EXTENDED RANKINE FAILURE CRITERIA FOR CONCRETE

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As possible failure criteria MC2010 refers to those of Rankine, Drucker-Prager and Mohr-Coulomb, respectively, and to modifications or combinations of them. After a brief overview of these failure criteria the characteristics of Mohr-circle are described. Next chapter discusses why the Modified Mohr–Coulomb failure criterion fails in case of concrete. Displaying the Mohr-circles of different bi- and triaxial test series reveal that neither the straight nor the parabolic failure criteria fit. The failure patterns of uniaxial compressive and tensile test specimens prove that concrete fails due to principal tensile stresses/strains in separation and concrete is not a frictional material. The proposed Extended Rankine failure criteria are based directly on the principal stresses.

The criteria limit the greatest- and the actual smallest principal stress, respectively. This latter is function of the two other principal stresses. It is shown that the Ultimate Strength Surface cannot be properly described as stated by CEB Bulletin Nr. 156

Keywords: failure criterion, Rankine, Mohr-circle, Mohr-Coulomb criterion, friction, sliding, separation, principal stress, shear stress

1. INTRODUCTION

MC2010 5.1.8.3 gives the following guidelines about the possible yield functions of concrete under multiaxial states of stresses:

"Basically, yield functions *f* and plastic potentials *g* can be chosen based on multi-axial failure criteria for concrete. These criteria should depend not only on shear stresses, but also on the first invariant I_1 of the stress tensor to consider the influence of the hydrostatic pressure on the ductility of the material. Thus, formulations as the

- Rankine criterion, where tensile failure occurs when the maximum principal stress reaches the uniaxial tensile strength f_{ct}; refer to Rankine, W.J.M., "A Manual of Applied Mechanics", (London, 1868)
- Drucker-Prager criterion, which is the modification of von Mises criterion including the influence of hydrostatic pressure on yielding; refer to Drucker, D.C.; Prager, W., "Soil mechanics and plastic analysis of limit design" (Quarterly of Applied Mechanics, Vol. 10, 1952),
- Mohr-Coulomb criterion, where the maximum shear stress is the decisive measure of yielding, and the critical shear stress value depends on hydrostatic pressure; refer to Mohr, O., "Scientific paper on the area of technical mechanics " (Ernst & Sohn, Berlin, 1906; in German),

and modifications or combinations of them can be used in concrete plasticity models."

After a brief overview about the different failure criteria and about the characteristics of Mohr-circle we discuss why the modified Mohr–Coulomb failure criterion does not fit in case of concrete. Thereafter an extended Rankine failure criterion is proposed. This criterion is based directly on the principal stresses for both, the tensile and the compression failure and is valid in case of triaxial states of stress, too.

2. BRIEF PRELIMINARY OVERVIEW

2.1 Failure criteria

Here are the relevant failure criteria briefly recalled.
Coulomb presented in 1776 his frictional hypothesis: failure often occurs along certain sliding planes, the resistance of which is determined by a parameter termed the cohesion and an internal friction, the magnitude of which depends on the normal stress in the sliding plane (*Figure 1*).

Note: Coulomb dealt with friction between two independent solid bodies separated already through a plane surface. His hypothesis does not apply to separate (fail) a solid body into two parts. After the force parallel to the contact surface has overcome the resting frictional force, the two bodies slide on each other.

Rankine stated in his 1876 published model that a body fails when any of the three principal stresses exceeds the ultimate tensile strength, regardless of the magnitude of the other principal stresses.

In 1868 Tresca postulated for mild steel that failure occurs when the maximum value of the shear stress is exceeded.

In 1882 Mohr assumed that failure occurs when the stresses in a section satisfy the condition:

$$f(\sigma, \tau) = 0 \qquad (1)$$

This function displaced in the σ, τ coordinate system yields Mohr's failure envelop (*Figure 2*). Its simplest form can be got as it just touches the Mohr's circles corresponding to the uniaxial tensile strength (σ_1, 0) and uniaxial compressive strength (σ_3, 0).

Leon (1935) realized that some test results of Mörsch on concrete specimens were in contradiction to the failure crite-

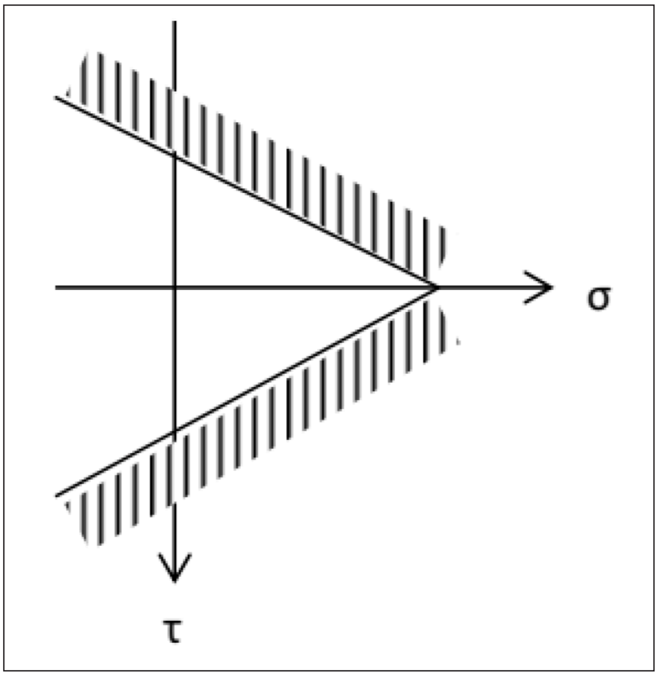


Figure 1: Coulomb´s frictional hypothesis

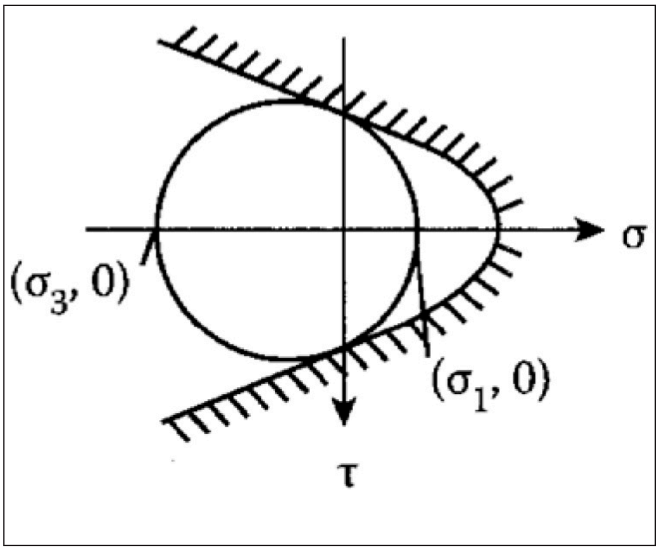


Figure 2: Mohr´s failure envelope

rion of Mohr with the straight envelop. This would anticipate that the failure section at tensile test occurs under the same inclination φ/2 as in case of compression, which is not the case. A parabolic envelop was introduced: it is symmetric to σ-axis, its apex touches the circle corresponding to the uniaxial tensile test at f_t. In addition, the parabola tangents the Mohr-circle of the uniaxial compression test. *Figure 3* shows this parabola corresponding to the ratio $f_c/f_t = 14$, and three circles representing some tests of Mörsch.

The equation of the envelop is:

$$\sigma(\tau) = -\frac{1}{2f_t + f_c - \sqrt{f_t(f_t + f_c)}} \cdot \tau^2 + f_t \qquad (2)$$

Both, Coulomb's hypothesis and Leon's parabola are defined by two parameters only, which generally are not sufficient to fit the experimental data. The simplest extension (one more parameter) is the tension cutoff: the greatest tensile principal stress is limited: a combination of Coulomb's friction hypothesis with Rankine's principal stress hypothesis to describe a separation failure (*Figure 4*). An advantage of this

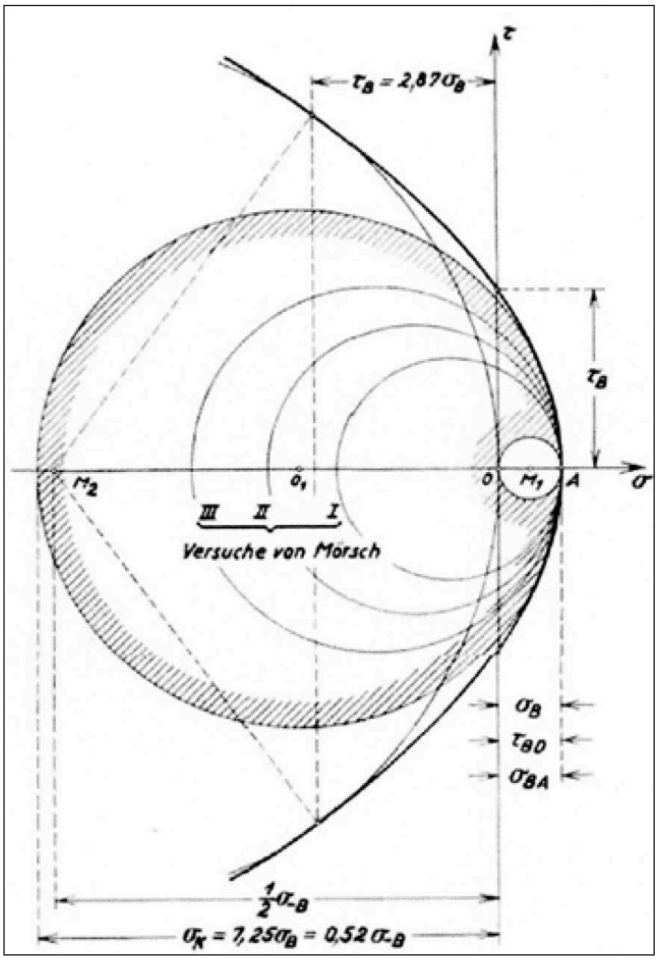


Figure 3: Parabolic rupture criterion for concrete in plane stress coordinate system as presented by Leon

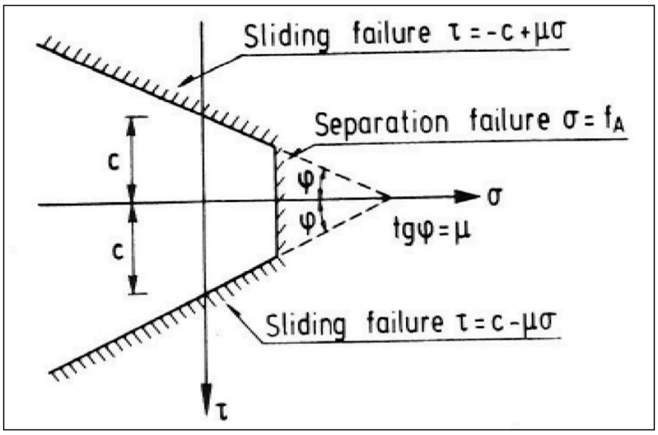


Figure 4: Modified Coulomb rupture criterion

Modified Coulomb Criteria is that the tensile strength (the third parameter) may be varied independently of the two others.

A Modified Coulomb Material can reveal sliding failure and separation failure (*Figure 4*). At sliding failure a motion parallel to the failure surface occurs, while the relative motion at separation failure is perpendicular to the failure surface.

2.2 Mohr´s Circle

As Mohr's circle plays a central role in this proposal some of its characteristics shall be brush up here.

Mohr's circle is a two-dimensional graphical representation of the transformation law for the Cauchy stress tensor. It is used for calculating stresses at a particular material point with respect to a coordinate system in many planes by reducing them to vertical and horizontal components. These are called principal planes in which principal stresses are calculated; Mohr's circle can also be used to find the principal planes and the principal stresses in a graphical representation and is one of the easiest ways to do so.

The abscissa and ordinate (σ, τ) of each point on the circle are the magnitudes of the normal stress and shear stress components, respectively, acting on the rotated coordinate system. In other words, the circle is the focus of points that represent the state of stress on individual planes at all their orientations, where the axes represent the principal axes of the stress elements.

Note: for a Mohr's circle for plane stress state three stress components are needed: σ_x, σ_y and τ_{xy}, i.e. two (σ, τ) are not enough.

Note: Having a Mohr-circle which pertains to a failure then in principle each point on this circle represents the failure. The point where the circle touches the (correct?) Modified Coulomb border line (which needs not to be straight) has no privileged importance. Concrete does not fail in shear/sliding but in all cases in tension/tensile cracking/separation.

Note: the 'shear stress' is an auxiliary parameter only, a stress component. It characterizes the stress in the structural material at a given point only in conjunction with the normal stress components. Concrete has no shear strength. The loading pattern 'pure shear' results in two principal stresses of the same size as the shear stress, acting in ±45° planes. The separation failure occurs when the size of principal tensile stress reaches the actual tensile strength. No sign of shear failure or of sliding!

For a general three-dimensional case of stresses at a point, the values of the principal stresses (σ_1, σ_2, σ_3) and their principal directions (n_1, n_2, n_3) must be first evaluated. Based on the three principal stresses the three Mohr-circles having their centers on the σ_n axis between the points with the abscissas σ_1 vs. σ_2, σ_1 vs. σ_3 and σ_2 vs. σ_3 resp. can be drawn.

3. WHY THE MODIFIED MOHR–COULOMB FAILURE CRITERION FAILS IN CASE OF CONCRETE?

It is generally accepted that the failure criterion of concrete is well described as a Modified Coulomb Material. The Mohr–Coulomb failure criterion represents the linear envelope that is obtained from a plot of the shear strength of a material versus the applied normal stress. This relation is expressed as

$$\tau = \sigma \cdot tan\varphi + c \tag{3}$$

where τ is the shear stress, σ is the normal stress, c is the intercept of the failure envelope with the τ axis, and tanφ is the slope of the failure envelope. The quantity c is often called the cohesion and the angle φ is called the angle of internal friction. Compression is assumed to be positive in the following discussion. (If compression is assumed to be negative then σ should be replaced with – σ)

If φ = 0, the Mohr–Coulomb criterion reduces to the Tresca criterion. On the other hand, if φ = 90° the Mohr–Coulomb model is equivalent to the Rankine model. Higher values of φ are not allowed.

Important note: in case of φ < 90° sliding failure should occur whereas at φ = 90° separation failure appears! An odd change!

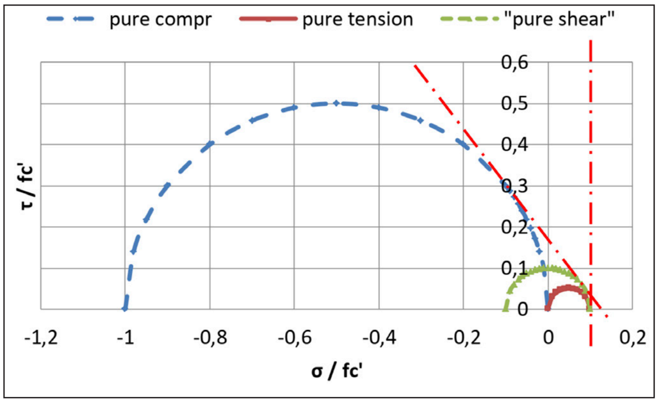


Figure 5: Normalized Mohr-circles define the Modified Coulomb failure criteria

Figure 5 presents the Mohr-circles (normalized with f_c') for uniaxial compression, uniaxial tension and 'pure shear' of a concrete with f_{ct} / f_c' = 0.1 (approx. C30) and the lines corresponding to 'separation' and 'sliding', resp. The figure reveals that the "cohesion" – implied at Coulomb's friction law – is in case of concrete not a material characteristic. It is simply only the length of a section along the shear – axis cut by a trend line (which is maybe even not straight) which cannot be determined in a test. Please note that – as the ratio f_{ct}/f_c' changes acc. to the class of concrete – both lines of the Modified Coulomb failure criteria (the abscissa of separation and the inclination, tanϕ, of segregation) differ acc. to the concrete class.

The use of Modified Coulomb Material is forced by researchers of shear and torsion design who apply in their models stress fields with uniaxial pressure which fail in inclined sliding. They must adhere to the existence of inclined cracks to load direction (see *Figure 6*), even in case of uniaxial tension (besides 'normal' separation cracks *Figure 7*).

Note: inclined cracks to load direction can have two sources:

- Some 'subversive' influences e.g. friction between loading plate and specimen: the tensile crack develops well perpendicular to the real tensile principal stresses, or
- Tensile cracks developed during an independent, previous loading situation. Here we remind you that the theory of plasticity is valid only if all external loads increase in proportion to one another, one-parametric loading!

The appearance of a sliding failure along a surface inclined to the axis of the in compression loaded specimen may have arisen at ordinary loading tests where the friction between the metal loading plate and the specimen hindered the lateral deformation of the specimen thus producing a shear-like supplementary loading which turned the direction of the principal stresses. Remember: in order to minimize this friction Kupfer (1973) loaded his test specimens through a metal brush thus reduced the shear stresses there. The failures of his specimens occurred as tensile (separation) failures in form of discrete cracks parallel to the plane of loading.

From Mohr's circle we have:

$$\sigma = \sigma_m - \tau_m \cdot sin\varphi; \qquad \tau = \tau_m \cdot cos\varphi \tag{4}$$

where

$$\tau_m = \frac{\sigma_1 - \sigma_3}{2} \; ; \; \sigma_m = \frac{\sigma_1 + \sigma_3}{2} \tag{5}$$

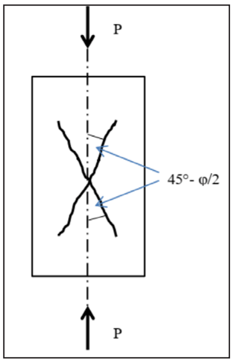


Figure 6: Sliding failure in case of pure compression as presented by Nielsen (2011)

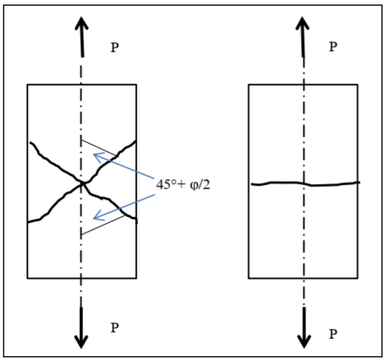


Figure 7: Sliding (left) and separation (right) failure, both might occur in case of pure tension acc. to Nielsen, (2011)

and σ_1 is the maximum principal stress and σ_3 is the minimum principal stress.

Note: σ_m is the abscissa of the center of Mohr-circle whereas τ_m is its radius.

In the following the Mohr-circles of well-known results of Kupfer's biaxial tests should be presented in Modified Coulomb rupture criterion for concrete, see Fig. 8.

Kupfer's tests can be classified into three groups:

- tension-tension
- tension-compression
- compression-compression

Please note: there is no word about 'pure shear' or shear strength.

With reference to the χ-ratio proposed by Windisch (2021) the relationship f_{ct}* vs. f_c* can be reasonably well described with

$$\left(\frac{f_c'^*}{f_c'}\right)^2 + \left(\frac{f_{ct}^*}{f_{ctm}}\right)^2 = 1 \qquad (6)$$

Applying $\chi = f_c' / f_{ctm}$ we get

$$\left(\frac{f_c'^*}{f_c'}\right)^2 + \left(\frac{\chi \cdot f_{ct}^*}{f_c'}\right)^2 = 1 \qquad (6a)$$

or

$$(f_c'^*)^2 + (\chi \cdot f_{ct}^*)^2 = f_c'^2 \qquad (6b)$$

Figure 8 shows the Mohr-circle representations of the tension-compression region of Kupfer's (1973) biaxial tests. It is clear: here separation failures occur.

Note: the sizes of the biaxial tensile strengths depend on the size of the compressive stress hence the three Mohr-circles intersect the $+\sigma_1$ axis at different abscissas.

Figure 9 presents the Mohr-circles of the compression-compression region of Kupfer's biaxial tests. Here it becomes clear that all three principal stresses (σ_1= 0, too) need to be taken into account for the Mohr-circle representation otherwise the Mohr-circle of the biaxial compressive stress state will shrink to a point.

Showing the Mohr-circles for different $\boldsymbol{\sigma}_1/f_c'$ vs. $\boldsymbol{\sigma}_3/f_c'$ ratios *Figure 9* reveals that in the compression-compression region the Modified Coulomb rupture criterion does not apply (no common tangent line is possible). In fact, in all loading cases not sliding failures but separation failures occur.

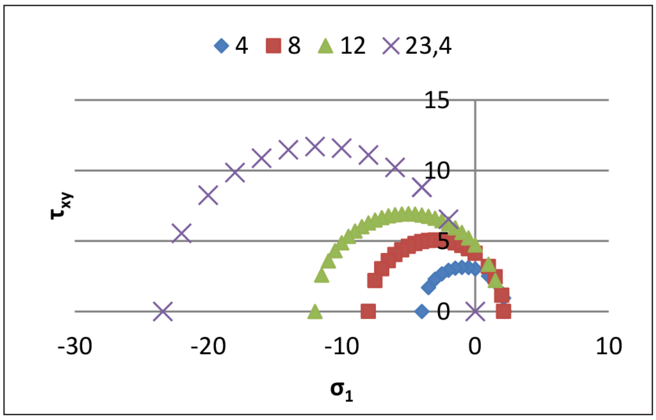


Figure 8: Mohr-circles of biaxial tension-compression and $\sigma_1 = \sigma_2$ compression-compression tests with different σ_{2u}-values as tested by Kupfer ($f_c' = 20$ N/mm²)

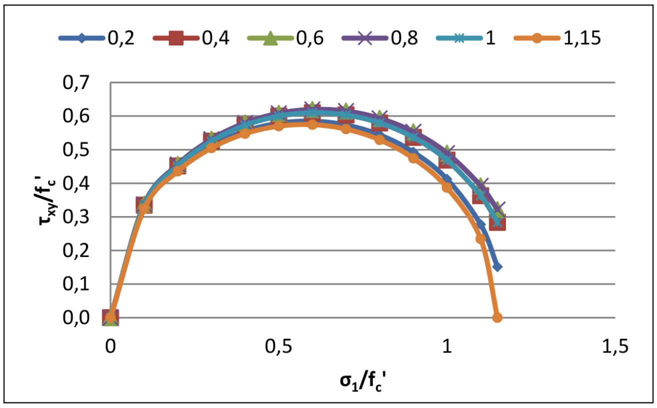


Figure 9: Normalized Mohr-circles of compression-compression tests with different σ_2/f_c' ratios by Kupfer ($f_c' = 20$ N/mm²)

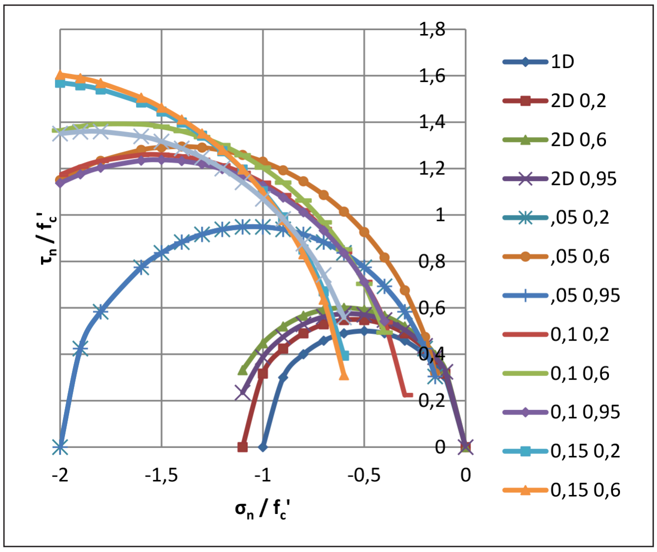


Figure 10: Mohr-circles of compression-compression bi- and triaxial tests of Speck ($f_c' = 72$ N/mm²)

Figure 10 shows the $\sigma_n / f_c' \geq -2$ region ($f_c' = 72$ N/mm²) of the Mohr-circles as measured by Speck (2007) in her 1D - 3D tests. The markings in Fig. 10 mean: 1D: uniaxial; 2D 0.2: biaxial, $\sigma_{2n}/\sigma_{3u} = 0.2$; .05 0.2: 3D, $\sigma_{1n}/\sigma_{3u} = 0.05$, $\sigma_{2n}/\sigma_{3u} = 0.2$. The positions of the circles reveal that neither the Mohr-Coulomb criterion (straight tangenting each circle) nor the Leon criterion (parabola with its tip at the tensile strength) could function as general failure criterion. It must be once more be emphasized that both, Mohr-Coulomb and Leon with their sliding failures are valid when the solid body is already separated in two parts through cracking, that contradicts the fundamental assumption (one parametric loading) of plastic theory.

4. EXTENDED RANKINE FAILURE CRITERIA FOR CONCRETE

The proposed Extended Rankine failure criteria are based directly on the principal stresses.

The Mohr-circle is applied acc. to its original purpose: a transparent geometrical method to determine the size and direction (inclination) of the principal stresses from the stress components given in a global coordinate system.

The Extended Rankine failure criteria are

- the greatest (>0) principal stress, σ_1, cannot be greater than the actual tensile strength (its size depends on the size of the σ_2 and σ_3 principal stresses if at least one of them is compressive stress. (In case of the original Rankine criterion one (fix) tensile strength governed.)
- the triple of the compressive principal stresses $\sigma_3 = \Phi(\sigma_1, \sigma_2)$ cannot be smaller than the actual smallest principal strength, $\sigma_{3,}$ which is function of the two other principal stresses:

$$\sigma_3 = \Phi(\sigma_1, \sigma_2) \quad (7)$$

Note: In this paper the data measured by Speck (2007) in her multiaxial compressive tests were evaluated. The results of the bi- and triaxial compression tests of the literature are evaluated and discussed in a next paper of the author.

It is known that lateral compressive stresses and/or hindered transverse deformations (confinement) let increase the compressive strength in the longitudinal direction. Ottosen (1977) and other researcher (CEB 1983) who use the hydrostatic- and deviatory stresses (octahedral normal- and shear stresses) claim that the form of the Ultimate Strength Surface (USS) can be calibrated with four strength values: the uniaxial tensile strength, the uniaxial compressive strength (point on the compressive meridian), the biaxial compressive strength (point on the tensile meridian), and a triaxial compressive strength at one point on the compressive meridian ($\sigma_1 = \sigma_2 > \sigma_3$).

Figure 11 should reveal that the three a.m. calibrating strength values (marked with the red arrows) cannot properly characterize the monotone increasing, continuous function of the USS (as deduced from the test results of Speck, 2007).

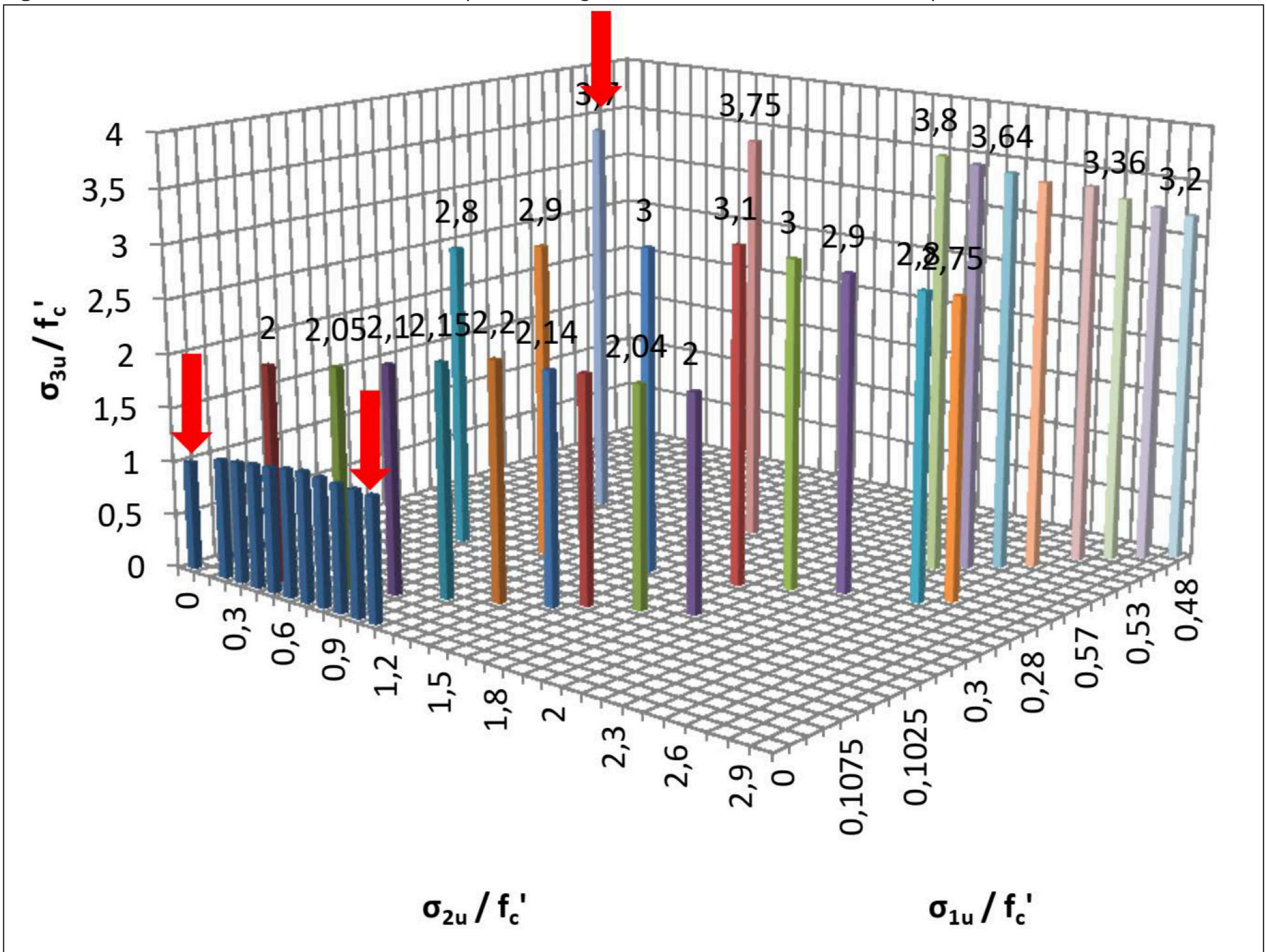
Figure 11 shows the local ordinates of normalized $\sigma_3/f_c' = \Phi(\sigma_1/f_c', \sigma_2/f_c')$ function as deduced from the bi- and triaxial tests measured by Speck on $f_c' = 72$ N/mm² test specimens.

Figure 12 presents the USS in another form: the abscissa is $\gamma = \sigma_1 / \sigma_3$, the ordinate is $\lambda = \sigma_2 / \sigma_3$. The USS increases monotonic in the γ-direction, whereas increases up to λ = 0.5 ~ 0.6 then decreases moderately.

Having determined the relevant failure causing principal stresses the corresponding normal- and shear stress components in the global coordinate system can be determined using Mohr-circles of the tensor calculus. Note: the global shear stress components calculated from the failure causing principal stresses refer by no means to shear failure!

Concrete fails due to the tensile deformations perpendicular to the axis with the maximum principal stress (smallest compressive stress).

Figure 11: Relative increase of the bi- and triaxial compressive strength as function of both other relative compressive stresses



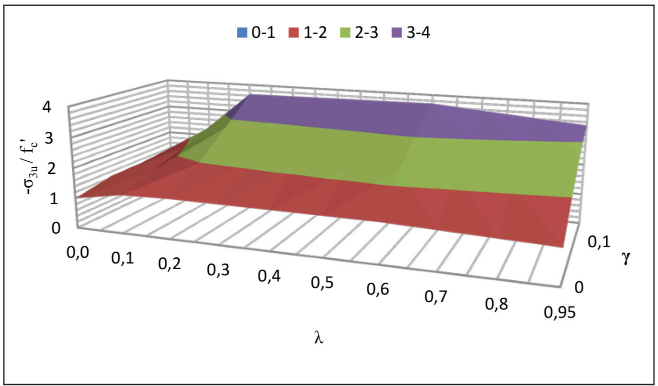


Figure 12: The Ultimate Strength Surface compiled from the data measured by Speck (2007)

The type of failure is in all cases separation, i.e. discrete cracks occur perpendicular to the direction of the greatest and possibly of the intermediate principal stress, resp.

In biaxial compression tests the direction of the transverse tensile deformations is given: the "third" axis (here it should be marked as axis 1). In case of Kupfer's biaxial panel tests the cracks developed perpendicular to the panel and parallel with the middle plane, resp.

Speck, who paid great attention to reduce the friction between the loading device and the test specimens, in all loading cases –whether uni,- bi- or triaxial loading, compression or tension, or mixed- found always separation failures (*Figure 13*), and she is right.

Separation failure (in form of discrete tensile cracks) occurs when the greatest of the principal stresses equals with the actual tensile strength or the smallest (greatest negative) principal stress reaches the actual 2D or 3D compressive strength. The failure occurs along the principal plane which is at an angle ϕ with axis x (this is the third component of the Mohr-circle besides the two principal stresses.

The Mohr-Coulomb failure criteria are not valid for concrete. Concrete is definitely not a frictional material.

The author hopes that with introduction of the failure criterion based on the principal stresses the wrong ways caused by the central reference to the sliding failures and ultimate shear stresses, a better and material appropriate description of the damage-theory etc. can be developed.

5. CONCLUSIONS

The Mohr-Coulomb failure criteria are inadequate to describe the bi- and triaxial failure characteritics of concrete.

Applying them for the evaluation of the test results given in the literature proves that neither a straight nor a curved line can be fitted to the Mohr-circles representing the 2D and 3D strength values.

Further problems are:

- the Coulomb-criteria refers to already existing failure surfaces only.
- The Mohr-circles are a useful and visually attractive geometrical interpretation of the stress-transformation between different axis-systems, moreover in 2D only, not more.
- The Mohr failure criterion does not consider the intermediate stress component which is an important influencing factor of the ultimate failure strength
- The "shear stress" is "produced" by our hugging to the global coordinate system. Nature and concrete do not "know" it.

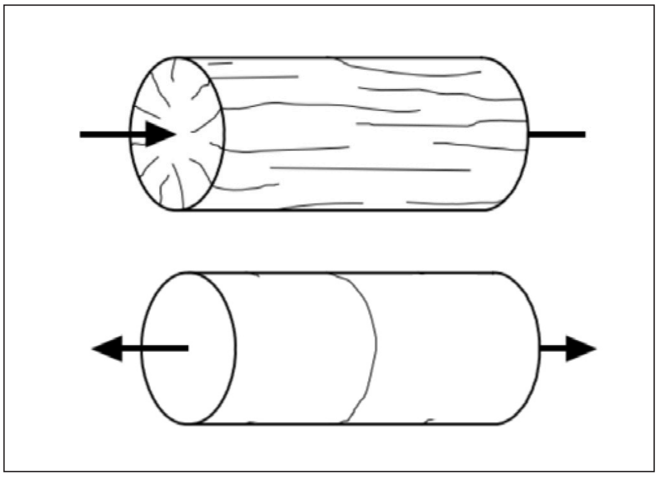


Figure 13: Separation failures in case of uniaxial compression- and tensile tests as presented by Speck (2007)

- Concrete obeys principal stresses only.
- Concrete fails in tension, in form of separation cracks.
- Concrete has no shear strength!
- The researchers were misled by the failure patterns of the specimens loaded with friction between its surface and the loading plate.

Summarizing: for concrete the Mohr-Coulomb failure criteria cannot be applied and concrete is definitely not a frictional material.

Extended Rankine failure criteria based directly on the principal stresses are proposed:

- the greatest (>0) principal stress, σ_1, cannot be greater than the actual tensile strength (its size depends on the size of the σ_2 and σ_3 principal stresses if at least one of them is compressive stress. (In case of the original Rankine criterion one (fix) tensile strength governed.)
- the triple of the compressive principal stresses $\sigma_3 = \Phi(\sigma_1, \sigma_2)$ cannot be smaller than the actual smallest principal strength, $\sigma_{3,}$ which is function of the two other principal stress components, $\sigma_3 = \Phi(\sigma_1, \sigma_2)$.

The author hopes that with introduction of the Extended Rankine failure criteria based on the principal stresses a material-appropriate description of the damage-theory etc. can be developed.

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7. NOTATIONS

c	cohesion
f	yield function (MC2010)
f_c, f_c'	concrete compressive strength
f_c*, f_{ct}*	ultimate strength (compression and tension resp.) in 2D and/or 3D loading
f_{ct}, f_t, f_A	tensile strength
f_{ctm}	mean tensile strength
g	plastic potential (MC2010)
φ	inclination of the sliding surface
ϕ	angle of internal friction
$\gamma = \sigma_1 / \sigma_3$	loading parameter
$\lambda = \sigma_2 / \sigma_3$	loading parameter
$\chi = f_{ck}/f_{ctm}$	ratio of characteristic concrete compressive strength to mean tensile strength
σ_1, σ_2, σ_3	principal stresses ($\sigma_1 \geq \sigma_2 \geq \sigma_3$)
σ_{3u}	ultimate strength measured in test
σ_{com}	hydrostatic normal stress
μ	friction coefficient
σ, σ_n	normal stress component
τ, τ_n	shear stress component

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