{t-1}{s-1}g_{s-r+1}\right]\right].$$

In particular, if $r=1$, we obtain

$$g_{n} = \sum_{t=1}^{n}\left[\binom{n-1}{t-1}\left[\sum_{s=1}^{t}(-1)^{t+s}\binom{t-1}{s-1}g_{s}\right]\right].$$

Let I_n be the identity matrix of order n, and let L_k be the $k\times k$ lower triangular matrix as follows:

$$L_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0\\ 1 & 1 & 0 & 0 & \cdots & 0\\ 1 & 0 & 1 & 0 & \cdots & 0\\ 1 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Set $S_l = L_{k+1}\oplus I_l, l=1,2,\ldots$ Further we define n by n matrices $\overline{\mathscr{F}}_n^{(k)} = [1]\oplus \mathscr{F}_{n-1}^{(k)}$, $G_1 = I_n, G_2 = I_{n-2}\oplus L_2, G_3 = I_{n-3}\oplus L_3, \ldots, G_k = I_{n-k}\oplus L_k\ G_{k+1} I_{n-k-1}\oplus L_{k+1}$, and, for $k+2\leqslant l\leqslant n, G_l = I_{n-l}\oplus S_{l-k-1}$. In particular $S_0 = L_{k+1}$ and $G_n = S_{n-k-1}$. In [10], the authors gave the factorization of the k-order Fibonacci matrix as: $\mathscr{F}_n^{(k)} = G_1G_2G_3\cdots G_n$.

For $k\geqslant 2$, we define the $n\times n$ matrices $H_n^{(k)}$ and $\overline{H}_i$ by

$$H_n^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & 0 & \cdots & 0\\ -1 & 1 & 1 & 0 & \cdots & 0\\ -1 & 1 & 1 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ -1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix},\quad H_n^{(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0\\ -1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0\\ -2 & 1 & 1 & 1 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ 1-k & 1 & 1 & 1 & 1 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 1-k & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \end{bmatrix},$$

$\overline{H}_i = I_{n-i}\oplus H_i^{(k)}$. From the definition of $\overline{H}_i$, we have $\overline{H}_1 = \overline{H}_2 = I_n$. Thus, we have the following lemma by the definition of the matrix product.

Lemma 1. *The matrix $\mathscr{L}_n^{(k)}$ can be factored by the $\overline{H}_i$'s as follows:*

$$\mathscr{L}_n^{(k)} = \overline{H}_n\overline{H}_{n-1}\overline{H}_{n-2}\cdots\overline{H}_2\overline{H}_1.$$

By virtue of Lemma 1, we have the following theorem.

Theorem 2. *For the $n\times n$ Pascal matrix, $\mathscr{P}_n$, we have*

$$\mathscr{P}_n=\prod_{i=1}^{n}G_i\prod_{i=1}^{n}\overline{H}_{n-i+1}.$$

3. Second factorization of the Pascal matrix

In this section, we consider the second factorization of the Pascal matrix by means of the k-order Fibonacci matrix.

Definition 2. For $n\in\mathbb{N}$, the $n\times n$ matrix $\mathscr{R}_n^{(k)}=[r_{ij}]$ is defined by

$$r_{ij}=\binom{i-1}{j-1}-\sum_{t=1}^{k}\binom{i-1}{j+t-1}. \tag{3.1}$$

From Definition 2, we observe that $r_{11}=1$, $r_{1j}=0$, for $j\geqslant 2$, $r_{21}=0$, $r_{22}=1$, $r_{2j}=0$, for $j\geq 3$, $r_{31}=-2$, $r_{32}=1$, $r_{33}=1$, $r_{3j}=0$, for $j\geq 4$ and

$$r_{i1}=\begin{cases}2-2^{i-1}, & 2\leqslant i\leqslant k+1,\\ 1-\sum_{t=1}^{k}\binom{i-1}{t}, & i>k+1.\end{cases}$$

Moreover, for $i,j\geqslant 2$ we have the recursion $r_{ij}=r_{i-1,j-1}+r_{i-1,j}$. For example, if we take $n=8$ and $k=5$, we get

$$\mathscr{R}_8^{(5)}=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\-2&1&1&0&0&0&0&0\\-6&-1&2&1&0&0&0&0\\-14&-7&1&3&1&0&0&0\\-30&-21&-6&4&4&1&0&0\\-61&-51&-27&-2&8&5&1&0\\-118&-112&-78&-29&6&13&6&1\end{bmatrix}.$$

By virtue of the equations (1.1), (1.3) and (3.1), we can derive the following theorem.

Theorem 3. *Let $\mathscr{R}_n^{(k)}$ be the matrix as in (3.1). For the Pascal matrix $\mathscr{P}_n$ and the k-order Fibonacci matrix $\mathscr{F}_n^{(k)}$, we have*

$$\mathscr{P}_n=\mathscr{R}_n^{(k)}\mathscr{F}_n^{(k)}.$$

Proof. Since the matrix $\mathscr{F}_n^{(k)}$ is invertible, we will prove that $\mathscr{P}_n\left(\mathscr{F}_n^{(k)}\right)^{-1}=\mathscr{R}_n^{(k)}$.

Let $\left(\mathscr{F}_n^{(k)}\right)^{-1}=[f'_{ij}]$ be the inverse of $\mathscr{F}_n^{(k)}$. Since $f'_{1j}=0$ for $j\geqslant 2$, $p_{11}f'_{11}=1$ and $r_{11}=1=\sum_{t=1}^{n}p_{1t}f'_{t1}$. Since $p_{1j}=0$ and $f'_{1j}=0$ for $j\geqslant 2, \sum_{t=1}^{n}p_{1t}f'_{tj}=0=r_{1j}$ for $j\geqslant 2$. Since $f'_{2j}=0$ for $j\geqslant 3, f'_{21}=-1$ and $f'_{22}=1$, we have $\sum_{t=1}^{n}p_{2t}f'_{t1}=r_{21}=0$. By virtue of (1.4), we have, for $i=3,4,\ldots,n,\sum_{t=1}^{n}p_{it}f'_{t1}=r_{i1}$.

Now, we consider $i \geqslant 3$ and $j \geqslant 2$. From (1.4) and the recurrence relation of r_{ij}, we have $\sum_{t=1}^{n} p_{it} f'_{tj} = r_{ij}$. Therefore, we have $\mathscr{P}_n \left(\mathscr{F}_n^{(k)}\right)^{-1} = \mathscr{R}_n^{(k)}$, and the proof is completed. □

By virtue of Theorem 3, we have the following Corollary.

Corollary 2. *For* $1 \leqslant s \leqslant n$, *we have*

$$\binom{n-1}{s-1} = \sum_{t=s}^{n} \left[\binom{n-1}{t-1} - \sum_{m=1}^{k} \binom{n-1}{t+m-1} \right] g_{t-s+1}. \tag{3.2}$$

Proof. Since $\mathscr{P}_n = \mathscr{R}_n^{(k)} \mathscr{F}_n^{(k)}$, we have $p_{ns} = \sum_{t=s}^{n} r_{n,t} g_{t-s+1}$ which is equal to (3.2). In particular, if we take $s = 1$, we have

$$p_{n1} = r_{n1}g_1 + r_{n2}g_2 + r_{n3}g_3 + \cdots + r_{n,n-1}g_{n-1} + r_{n,n}g_n$$

By virtue of (3.1), we get

$$\begin{aligned} 1 = g_1 &\left[\binom{n-1}{0} - \binom{n-1}{1} - \cdots - \binom{n-1}{k} \right] \\ &+ g_2 \left[\binom{n-1}{1} - \binom{n-1}{2} - \cdots - \binom{n-1}{k+1} \right] + \cdots \\ &+ g_{n-1} \left[\binom{n-1}{n-2} - \binom{n-1}{n-1} \right] + g_n \left[\binom{n-1}{n-1} \right]. \end{aligned}$$

By simplifying the above expressions, we find that

$$g_n = 1 - (n-2)g_{n-1} - \left(\frac{n^2 - 5n + 2}{2} \right) g_{n-2} - \sum_{t=1}^{n-3} \left[\binom{n-1}{t-1} - \sum_{m=1}^{k} \binom{n-1}{t+m-1} \right] g_t.$$

Hence the proof is completed. □

From (3.1), we can find the inverse of the matrix $\left(\mathscr{R}_n^{(k)}\right)^{-1} = [r'_{ij}]$ as

$$r'_{ij} = \sum_{t=j}^{i} (-1)^{j+t} \binom{t-1}{j-1} g_{i-t+1}. \tag{3.3}$$

Since $\mathscr{F}_n^{(k)} = \left(\mathscr{R}_n^{(k)}\right)^{-1} \mathscr{P}_n$, we have

$$f_{ns} = r'_{n1}p_{1s} + r'_{n2}p_{2s} + r'_{n3}p_{3s} + \cdots + r'_{n,n-1}p_{n-1,s} + r'_{nn}p_{ns}.$$

By virtue of (1.1) and (3.3), we get

$$g_{n-s+1} = \sum_{t=s}^{n} \left[\left[\sum_{m=t}^{n} (-1)^{t+m} \binom{m-1}{t-1} g_{n-m+1} \right] \binom{t-1}{s-1} \right].$$

In particular, if $s = 1$ then,

$$g_n = \sum_{t=1}^{n} \left[\sum_{m=t}^{n} (-1)^{t+m} \binom{m-1}{t-1} g_{n-m+1} \right].$$

For $k \geqslant 2$, we define the $n \times n$ matrices $W_n^{(k)}$ and $\overline{W}_i$ by

$$W_n^{(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -6 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 2-2^k & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\ 1-\sum_{t=1}^{k}\binom{k+1}{t} & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1-\sum_{t=1}^{k}\binom{n-1}{t} & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$\overline{W}_i = I_{n-i} \oplus W_i^{(k)}$. From the definition of $\overline{W}_i$, we have $\overline{W}_1 = \overline{W}_2 = I_n$. Thus, we have the following lemma by the definition of the matrix product.

Lemma 2. *The matrix $\mathscr{R}_n^{(k)}$ can be factored by the $\overline{W}_i$'s as follows:*

$$\mathscr{R}_n^{(k)} = \overline{W}_n \overline{W}_{n-1} \overline{W}_{n-2} \cdots \overline{W}_2 \overline{W}_1.$$

By virtue of Lemma 2, we have the following theorem.

Theorem 4. *For the $n \times n$ Pascal matrix, $\mathscr{P}_n$, we have*

$$\mathscr{P}_n = \prod_{i=1}^{n} \overline{W}_{n-i+1} \prod_{i=1}^{n} G_i.$$

4. STIRLING NUMBER OF THE SECOND KIND

In this section, we consider relationships between Stirling matrices of the second kind and k-order Fibonacci matrices.

We define an $n \times n$ matrix $\mathscr{M}_n^{(k)} = [m_{ij}]$ by using the Stirling numbers of the second kind as follows:

$$m_{ij} = S(i,j) - \sum_{t=1}^{k} S(i-t,j). \tag{4.1}$$

From (4.1), we observe that $m_{11} = 1, m_{1j} = 0$ for $j \geqslant 2, m_{21} = 0, m_{22} = 1$ $m_{2j} = 0$ for $j \geqslant 3$,

$$m_{i1} = \begin{cases} 2-i, & 1 \leqslant i \leqslant k, \\ 1-k, & i > k, \end{cases} \tag{4.2}$$

and, for $i,j \geqslant 2, m_{ij} = m_{i-1,j-1} + jm_{i-1,j}$. From the definition of $\mathscr{S}_n^{(2)}, \mathscr{F}_n^{(k)}$ and $\mathscr{M}_n^{(k)}$, we have the following theorem.

Theorem 5. *Let $\mathscr{M}_n$ be the $n \times n$ matrix as in (4.1). For the Stirling matrix of the second kind $\mathscr{S}_n^{(2)}$ and the Fibonacci matrix $\mathscr{F}_n^{(k)}$, we have $\mathscr{S}_n^{(2)} = \mathscr{F}_n^{(k)} \mathscr{M}_n^{(k)}$.*

Proof. Since the matrix $\mathscr{F}_n^{(k)}$ is invertible, we will prove that $\left(\mathscr{F}_n^{(k)}\right)^{-1} \mathscr{S}_n^{(2)} = \mathscr{M}_n^{(k)}$.

Let $\left(\mathscr{F}_n^{(k)}\right)^{-1} = [f'_{ij}]$ be the inverse of $\mathscr{F}_n^{(k)}$. Since $f'_{1j} = 0$ for $j \geqslant 2, f'_{11}S_{11} = 1 = m_{11}$. As $S_{1j} = 0$ and $f'_{1j} = 0$ for $j \geqslant 2, \sum_{t=1}^{n} f'_{1t}S_{tj} = 0 = m_{1j}$ for $j \geqslant 2$. As $f'_{2j} = 0$ for $j \geqslant 3, f'_{21} = -1$ and $f'_{22} = 1$, we have $\sum_{t=1}^{n} f'_{2t}S_{t1} = m_{21}$. By virtue of (1.4), we have, for $i = 3,4,\ldots,n$, $\sum_{t=1}^{n} f'_{it}S_{t1} = m_{i1}$ which is given in (4.2).

Now we consider $i \geqslant 3$ and $j \geqslant 2$. By (1.4) and (4.1), we have $\sum_{t=1}^{n} f'_{it}S_{tj} = m_{ij}$. Thus, we have $\mathscr{M}_n^{(k)} = \left(\mathscr{F}_n^{(k)}\right)^{-1} \mathscr{S}_n^{(2)}$, i.e. $\mathscr{S}_n^{(2)} = \mathscr{F}_n^{(k)} \mathscr{M}_n^{(k)}$ which completes the proof. □

For example, for $k = 5$ and $n = 8$, we have

$$\mathscr{S}_8^{(2)} = \mathscr{F}_8^{(5)} \mathscr{M}_8^{(5)}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 & 0 & 0 \\ 16 & 8 & 4 & 2 & 1 & 1 & 0 & 0 \\ 31 & 16 & 8 & 4 & 2 & 1 & 1 & 0 \\ 61 & 31 & 16 & 8 & 4 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & 5 & 1 & 0 & 0 & 0 & 0 \\ -3 & 4 & 18 & 9 & 1 & 0 & 0 & 0 \\ -4 & 5 & 58 & 54 & 14 & 1 & 0 & 0 \\ -4 & 6 & 179 & 274 & 124 & 20 & 1 & 0 \\ -4 & 8 & 543 & 1275 & 894 & 244 & 27 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 & 0 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 & 0 & 0 \\ 1 & 63 & 301 & 350 & 140 & 21 & 1 & 0 \\ 1 & 127 & 966 & 1701 & 1050 & 266 & 28 & 1 \end{bmatrix}.$$

Since $S_{nt} = S(n,t) = \sum_{r=1}^{n} f_{nr} m_{rt}$ and for $i > k$

$$m_{ij} = \frac{1}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left((j-l)^i - \sum_{t=1}^{k} (j-l)^{i-t} \right),$$

we have the following corollary.

Corollary 3. *For $k < j \leqslant n$, we have*

$$S(n,j) = \sum_{i=j}^{n} g_{n-i+1} \left[\frac{1}{j!} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \left((j-l)^i - \sum_{t=1}^{k} (j-l)^{i-t} \right) \right].$$

Lemma 3. *For the $(n-1) \times (n-1)$ Stirling matrix of the second kind, $\mathscr{S}_{n-1}^{(2)}$, we have*

$$\mathscr{M}_n^{(k)} = \mathscr{L}_n^{(k)} \left([1] \oplus \mathscr{S}_{n-1}^{(2)} \right).$$

Proof. Let $\Phi_n = [\varphi_{ij}] = \mathscr{L}_n^{(k)} \left([1] \oplus \mathscr{S}_{n-1}^{(2)} \right)$. By Theorem 1, we know that $l_{11} = 1 = m_{11}$, $l_{21} = 0 = m_{21}$ and $l_{22} = S(1,1) = 1 = m_{22}$. So, we have $\varphi_{ij} = m_{ij}$ for $i = 1,2$.

Now we consider the case $i \geqslant 3$. Since

$$\varphi_{ij} = \sum_{t=j-1}^{i-1} \left[\binom{i-1}{t} S(t,j-1) - \sum_{s=1}^{k} \binom{i-s-1}{t} S(t,j-1) \right]$$

and from (1.2), we have $\varphi_{ij} = S(i,j) - \sum_{t=1}^{k} S(i-t,j) = m_{ij}$.

Therefore, $\mathscr{M}_n^{(k)} = \mathscr{L}_n^{(k)} \left([1] \oplus \mathscr{S}_{n-1}^{(2)} \right)$. □

The following corollary is an immediate consequence of Lemma 3.

Corollary 4. *For $n \geqslant 2$, $\mathscr{S}_n^{(2)} = \mathscr{F}_n^{(k)} \mathscr{L}_n^{(k)} \left([1] \oplus \mathscr{S}_{n-1}^{(2)} \right)$.*

For $(s \times s)$ matrices $\mathscr{F}_s^{(k)}$ and $\mathscr{L}_s^{(k)}$, we define the $n \times n$ matrix $\overline{\mathscr{F}_n^{(k)} \mathscr{L}_n^{(k)}}$ by $\overline{\mathscr{F}_s^{(k)} \mathscr{L}_s^{(k)}} = I_{n-s} \oplus \mathscr{F}_s^{(k)} \mathscr{L}_s^{(k)}$. So, $\overline{\mathscr{F}_n^{(k)} \mathscr{L}_n^{(k)}} = \mathscr{F}_n^{(k)} \mathscr{L}_n^{(k)}$ and $\overline{\mathscr{F}_1^{(k)} \mathscr{L}_1^{(k)}} = I_n$. Therefore, we can give the following corollary.

Corollary 5. *The Stirling matrix of the second kind, $\mathscr{S}_n^{(2)}$, can be factorized by the matrices $\overline{\mathscr{F}_n^{(k)}\mathscr{L}_n^{(k)}}$'s as :*

$$\mathscr{S}_n^{(2)} = \overline{\left(\mathscr{F}_n^{(k)}\mathscr{L}_n^{(k)}\right)}\,\overline{\left(\mathscr{F}_{n-1}^{(k)}\mathscr{L}_{n-1}^{(k)}\right)}\cdots\overline{\left(\mathscr{F}_2^{(k)}\mathscr{L}_2^{(k)}\right)}\,\overline{\left(\mathscr{F}_1^{(k)}\mathscr{L}_1^{(k)}\right)}.$$

Since $P_n = \mathscr{F}_n^{(k)}\mathscr{L}_n^{(k)}$ and $\overline{P}_s = \overline{\mathscr{F}_s^{(k)}\mathscr{L}_s^{(k)}}$, we can give the following corollary as an immediate consequence of Corollary 5.

Corollary 6. *The Stirling matrix of the second kind, $\mathscr{S}_n^{(2)}$, can be factorized by the Pascal matrices $\overline{P}_n$'s as :*

$$\mathscr{S}_n^{(2)} = \overline{P}_n\overline{P}_{n-1}\cdots\overline{P}_2\overline{P}_1.$$

5. Stirling number of the first kind

In this section, we consider the relationships between the Stirling matrix of the first type $\mathscr{S}_n^{(1)}$ and k-order Fibonacci matrix $\mathscr{F}_n^{(k)}$.

We define an $n\times n$ matrix $\mathscr{C}_n^{(k)} = [c_{ij}]$ by using the Stirling number of the first kind as follows:

$$c_{ij} = s(i,j) - \sum_{t=1}^{k} s(i,j+t). \tag{5.1}$$

From (5.1), we see that $c_{11}=1$, $c_{1j}=0$, for $j\geqslant 2$, $c_{21}=-2$, $c_{22}=1$, $c_{2j}=0$, for $j\geqslant 3$, and for $i,j\geqslant 2$, $c_{ij}=c_{i-1,j-1}-(i-1)c_{i-1,j}$. By virtue of the definition of $\mathscr{S}_n^{(1)}$, $\mathscr{C}_n^{(k)}$ and $\mathscr{F}_n^{(k)}$, we can derive the following Theorem.

Theorem 6. *Let $\mathscr{C}_n^{(k)}$ be the $n\times n$ matrix as in (5.1). For the Stirling matrix of the first kind $\mathscr{S}_n^{(1)}$ and the Fibonacci matrix $\mathscr{F}_n^{(k)}$, we have $\mathscr{S}_n^{(1)} = \mathscr{C}_n^{(k)}\mathscr{F}_n^{(k)}$.*

Proof. As $\mathscr{F}_n^{(k)}$ is an invertible matrix, it suffices to prove that $\mathscr{C}_n^{(k)} = \mathscr{S}_n^{(1)} \cdot \left(\mathscr{F}_n^{(k)}\right)^{-1}$.

Let $\left(\mathscr{F}_n^{(k)}\right)^{-1} = [f'_{ij}]$ be the inverse of $\mathscr{F}_n^{(k)}$. Since $f'_{1j}=0$ for $j\geqslant 2, s_{11}f'_{11}=1 = c_{11}$. Since $s_{1j}=0$ and $f'_{1j}=0$ for $j\geqslant 2, \sum_{t=1}^n s_{1t}f'_{tj}=0=c_{1j}$ for $j\geqslant 2$. Since $f'_{2j}=0$ for $j\geqslant 3, f'_{21}=-1$ and $f'_{22}=1$, we have $\sum_{t=1}^n s_{2t}f'_{t1}=c_{21}=-2$ and $c_{22}=1$. By virtue of (1.4), we have, for $i=3,4,\ldots,n, \sum_{t=1}^n s_{it}f'_{t1}=c_{i1}$.

Now, we consider $i\geqslant 3$ and $j\geqslant 2$. From (1.4) and the recurrence relation $c_{ij} = c_{i-1,j-1}-(i-1)c_{i-1,j}$, we have $\sum_{t=1}^n s_{it}f'_{tj}=c_{ij}$. Therefore, we have $\mathscr{C}_n^{(k)} = \mathscr{S}_n^{(1)} \cdot \left(\mathscr{F}_n^{(k)}\right)^{-1}$, and the proof is completed. □

For example, for $k=4$ and $n=6$, we have

$$\mathscr{S}_6^{(1)} = \mathscr{C}_6^{(4)} \mathscr{F}_6^{(4)}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 4 & -4 & 1 & 0 & 0 & 0 \\ -12 & 16 & -7 & 1 & 0 & 0 \\ 48 & -76 & 44 & -11 & 1 & 0 \\ -239 & 428 & -296 & 99 & -16 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 \\ 15 & 8 & 4 & 2 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 & 0 \\ 24 & -50 & 35 & -10 & 1 & 0 \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}.$$

By virtue of Theorem 6, we can arrive at the following interesting identity:

$$(n-1)! = \sum_{j=1}^{n} (-1)^{n+1} \left(s(n,j) - \sum_{t=1}^{k} s(n,j+t) \right) g_j.$$

Furthermore, by analogy to the Corollaries 4, 5 and 6, we can derive the following theorem.

Theorem 7. *For the Stirling matrix $\mathscr{S}_n^{(1)}$ of the first kind,*

$$\mathscr{S}_n^{(1)} = \left([1] \oplus \mathscr{S}_{n-1}^{(1)} \right) \mathscr{F}_n^{(k)} \mathscr{L}_n^{(k)} = \overline{P}_1 \overline{P}_2 \cdots \overline{P}_n.$$

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