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BOUNDS FOR THE GENERALIZED ELLIPTIC INTEGRAL OF THE SECOND KIND

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Abstract. For $a \in (0,1)$ and $r \in (0,1)$, let $\mathcal{E}_a(r)$ be the generalized elliptic integral of the second kind and $R(a)$ be Ramanujan's constant. In this paper, we prove the following inequalities

$$\frac{\sin(\pi a)}{2(1-a)} + r'^2\left((1-a)\sin(\pi a)\log\left(\frac{e^{R(a)/2}}{r'}\right) - \sigma\right) < \mathcal{E}_a(r)$$

$$< \frac{\sin(\pi a)}{2(1-a)} + r'^2\left((1-a)\sin(\pi a)\log\left(\frac{e^{R(a)/2}}{r'}\right) - \delta\right)$$

with the best possible constants $\sigma = \frac{1}{4}\sin(\pi a)$ and $\delta = \frac{\sin(\pi a)}{2(1-a)} + (1-a)\sin(\pi a)\frac{R(a)}{2} - \frac{\pi}{2}$.

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1. Introduction

For real numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is defined by

$$_2F_1(a,b;c;x) \equiv \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \quad |x| < 1.$$

Here $(a,0) = 1$ for $a \neq 0$, and (a,n) is the *shifted factorial function*

$$(a,n) \equiv a(a+1)(a+2)\cdots(a+n-1)$$

for $n \in \mathbb{N} \equiv \{k : k \text{ is a positive integer}\}$.

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The generalized elliptic integrals can be represented by the Gaussian hypergeometric function as follows [2]: for $a \in (0,1)$,

$$\begin{cases} \mathcal{K}_a = \mathcal{K}_a(r) = \dfrac{\pi}{2} {}_2F_1\left(a, 1-a; 1; r^2\right), \\ \mathcal{K}_a' = \mathcal{K}_a'(r) = \mathcal{K}_a(r'), \\ \mathcal{K}_a(0) = \dfrac{\pi}{2}, \quad \mathcal{K}_a(1) = \infty, \end{cases}$$

and

$$\begin{cases} \mathcal{E}_a = \mathcal{E}_a(r) = \dfrac{\pi}{2} {}_2F_1\left(1-a, a-1; 1; r^2\right), \\ \mathcal{E}_a' = \mathcal{E}_a'(r) = \mathcal{E}_a(r'), \\ \mathcal{E}_a(0) = \dfrac{\pi}{2}, \quad \mathcal{E}_a(1) = 1. \end{cases}$$

Here and hereafter, we always let $r' = (1-r^2)^{1/2}$. It is easy to see that $\mathcal{K}_a$ is strictly increasing and $\mathcal{E}_a$ is strictly decreasing on $(0,1)$. The generalized elliptic integrals satisfy the following Legendre relation:

$$\mathcal{K}_a(r)\mathcal{E}_a'(r) + \mathcal{K}_a'(r)\mathcal{E}_a(r) - \mathcal{K}_a(r)\mathcal{K}_a'(r) = \frac{\pi \sin(\pi a)}{4(1-a)}.$$

For $a = 1/2$, the generalized elliptic integrals reduce to the classical complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$, respectively. It is well known that the complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ have many applications in several fields of mathematics as well as in physics and engineering. Numerous properties have been obtained for $\mathcal{K}$ and $\mathcal{E}$ (for instance, see [1, 4–7, 9, 18]).

In 2000, Anderson, et al [2] investigated certain combinations of the generalized elliptic integrals which occur in Ramanujan's modular equations and approximations to π. They showed the monotonicity and convexity properties of these quantities and obtained various sharp inequalities for them. Recently, the generalized elliptic integrals have attracted the attention of many mathematicians. In particular, many remarkable properties and inequalities for the generalized elliptic integrals can be found in the literature [2, 6, 8, 10–13, 15, 17, 19].

During the past few decades, many authors obtained various sharp elementary estimates for $\mathcal{E}$ and $\mathcal{E}_a$. For instance, in 2004, Alzer and Qiu [1] proved the inequality

$$\frac{\pi}{2}\left(\frac{1+r'^{\eta}}{2}\right)^{\eta} < \mathcal{E}(r) < \frac{\pi}{2}\left(\frac{1+r'^{\theta}}{2}\right)^{\theta},$$

where $\eta = 3/2$ and $\theta = \log 2/\log(\pi/2)$ are best possible. This result improves previous known bounds and gives symmetric upper and lower bounds.

In 2011, Wang et al. [13, Theorem 1.2] obtained the following sharp bounds for $\mathcal{E}(r)$:

$$\frac{\pi}{2}-\frac{3\pi}{16}\frac{r-r'^2\,\mathrm{arth}r}{r}<\mathcal{E}(r)<\frac{\pi}{2}-\left(\frac{\pi}{2}-1\right)\frac{r-r'^2\,\mathrm{arth}r}{r} \tag{1.1}$$

and

$$\frac{\pi}{2}-\log r'-r\,\mathrm{arth}r<\mathcal{E}(r)<\frac{\pi}{2}-\left(2-\frac{\pi}{4}\right)\log r'-r\,\mathrm{arth}r. \tag{1.2}$$

The purpose of this paper is to study some monotonicity properties for the generalized elliptic integrals of the second kind and obtain the following theorem.

Theorem 1. *Given $a\in(0,1)$, let $R(a)$ be Ramanujan's constant defined as (2.1). Then, for all $r\in(0,1)$,*

$$\frac{\sin(\pi a)}{2(1-a)}+r'^2\left((1-a)\sin(\pi a)\log\left(\frac{e^{R(a)/2}}{r'}\right)-\sigma\right)<\mathcal{E}_a(r)$$
$$<\frac{\sin(\pi a)}{2(1-a)}+r'^2\left((1-a)\sin(\pi a)\log\left(\frac{e^{R(a)/2}}{r'}\right)-\delta\right),$$

where the constants $\sigma=\frac{1}{4}\sin(\pi a)$ and $\delta=\frac{\sin(\pi a)}{2(1-a)}+(1-a)\sin(\pi a)\frac{R(a)}{2}-\frac{\pi}{2}$ are best possible.

As an application, we obtain a sharp functional inequality for the generalized Hersch-Pfluger distortion function

$$\varphi_k^a(r)<r^{1/K}\exp\left\{(1-\frac{1}{K})\frac{R(a)}{\pi-2}r'^2(\frac{1}{2}\log\frac{2}{r'}+\frac{\pi}{2}-1)\right\}.$$

The definition of the generalized Hersch-Pfluger distortion function is in Section 3.

2. Proof of main result

Throughout this paper, let $\gamma=0.577215\ldots$ be the Euler constant and let

$$R(a,b)=-2\gamma-\psi(a)-\psi(b),\qquad R(a)=R(a,1-a),\qquad R(1/2)=\log 16 \tag{2.1}$$

be Ramanujan's constant, where ψ is the classical psi function.

The functions $\mathcal{K}_a$ and $\mathcal{E}_a$ satisfy the following derivative formulas [2]:

$$\frac{d\mathcal{K}_a}{dr}=\frac{2(1-a)}{rr'^2}(\mathcal{E}_a-r'^2\mathcal{K}_a),$$
$$\frac{d\mathcal{E}_a}{dr}=\frac{-2(1-a)}{r}(\mathcal{K}_a-\mathcal{E}_a),$$
$$\frac{d(\mathcal{E}_a-r'^2\mathcal{K}_a)}{dr}=2ar\mathcal{K}_a,$$
$$\frac{d(\mathcal{K}_a-\mathcal{E}_a)}{dr}=\frac{2(1-a)r\mathcal{E}_a}{r'^2}.$$

We shall prove monotonicity properties of some functions defined by the generalized elliptic integrals.

Lemma 1. *The function*

$$v(a) = (1-a)^2(4a^2-8a)\pi + 8(1-a)\sin(\pi a)$$

is positive on (0,1).

Proof. Let $w(a) = v(a)/(1-a) = (1-a)(4a^2-8a)\pi + 8\sin(\pi a)$. We need to prove $w(a)$ is positive on $(0,1)$. Simple computations lead to

$$w(0) = 0 = w(1). \tag{2.2}$$

By differentiation, we have

$$w'(a) = (1-a)(8a-8)\pi - (4a^2-8a)\pi + 8\pi\cos(\pi a)$$

with

$$w'(0+) = 0, \quad w'(1-) = -4\pi, \tag{2.3}$$

$$w''(a) = (24-24a)\pi - 8\pi^2\sin(\pi a)$$

with

$$w''(0+) = 24\pi, \quad w''(1-) = 0, \tag{2.4}$$

and

$$w'''(a) = -24\pi - 8\pi^3\cos(\pi a)$$

with

$$w'''(0+) = -24\pi - 8\pi^3 < 0, \quad w'''(1-) = -24\pi + 8\pi^3 > 0. \tag{2.5}$$

Since $w'''(a)$ is strictly increasing in a, we see from (2.5) that there exists $a_0 \in (0,1)$ such that $w'''(a) < 0$ for $a \in (0,a_0)$ and $w'''(a) > 0$ for $a \in (a_0,1)$. Thus $w''(a)$ is strictly decreasing on $(0,a_0)$ and strictly increasing on $(a_0,1)$. The limiting values (2.4) and the piecewise monotonicity of $w''(a)$ imply that there exists $a_1 \in (0,a_0)$ such that $w''(a) > 0$ for $a \in (0,a_1)$ and $w''(a) < 0$ for $a \in (a_1,1)$. Then $w'(a)$ is strictly increasing on $a \in (0,a_1)$ and strictly decreasing on $(a_1,1)$. It follows from (2.3) together with the piecewise monotonicity of $w'(a)$ that there exists $a_2 \in (a_1,1)$ such that $w'(a) > 0$ for $a \in (0,a_2)$ and $w'(a) < 0$ for $a \in (a_2,1)$. Thus $w(a)$ is strictly increasing on $(0,a_2)$ and strictly decreasing on $(a_2,1)$. The limiting values (2.2) and the piecewise monotonicity of $w(a)$ imply that $w(a) > 0$ for $a \in (0,1)$ and hence $v(a) > 0$. □

Lemma 2. *The function*

$$p(a) = (1-(1-a)^2)\pi - \frac{\sin(\pi a)}{1-a} - (1-a)\sin(\pi a)$$

is negative on (0,1).

Proof.

$$
\begin{aligned}
p(a) &= \pi\left(1-(1-a)^2-\frac{1+(1-a)^2}{\pi(1-a)}\sin(\pi a)\right)\\
&= \pi\left(1-(1-a)^2-\frac{1+(1-a)^2}{\pi(1-a)}\sin(\pi(1-a))\right)\\
&< \pi\left(1-(1-a)^2-(1+(1-a)^2)\frac{1-(1-a)^2}{1+(1-a)^2}\right)=0,
\end{aligned}
$$

where the inequality follows from the well-known Redheffer inequality [16]:

$$\frac{\sin(\pi x)}{\pi x} > \frac{1-x^2}{1+x^2} \quad \text{for} \quad x \in (0,1).$$

□

Lemma 3. *For given* $a \in (0,1)$*, the function*

$$u(r) = -2(1-a)(\mathcal{K}_a(r)-\mathcal{E}_a(r))r'^2 + 2r^2\mathcal{E}_a(r) - \frac{1+(1-a)^2r'^2}{1-a}r^2\sin(\pi a)$$

is negative on $(0,1)$.

Proof. We denote f and g respectively the functions

$$f(r) = 4(1-(1-a)^2)\mathcal{E}_a(r) - \frac{2\sin(\pi a)}{1-a} - 2(1-a)\sin(\pi a)(r'^2-r^2)$$

and

$$g(r) = -8(1-(1-a)^2)(1-a)\frac{\mathcal{K}_a(r)-\mathcal{E}_a(r)}{r^2} + 8(1-a)\sin(\pi a).$$

Applying the derivative formulas, we obtain

$$u'(r) = rf(r) \quad \text{and} \quad f'(r) = rg(r).$$

Since the function $r \to (\mathcal{K}_a(r)-\mathcal{E}_a(r))/r^2$ is strictly increasing from $(0,1)$ onto $(\pi(1-a)/2,\infty)$ (see [2]), the function g is strictly decreasing from $(0,1)$ onto $(-\infty,(a-1)^2(4a^2-8a)\pi+8(1-a)\sin(\pi a))$. It follows from Lemma 1 that $(a-1)^2(4a^2-8a)\pi+8(1-a)\sin(\pi a)>0$. We see that there exists a number $r_0 \in (0,1)$ such that $rg(r)$ is positive on $(0,r_0)$ and negative on $(r_0,1)$. Hence, the function f is strictly increasing on $(0,r_0)$ and decreasing on $(r_0,1)$. Since $f(0^+) = 2(1-(1-a)^2)\pi - 2\sin(\pi a)/(1-a) - 2(1-a)\sin(\pi a) < 0$ by Lemma 2 and $f(1^-) = 0$, we conclude that there exists a number $r_1 \in (0,1)$ such that $rf(r)$ is negative on $(0,r_1)$ and positive on $(r_1,1)$. Therefore the function u is strictly decreasing on $(0,r_1)$ and increasing on $(r_1,1)$. Considering the fact $u(0^+) = 0 = u(1^-)$, we conclude that $u(r) < 0$ for $r \in (0,1)$. □

Proof of Theorem 1. Given $a \in (0,1)$. For $r \in (0,1)$, we construct a function

$$H(r) = \frac{1}{r'^2}\left(\mathcal{E}_a(r) - \frac{\sin(\pi a)}{2(1-a)}\right) - (1-a)\sin(\pi a)\log\frac{e^{R(a)/2}}{r'}.$$

We aim to prove that the function H is strictly decreasing from $(0,1)$ onto $\left(-\frac{1}{4}\sin(\pi a), -\delta\right)$, where $\delta = \frac{\sin(\pi a)}{2(1-a)} + (1-a)\sin(\pi a)\frac{R(a)}{2} - \frac{\pi}{2}$. In fact, differentiation yields that

$$\begin{aligned} rr'^4H'(r) = &-2(1-a)(\mathcal{K}_a(r) - \mathcal{E}_a(r))r'^2 + 2r^2\mathcal{E}_a(r) \\ &- \frac{r^2\sin(\pi a)}{1-a} - (1-a)\sin(\pi a)r^2r'^2 < 0 \end{aligned}$$

by Lemma 3. Hence, the function H is strictly decreasing on $(0,1)$. For the limiting values, it is clear that

$$H(0^+) = -\frac{\sin(\pi a)}{2(1-a)} - (1-a)\sin(\pi a)\frac{R(a)}{2} + \frac{\pi}{2} = -\delta.$$

Write

$$\begin{aligned} h_1(r) &= \mathcal{E}_a(r) - \frac{\sin(\pi a)}{2(1-a)} - (1-a)r'^2\sin(\pi a)\log\left(\frac{e^{R(a)/2}}{r'}\right), \\ h_2(r) &= r'^2. \end{aligned}$$

Then $h_1(1^-) = 0 = h_2(1^-)$, and by l'Hôpital's rule

$$\begin{aligned} H(1^-) &= \lim_{r\to 1^-}\frac{h_1'(r)}{h_2'(r)} \\ &= \lim_{r\to 1^-}\frac{2(1-a)}{2r^2}(\mathcal{K}_a(r) - \mathcal{E}_a(r)) - (1-a)\sin(\pi a)\log(\frac{e^{R(a)/2}}{r'}) + \frac{1}{4}\sin(\pi a) \\ &= \lim_{r\to 1^-} -\frac{1}{2}\sin(\pi a) + (1-a)(\mathcal{K}_a(r) - \sin(\pi a)\log(\frac{e^{R(a)/2}}{r'})) + \frac{1}{4}\sin(\pi a) \\ &= -\frac{1}{2}\sin(\pi a) + \frac{1}{4}\sin(\pi a) \\ &= -\frac{1}{4}\sin(\pi a), \end{aligned}$$

where we used the fact

$$\lim_{r\to 1^-}\mathcal{K}_a(r) - \sin(\pi a)\log(\frac{e^{R(a)/2}}{r'}) = 0.$$

The desired inequalities follow from the monotonicity of the function $H(r)$ and its limiting values. □

Setting $a = 1/2$ in Theorem 1, we have the following inequality for the complete elliptic integral of the second kind.

Corollary 1. *For $r \in (0,1)$, then*

$$1 + r'^2\left(\frac{1}{2}\log\frac{4}{r'} - \gamma\right) < \mathcal{E}(r) < 1 + r'^2\left(\frac{1}{2}\log\frac{4}{r'} - \delta\right),$$

where the constants $\gamma = \frac{1}{4}$ and $\delta = 1 + \log 2 - \frac{\pi}{2}$ are best possible.

Remark 1. Computational and numerical experiments show that, on the whole interval $(0,1)$, neither the upper bound nor lower bound in Corollary 1 is comparable with the bounds in (1.1) and (1.2).

3. Application to the generalized Hersch-Pfluger function

Three related functions μ_a, m_a, and φ_K^a are defined as follows [2]: for $a \in (0,1/2)$,, $r \in (0,1)$, and $K \in (0,\infty)$,

$$\mu_a(r) = \frac{\pi}{2\sin\pi a}\frac{\mathcal{K}_a'(r)}{\mathcal{K}_a(r)},$$
$$m_a(r) = \frac{2}{\pi\sin\pi a} r'^2 \mathcal{K}_a(r)\mathcal{K}_a'(r),$$
$$\varphi_K^a(r) = \mu_a^{-1}(\mu_a(r)/K).$$

The function φ_K^a is called the generalized Hersch-Pfluger function. When $a = 1/2$, the function reduces to the Hersch-Pfluger distortion function. These functions have many important applications in the distortion theory of quasiconformal mappings and Ramanujan's modular equations [2–6, 12, 14].

We have the following sharp functional inequalities (see [13] and [10], respectively): for all $r \in (0,1)$,

$$2(\mathcal{E}(r) - 1) < m_a(r) + \log r < \frac{R(a)}{\pi - 2}(\mathcal{E}(r) - 1), \tag{3.1}$$

and

$$\varphi_K^a(r) < r^{1/K}\exp\left\{\left(1 - \frac{1}{K}\right)(m_a(r) + \log r)\right\} \tag{3.2}$$

for $K \in [1,\infty)$.

Combining the above inequalities (3.1), (3.2) and Corollary 1, we obtain a sharp functional inequalities for the generalized Hersch-Pfluger function.

Corollary 2. *For $a \in (0,1/2]$ and $K \in [1,+\infty)$, the inequality*

$$\varphi_K^a(r) < r^{1/K}\exp\left\{\left(1 - \frac{1}{K}\right)\frac{R(a)}{\pi - 2} r'^2\left(\frac{1}{2}\log\frac{2}{r'} + \frac{\pi}{2} - 1\right)\right\}$$

holds for all $r \in (0,1)$.

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