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GLOBAL DYNAMICAL BEHAVIOURS AND PERIODICITY OF A CERTAIN QUADRATIC-RATIONAL DIFFERENCE EQUATION WITH DELAY

E. TAŞDEMIR, M. GÖCEN, AND Y. SOYKAN

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Abstract. Our aim in this paper is to deal with the dynamics of following higher order difference equation

$$x_{n+1} = A + B\frac{x_{n-m}}{x_n^2}$$

where $A, B > 0$, and initial values are positive, and $m = \{1, 2, ...\}$. Furthermore, we discuss the periodicity, boundedness, semi-cycles, global asymptotic stability of solutions of these equations. We also handle the rate of convergence of solutions of these difference equations.

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Keywords: Difference equations, periodicity, boundedness, semi-cycle, global asymptotic stability, rate of convergence, delay.

1. INTRODUCTION

Last few decades, rational difference equations and their systems have attracted the interest of many researchers for varied reasons. One of the reason of this rapid growth of interest is, these equations provided a natural description of many discrete mathematical models. Such discrete mathematical models are often scrutinized in various fields of science and technology for instance, biology, ecology, physiology, physics, engineering, economics, probability theory, genetics, psychology, resource management and population dynamics. We believe that the interest of studying rational difference equations will increase in future years as more fascinating and intresting results are obtaining. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions. Furthermore, higher-order rational difference equations and systems of rational equations have also been widely studied but still have many aspects to be investigated. There are many papers related to the rational difference equations and higher-order rational difference equations (see, for example, [2–4, 11, 13] and references therein).

In [5], Amleh et al. discussed the stability, boundedness and periodic character of solutions of difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n},$$

where the initial values are positive numbers, and $\alpha \geq 0$.

In [9], Devault et al. studied periodicity, global stability and the boundedness of solutions of the following higher order difference equation

$$x_{n+1} = p + \frac{x_{n-k}}{x_n},$$

where the initial conditions are positive numbers, and $p > 0$.

In [16], Saleh et al. handled the dynamical behaviours of following higher order difference equation

$$y_{n+1} = A + \frac{y_{n-k}}{y_n}, \tag{1.1}$$

with $k \in \{2, 3, \cdots\}$ and A is positive. The authors especially discussed the global asymptotic stability, semi-cycle analysis and periodicity of the unique positive equilibrium of Eq.(1.1).

In [1], Abu-Saris et al. dealt with the global asymptotic stability of positive equilibrium point of difference equations

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \tag{1.2}$$

where $k \in \{2, 3, \cdots\}$ and A is positive. Moreover, in [17], Saleh et al. studied the global stability of the negative equilibrium of the difference equation (1.2) where $k \in \{1, 2, \cdots\}$ and $A < 0$.

In [12], Hamza et al. discussed the dynamics of following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}, \tag{1.3}$$

where α, k and the initial values are positive real numbers. The authors dealt with the boundedness, oscillation behaviours and stability analysis of unique equilibrium point of Eq.(1.3).

In [20], Yalçınkaya handled the oscillation behaviours, bounded solutions, periodic solutions and global stability of solutions of difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k},$$

where the initial values are positive real numbers and $\alpha, k > 0$.

In [18], Stevic investigated the dynamical properties of difference equation

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)},$$

where $x_{-1}, x_0 > 0$.

In [6], Bešo et al. showed the Neimark–Sacker bifurcation, boundedness and global attractivity of following difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2},$$

with the initial conditions and γ, δ are positive real numbers.

In [19], Taşdemir investigated the boundedness, rate of convergence, global asymptotic stability and periodicity of the following higher order difference equations

$$x_{n+1} = A + B \frac{x_n}{x_{n-m}^2}, \tag{1.4}$$

where the initial conditions and A, B are positive real numbers and $m \in \{2, 3, \cdots\}$.

Our aim in this work is to deal with the dynamics of following higher order difference equation

$$x_{n+1} = A + B \frac{x_{n-m}}{x_n^2}, \tag{1.5}$$

with $m = \{1, 2, ...\}$, and the initial conditions are positive numbers, and $A, B > 0$. We first handle the periodicity, boundedness and oscillation behaviors of solutions of Eq.(1.5). Moreover, we analyze the local and global asymptotic stability of the solutions of Eq.(1.5). Finally, we study the rate of convergence of Eq.(1.5) and we present some numerical examples to verify our theoretical results.

Here, we summarize the significant results and definitions on the theory of difference equations. For more information, see [8, 10, 14] and references therein.

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \quad n = 0, 1, \cdots. \tag{1.6}$$

A solution of Eq.(1.6) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(1.6) for all $n \geq -k$.

Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, \cdots, \bar{x}), \text{ for } i = 0, 1, \cdots, k$$

denote the partial derivative of $f(u_0, u_1, \cdots, u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(1.6).

Definition 1. The equation

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \cdots + q_k z_{n-k}, \; n = 0, 1, \cdots, \tag{1.7}$$

is called the linearized equation of Eq.(1.6) about the equilibrium point $\bar{x}$.

Theorem 1 (Clark's Theorem)**.** *Consider Eq.(1.7). Then,*

$$\sum_{i=0}^{k} |q_i| < 1.$$

is a sufficient condition for the locally asymptotically stability of Eq.(1.6).

Consider the scalar kth-order linear difference equation

$$x_{n+k} + p_1(n)x_{n+k-1} + \cdots + p_k(n)x_n = 0, \tag{1.8}$$

where k is a positive integer and $p_i : \mathbb{Z}^+ \to \mathbb{C}$ for $i = 1, \cdots, k$. Assume that

$$q_i = \lim_{k\to\infty} p_i(n), i = 1, \cdots, k, \tag{1.9}$$

exist in $\mathbb{C}$. Consider the limiting equation of (1.8):

$$x_{n+k} + q_1 x_{n+k-1} + \cdots + q_k x_n = 0. \tag{1.10}$$

Theorem 2 (Poincaré's Theorem)**.** *Consider (1.8) subject to condition (1.9). Let $\lambda_1, \cdots, \lambda_k$ be the roots of the characteristic equation*

$$\lambda^k + q_1\lambda^{k-1} + \cdots + q_k = 0 \tag{1.11}$$

of the limiting equation (1.10) and suppose that $|\lambda_i| \neq |\lambda_j|$ for $i \neq j$. If x_n is a solution of (1.8), then either $x_n = 0$ for all large n or there exists an index $j \in \{1, \cdots, k\}$ such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_j.$$

The following results were obtained by Perron, and one of Perron's results was improved by Pituk, see [15].

Theorem 3. *Suppose that (1.9) holds. If x_n is a solution of (1.8), then either $x_n = 0$ eventually or*

$$\lim_{n\to\infty} \sup \left(|x_j(n)|\right)^{1/n} = \lambda_j.$$

where $\lambda_1, \cdots, \lambda_k$ are the (not necessarily distinct) roots of the characteristic equation (1.11).

Theorem 4 (See [7])**.** *Let $n \in N^+_{n_0}$ and $g(n,u,v)$ be a nondecreasing function in u and v for any fixed n. Suppose that, for $n \geq n_0$, the inequalities*

$$\begin{aligned} y_{n+1} &\leq g(n, y_n, y_{n-1}), \\ u_{n+1} &\geq g(n, u_n, u_{n-1}) \end{aligned}$$

hold. Then

$$\begin{aligned} y_{n_0-1} &\leq u_{n_0-1}, \\ y_{n_0} &\leq u_{n_0} \end{aligned}$$

implies that

$$y_n \leq u_n, n \geq n_0.$$

Firstly, we are in a position to study the dynamics of the higher order difference equation (1.5). The Eq.(1.5) which by the change of variables

$$y_n = \frac{x_n}{A},$$

reduces to the following difference equation

$$y_{n+1} = 1 + p\frac{y_{n-m}}{y_n^2}, \tag{1.12}$$

where $p = \frac{B}{A^2}$. Henceforth, we consider the difference equation (1.12). Note that Eq.(1.12) has an unique positive equilibrium point such that

$$\bar{y} = \frac{1+\sqrt{1+4p}}{2}.$$

2. Boundedness of Solutions of Eq.(1.12)

Now, we handle the bounded solutions of Eq.(1.12). We also find out that Eq.(1.12) has bounded solutions.

Theorem 5. *If $p > 0$, then $y_n > 1$ for $n \geq 1$. Moreover, if $0 < p < 1$, then*

$$1 < y_n \leq \frac{1}{1-p} + \sqrt[m+1]{p^n} \sum_{j=1}^{m+1} c_j \left(e^{\frac{2\pi i}{m+1}(j-1)}\right)^n,$$

where c_j, $j = 1,2,\cdots,m+1$, are arbitrary constants, and $n \in \{0,1,2,\cdots\}$, and $\sqrt[m+1]{p}$ is one of the $(m+1)$th roots of p.

Proof. Let $p > 0$, and $\{y_n\}_{n=-m}^{\infty}$ be a positive solution of Eq.(1.12). Then, we obtain from Eq.(1.12)

$$\begin{aligned} y_1 &= 1 + p\frac{y_{-m}}{y_0^2} > 1, \\ y_2 &= 1 + p\frac{y_{1-m}}{y_1^2} > 1. \end{aligned}$$

Hence, by induction, we get $y_n > 1$ for $n \geq 1$.

Now, we handle the other case. We have from Eq.(1.12)

$$y_{n+1} = 1 + p\frac{y_{n-m}}{y_n^2} \leq 1 + py_{n-m}.$$

According to Theorem 4, we consider a sequence $\{u_n\}_{n=0}^{\infty}$, and $y_n \leq u_n$, $n = 0,1,\cdots$, and

$$u_{n+1} = 1 + pu_{n-m}, n \geq 1, \tag{2.1}$$

such that

$$u_{s+i} = y_{s+i}, s \in \{-m, -m+1, \cdots\}, i = \{0,1,2,\cdots\}, n \geq s. \tag{2.2}$$

The characteristic polynomial to Eq.(2.1) is

$$P_{m+1}(\lambda) = \lambda^{m+1} - p.$$

Thus, we have the roots of characteristic polynomial as follows:

$$\lambda_j = \sqrt[m+1]{p} e^{\frac{2\pi i}{m+1}(j-1)},$$

where $j = 1, 2, \cdots, m+1$. The homogeneous solution of Eq.(2.1) is

$$u_h = \sum_{j=1}^{m+1} c_j \sqrt[m+1]{p^n} \left(e^{\frac{2\pi i}{m+1}(j-1)} \right)^n,$$

where c_j are arbitrary constants for $j = 1, 2, \cdots, m+1$. Now, we handle the equilibrium solution of Eq.(2.1). From Eq.(2.1), we get that

$$\bar{u} = \frac{1}{1-p}.$$

Therefore, the solution of Eq.(2.1) is

$$u_n = \frac{1}{1-p} + \sum_{j=1}^{m+1} c_j \sqrt[m+1]{p^n} \left(e^{\frac{2\pi i}{m+1}(j-1)} \right)^n, \tag{2.3}$$

where c_j are arbitrary constants for $j = 1, 2, \cdots, m+1$. Furthermore, we have from (2.2) and (2.3)

$$y_{n+1} - u_{n+1} \le p (y_n - u_n),$$

where $n > s$, and $p \in (0, 1)$. Therefore, we get $y_n \le u_n$, $n > s$. So, the proof completed. □

3. Oscillation Behaviors of Eq.(1.12)

In this section, we discuss the semi-cycles of Eq.(1.12). We also reveal the oscillation behaviours of solutions of Eq.(1.12) in detail.

Theorem 6. *Let $\{y_n\}_{n=-m}^{\infty}$ be a positive solution of Eq.(1.12). Then, the following statements are true:*
(i) The every semi-cycle at most m terms.
(ii) Every solution of Eq.(1.12) oscillates about the positive equilibrium $\bar{y}$.

Proof. We firstly handle the positive semi-cycle of solution of Eq.(1.12). The negative semi-cycle is similar and can be omitted. Assume that Eq.(1.12) has a positive semi-cycle with m terms. Suppose that y_N is the first term in this positive semi-cycle. Therefore, we get

$$y_N, y_{N+1}, \cdots, y_{N+m-1} > \bar{y}.$$

Hence, we obtain from Eq.(1.12)

$$y_{N+m} = 1 + p \frac{y_{N-1}}{y_{N+m-1}^2} < 1 + p \frac{y_{N-1}}{\bar{y}^2} < \bar{y}.$$

So, we have that a semi-cycle consists at most m terms. We also get that every solution of Eq.(1.12) oscillates about $\bar{y}$. The proof completed as desired. □

Theorem 7. *Let m be an odd number and let*

$$y_{-m}, y_{-m+2}, \cdots, y_{-1} \leq \bar{y} \text{ and } y_{-m+1}, y_{-m+3}, \cdots, y_0 > \bar{y}. \tag{3.1}$$

Then, every semi-cycle of Eq.(1.12) has lenght one. Additionally, the solution $\{y_n\}_{n=-m}^{\infty}$ *of Eq.(1.12) is oscillatory about unique positive equilibrium point* $\bar{y}$.

Proof. We will prove this theorem by induction method. Let $\{y_n\}_{n=-m}^{\infty}$ be a positive solution of Eq.(1.12). Assume that (3.1) holds. Hence, we obtain from Eq.(1.12),

$$\begin{aligned} y_1 &= 1 + p\frac{y_{-m}}{y_0^2} < \bar{y}, \\ y_2 &= 1 + p\frac{y_{-m+1}}{y_1^2} > \bar{y}, \\ y_3 &= 1 + p\frac{y_{-m+2}}{y_2^2} < \bar{y}. \end{aligned}$$

Therefore, we have by induction

$$y_{2n+1} = 1 + p\frac{y_{2n-m}}{y_{2n}^2} < \bar{y},$$

and

$$y_{2n} = 1 + p\frac{y_{2n-(m+1)}}{y_{2n-1}^2} > \bar{y}.$$

□

4. Periodicity of Eq.(1.12)

Now, we study the existence of periodic solutions of Eq.(1.12).

Theorem 8. *Assume that m is an even number. Then, Eq.(1.12) has no two periodic solution.*

Proof. Let m is an even number. We also suppose that Eq.(1.12) has two periodic solution such that

$$\cdots, \alpha, \beta, \alpha, \beta, \cdots$$

where α and β are positive numbers and $\alpha \neq \beta$. Hence, we get the followings

$$\begin{aligned} y_{2n+1} &= 1 + p\frac{y_{2n-m}}{y_{2n}^2} \Rightarrow \alpha = 1 + \frac{p}{\beta}, \\ y_{2n+2} &= 1 + p\frac{y_{2n+1-m}}{y_{2n+1}^2} \Rightarrow \beta = 1 + \frac{p}{\alpha}. \end{aligned} \tag{4.1}$$

where $n \geq 1$. Therefore, we obtain that

$$(\alpha - \beta)\left(1 - \frac{p}{\alpha\beta}\right) = 0.$$

Now, from our supposition, we have $\alpha \neq \beta$. Thus, we get

$$1 - \frac{p}{\alpha\beta} = 0 \Rightarrow p = \alpha\beta.$$

From (4.1), we have

$$\alpha\beta = \beta + p \Rightarrow \beta = 0.$$

So, we have a contradiction. The proof completed as desired. □

5. Stability of Solutions of Eq.(1.12)

In this section, we deal with the asymptotic stability of the solutions of Eq.(1.5). Firstly, we find the linearized equation associated with Eq.(1.12) about its positive equilibrium point.

The function f is continuously differentiable in some open neighborhood of unique positive equilibrium point $\bar{y}$ as follow:

$$f(y_n, y_{n-1}, \cdots, y_{n-m}) = 1 + p\frac{y_{n-m}}{y_n^2}$$

Thus, we obtain that

$$q_0 = \frac{\partial f}{\partial y_n}(\bar{y}, \bar{y}, \cdots, \bar{y}) = -\frac{2p}{\bar{y}^2},$$

$$q_1 = q_2 = \cdots = q_{m-1} = 0,$$

$$q_m = \frac{\partial f}{\partial y_{n-m}}(\bar{y}, \bar{y}, \cdots, \bar{y}) = \frac{p}{\bar{y}^2}.$$

Then the linearized equation of Eq.(1.12) about its unique positive equilibrium point $\bar{y}$ is:

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \cdots + q_k z_{n-k},$$

for $n = 0, 1, \cdots$. Therefore, we get

$$z_{n+1} + \frac{2p}{\bar{y}^2} z_n - \frac{p}{\bar{y}^2} z_{n-m} = 0. \tag{5.1}$$

Hence, the characteristic equation of Eq.(1.12) is as follows,

$$\lambda^{m+1} + \frac{2p}{\bar{y}^2}\lambda^m - \frac{p}{\bar{y}^2} = 0. \tag{5.2}$$

Theorem 9. *The positive equilibrium $\bar{y}$ of Eq.(1.12) is locally asymptotically stable when $p \in \left(0, \frac{3}{4}\right)$.*

Proof. From Theorem 1, all roots of the characteristic equation of Eq.(5.1) lie in an open disc $|\lambda| < 1$, if

$$|q_0| + |q_1| + |q_2| + \cdots |q_m| < 1.$$

It follows from (5.2) that

$$|q_0| + |q_1| + |q_2| + \cdots |q_m| = \frac{3p}{\bar{y}^2}.$$

Note that

$$\frac{p}{\bar{y}^2} = \frac{2p+1-\sqrt{4p+1}}{2p}.$$

With many numerical calculations, we get that

$$\begin{aligned} |q_0| + |q_1| + |q_2| + \cdots |q_m| &= \frac{3p}{\bar{y}^2} < 1, \\ \frac{3\left(2p+1-\sqrt{4p+1}\right)}{2p} &< 1, \\ \frac{4p+3-3\sqrt{4p+1}}{2p} &< 0. \end{aligned}$$

Hence, we obtain from $p > 0$,

$$\left(\sqrt{4p+1}-1\right)\left(\sqrt{4p+1}-2\right) < 0.$$

Therefore, we get that $0 < p < \frac{3}{4}$. And the proof is complete. □

Theorem 10. *The equilibrium point $\bar{y}$ of Eq.(1.12) is globally asymptotically stable if $0 < p < \frac{1}{2}$.*

Proof. From Theorem 5, we know that there exist I and S such that

$$1 < I = \lim_{n\to\infty} \inf y_n \le S = \lim_{n\to\infty} \sup y_n.$$

Hence, we get from Eq.(1.12)

$$I \ge 1 + p\frac{I}{S^2} \text{ and } S \le 1 + p\frac{S}{I^2}.$$

Therefore, we obtain that

$$S + p\frac{I}{S} \le IS \le I + p\frac{S}{I}.$$

Thus, we have

$$(S-I)\left(1 - p\left(\frac{1}{S} + \frac{1}{I}\right)\right) \le 0.$$

From $S \ge I > 1$ and $0 < p < \frac{1}{2}$, we also get

$$1 - p\left(\frac{1}{S} + \frac{1}{I}\right) > 0.$$

So, we have $S \leq I$ which the result follows. Hence, the equilibrium point $\bar{y}$ of Eq.(1.12) is globally asymptotically stable if $0 < p < \frac{1}{2}$. □

Conjecture 1. *Many numerical simulations show that If $\frac{1}{2} \leq p < \frac{3}{4}$, then the equilibrium point $\bar{y}$ of Eq.(1.12) is globally asymptotically stable.*

6. Rate of Convergence of Eq.(1.12)

Here, we investigate the rate of convergence of solutions of Eq.(1.12).

Theorem 11. *Let λ_j be roots of characteristic equation (5.2) where $j \in \{1, \cdots, k\}$. Then, every solution of Eq.(1.12) ensures the following relations:*

$$\lim_{n\to\infty} \left| \frac{y_{n+1} - \bar{y}}{y_n - \bar{y}} \right| = \left|\lambda_j\right|,$$

and

$$\lim_{n\to\infty} \sup \left(|y_n - \bar{y}|\right)^{1/n} = \left|\lambda_j\right|.$$

Proof. According to Eq.(1.12), we have that

$$y_{n+1} - \bar{y} = -\frac{p\left(y_n + \bar{y}\right)}{\bar{y}y_n^2}\left(y_n - \bar{y}\right) + \frac{p}{y_n^2}\left(y_{n-m} - \bar{y}\right).$$

Now, we consider $e_n = y_n - \bar{y}$. Then, we obtain

$$e_{n+1} + p_n e_n + q_n e_{n-m} = 0,$$

such that

$$p_n = -\frac{p\left(y_n + \bar{y}\right)}{\bar{y}y_n^2},$$

and

$$q_n = \frac{p}{y_n^2}.$$

Therefore, we get from globally asymptotic stability

$$\lim_{n\to\infty} p_n = -\frac{2p}{\bar{y}^2},$$

and

$$\lim_{n\to\infty} q_n = \frac{p}{\bar{y}^2}.$$

So, the proof is completed. □

7. Numerical Simulations

In an attemp to support our theoretical results, we handle three numerical examples. These examples include three figures which drawn by Mathematica.

Example 1. With $A=\sqrt{5}$, $B=4$ and $m=3$, we handle Eq.(1.5). Thus, we get following fourth order difference equation

$$x_{n+1}=\sqrt{5}+4\frac{x_{n-3}}{x_n^2}. \tag{7.1}$$

Here, we apply the following change of the variables for Eq.(7.1) $y_n=\frac{x_n}{\sqrt{5}}$. Therefore, we have that $p=\frac{B}{A^2}=0.8$, and we obtain the following difference equation

$$y_{n+1}=1+0.8\frac{y_{n-3}}{y_n^2}. \tag{7.2}$$

Let the initial conditions are $y_{-3}=8$, $y_{-2}=5$, $y_{-1}=4$ and $y_0=2$. Then, every solution of Eq.(7.2) oscillates about the equilibrium point $\bar{y}$. Additionally, every solution of Eq.(7.2) has bounded from below and above. Figure 1 shows the first 300 terms of Eq.(7.2).

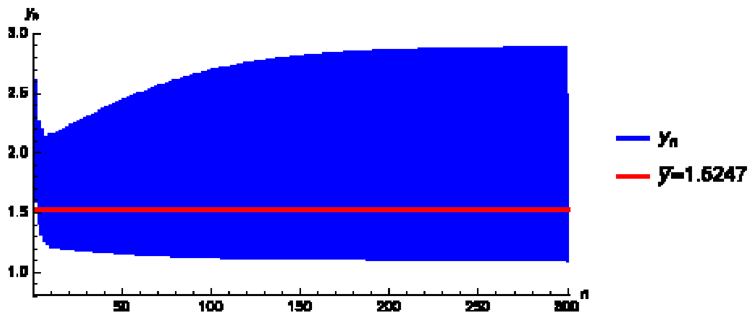

FIGURE 1. Plot of Eq.(1.5) with $A=\sqrt{5}$, $B=4$ and $m=3$.

Example 2. Let the Eq.(1.5) with $A=\sqrt{20}$, $B=9$ and $m=2$. Then we obtain the following third order quadratic-rational difference equation

$$x_{n+1}=\sqrt{20}+9\frac{x_{n-2}}{x_n^2}. \tag{7.3}$$

Hence, we get $p=\frac{B}{A^2}=0.45$ and we obtain the difference equation

$$y_{n+1}=1+0.45\frac{y_{n-2}}{y_n^2}. \tag{7.4}$$

Assume that the initial conditions are $y_{-2}=200$, $y_{-1}=0.1$ and $y_0=0.07$. Hence, the positive equilibrium point $\bar{y}=1.3367$ of Eq.(7.4) is globally asymptotically stable.

Furthermore, the every solution of Eq.(7.4) bounded, and also oscillate about positive equilibrium point. Figure 2 presents the first 100 terms of Eq.(7.4).

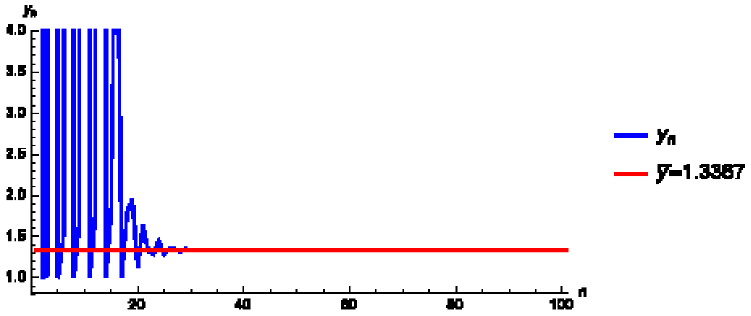

FIGURE 2. Plot of Eq.(1.5) with $A = \sqrt{20}$, $B = 9$ and $m = 2$.

Example 3. With $A = 10$, $B = 39$ and $m = 5$, we consider Eq.(1.5). So, we obtain the sixth order difference equation

$$x_{n+1} = 10 + 39\frac{x_{n-5}}{x_n^2}. \tag{7.5}$$

Hence, we get that $p = \frac{B}{A^2} = 0.39$ and we obtain the following difference equation

$$y_{n+1} = 1 + 0.39\frac{y_{n-5}}{y_n^2}. \tag{7.6}$$

Let the initial values are $y_{-5} = 2$, $y_{-4} = 6$, $y_{-3} = 1$, $y_{-2} = 4$, $y_{-1} = 2$ and $y_0 = 3$. So, the positive equilibrium point $\bar{y} = 1.3$ of Eq.(7.6) is globally asymptotically stable. Furthermore, the every solution of Eq.(7.6) has bounded, and also oscillate about positive equilibrium point. Figure 3 demonstrates the first 100 terms of Eq.(7.6).

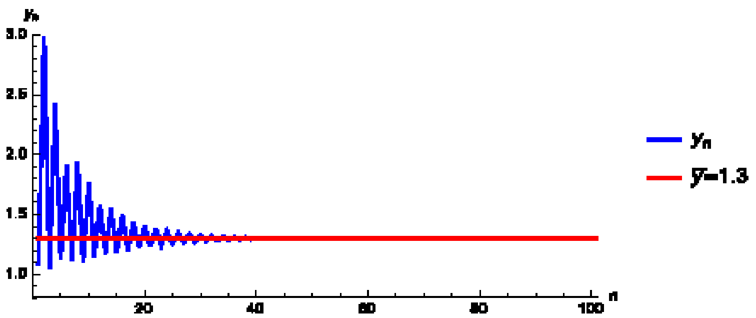

FIGURE 3. Plot of Eq.(1.5) with $A = 10$, $B = 39$ and $m = 5$.

8. Conclusion

In this paper, we firstly study the boundedness of solutions of Eq.(1.5). For the choice of p, if $p \in (0,1)$, then every solution of Eq.(1.5) has bounded from below and above. Moreover, we examine the oscillation behaviors of solutions of Eq.(1.5). Next, we handle the existence of periodic solutions of Eq.(1.5). According to this result, there are no two periodic solutions of Eq.(1.5) if m is an even number. We also investigate the global asymptotic stability and rate of convergence of Eq.(1.5). Lastly, we demonstrate three numerical simulations and graphed by using Mathematica.

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Authors' addresses

E. Taşdemir
(**Corresponding author**) Kırklareli University, Pınarhisar Vocational School, 39300, Kırklareli, Turkey
E-mail address: erkantasdemir@hotmail.com

M. Göcen
Zonguldak Bülent Ecevit University, Department of Mathematics, Art and Science Faculty, 67100, Zonguldak, Turkey
E-mail address: gocenm@hotmail.com

Y. Soykan
Zonguldak Bülent Ecevit University, Department of Mathematics, Art and Science Faculty, 67100, Zonguldak, Turkey
E-mail address: yuksel_soykan@hotmail.com