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Dynamic modeling of a simply supported beam with an overhang mass

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ABSTRACT

The aim is to derive an expression to calculate the natural frequencies and plot the mode shapes of a simply-supported beam with an overhang with an end overhang point mass by using the Euler-Bernoulli theory in the case of free transverse vibrations. The results are validated by finite element analysis. The importance of the system presented is that it can represent machine tool spindles or even machining tools like boring bars. The results are in good agreement with the results from the finite element analyses. The derived expression can be used in optimizing the value of the point mass and optimizing the support location for better performance of the system without the need to perform complex analysis to obtain the values of the natural frequencies and to plot the mode shapes.

KEYWORDS

beam, pinned-pinned-free, vibration, Bernoulli beam, transverse vibration

1. INTRODUCTION

Beams with different configurations can be essential elements in representing engineering systems, starting from truck axels [1] to sandwich beams structures [2] to machine-tool systems, mainly when it comes to machine-tool spindle systems. Being able to calculate the natural frequencies of beam systems help optimize them in terms of weight, performance and cost. Double span beam systems attract a lot of researchers' attention since it is the case, which is present in many engineering applications in addition to the traditional beam configurations. Euler-Bernoulli beam theory is widely used in the analysis of the mechanical vibrations of beams. Beams are analyzed using many methods and different techniques. The researcher He's Variational Iterational Method (VIM) was used by Alima and Desmond [3] to analyze a Euler-Bernoulli beam on an elastic support in the case of free mechanical vibrations. Also, Lai et al. used the Adomian Decomposition Method (ADM) as an innovative eigenvalue solver for free vibration of a Euler-Bernoulli beam under different supporting conditions [4]. A method called Differential Transforms Method (DTM) was used by Ozgumus and Kaya [5] to analyze flap-wise bending vibrations of a double tapered rotating Euler-Bernoulli beam. The natural frequencies and mode shapes of Euler-Bernoulli beams were determined by Yieh [6] using the singular value decomposition method. An approximate solution to the transverse vibration of the uniform Euler-Bernoulli beam under linearly varying axial force was derived by Naguleswaran [7].

In this study, transverse vibration analysis of a Continuous Pinned-Pinned-Free Beam (CPPFB) with a mass attached at the free end is carried out using the Euler-Bernoulli beam theory. Then the results of the natural frequency and mode shapes obtained by the analysis will be compared to the results of the Finite Element Analysis (FEA) of the same case.

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2. FREE VIBRATION OF A SIMPLY SUPPORTED BEAM WITH AN OVERHANGING

Figure 1 illustrates a pinned-pinned-free beam with an overhang with a mass attached to the free end. The beam is assumed to be slender and subjected to a transverse load. Since the interest of this research is about uniform beams and for free vibrations and using Euler-Bernoulli's beam theory the governing equation can be written as [8]:

$$EI\frac{\partial^4 w}{\partial x^4}(x,\,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,\,t) = 0, \tag{1}$$

where E is the Young's modulus; I is the moment of inertia of the beam cross section; w is the vertical deflection; t is time; ρ is the mass density of the beam and A is the cross-sectional area of the beam.

Assuming a solution of Eq. (1) by the help of the separation of variables method, the solution takes the following form:

$$w(x,\,t) = X(x)T(t), \tag{2}$$

where X is the solution related to space; T is the solution related to time.

The solution X can be written as:

$$X(x) = \begin{cases} X_1(x), & 0 \le x \le a, \\ X_2(x), & a \le x \le L. \end{cases} \tag{3}$$

$X_1(x)$ and $X_2(x)$ can be expressed in the general solution form as follows:

$$X_1(x) = a_1\sin(\beta x) + a_2\cos(\beta x) + a_3\sinh(\beta x) + a_4\cosh(\beta x), \tag{4}$$

$$X_2(x) = b_1\sin(\beta x) + b_2\cos(\beta x) + b_3\sinh(\beta x) + b_4\cosh(\beta x), \tag{5}$$

where β is the beam vibration eigenvalue.

2.1. Setting the boundary conditions

For the presented beam system in Fig. 1, the boundary conditions are to be determined at the first pin, the second pin and at the free end from left to right. For the first pin, the displacement and the moment are equal to zero, so:

$$X_1(0) = 0, \tag{6}$$

$$\frac{d^2X_1(0)}{dx^2} = 0. \tag{7}$$

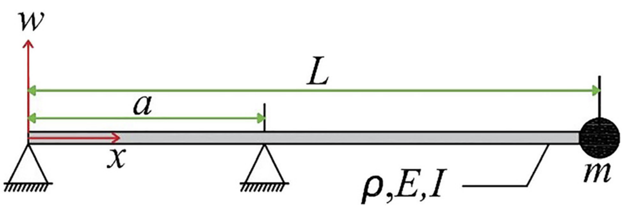

Fig. 1. Pinned-pinned-free beam with a mass attached to the free end

At the free end, the bending moment is equal to zero and the shear is represented in Eq. (9) [7], so:

$$\frac{d^2X_2(0)}{dx^2} = 0, \tag{8}$$

$$EI\frac{d^3X_2(L)}{dx^3} - \mu\frac{d^2T_2(L,\,t)}{dt^2} = 0, \tag{9}$$

where μ is the mass attached to the free end of the beam.

At the second pin, the following conditions are present:

$$X_1(a) = 0, \tag{10}$$

$$X_2(a) = 0, \tag{11}$$

$$\frac{dX_1(a)}{dx} = \frac{dX_2(a)}{dx}, \tag{12}$$

$$\frac{d^2X_1(a)}{dx^2} = \frac{d^2X_2(a)}{dx^2}. \tag{13}$$

2.2. Substituting in the general form solution

Starting with the first boundary condition at the first pin requires substituting Eq. (4) to Eq. (6) which gives:

$$X_1(0) = a_2 + a_4 = 0. \tag{14}$$

Then, after differentiating Eq. (4) two times and substituting the result in equation (7), it gives:

$$\frac{d^2X_1(0)}{dx^2} = -a_2 + a_4 = 0. \tag{15}$$

Moving to the free end, by differentiating Eq. (5) two times and substituting the result in Eq. (8), it gives:

$$\frac{d^2X_2(0)}{dx^2} = -b_1\sin(\beta L) - b_2\cos(\beta L) + b_3\sinh(\beta L) + b_4\cosh(\beta L) = 0. \tag{16}$$

Rearranging Eq. (1) by isolating for $d^2T_2(L,t)/dt^2$ and substituting to Eq. (9) gives:

$$\frac{d^3X_2(L)}{dx^3} + \frac{\mu}{\rho A}\frac{d^4X_2(L)}{dx^4} = 0. \tag{17}$$

After differentiating Eq. (5) four times and substituting to Eq. (17) it gives:

$$\frac{d^3X_2(L)}{dx^3} + \frac{\mu}{\rho A}\frac{d^4X_2(L)}{dx^4} =$$
$$b_1\Big(-\cos(\beta L) + \frac{\mu}{\rho A}\beta\sin(\beta L)) + b_2\Big(\sin(\beta L) + \frac{\mu}{\rho A}\beta\cos(\beta L)\Big)$$
$$+b_3\Big(\cosh(\beta L) + \frac{\mu}{\rho A}\beta\sinh(\beta L)) + b_4\Big(\sinh(\beta L) + \frac{\mu}{\rho A}\beta\cosh(\beta L)\Big) = 0. \tag{18}$$

Substituting Eq. (4) to Eq. (10) gives:

$$X_1(a) = a_1\sin(\beta a) + a_2\cos(\beta a) + a_3\sinh(\beta a) + a_4\cosh(\beta a) = 0. \tag{19}$$

Substituting Eq. (5) to Eq. (11) gives:

$$X_2(a) = b_1\sin(\beta a) + b_2\cos(\beta a) + b_3\sinh(\beta a) + b_4\cosh(\beta a) = 0. \quad (20)$$

Substituting the first derivatives of Eqs (4) and (5) to Eq. (12) gives:

$$\frac{dX_1(a)}{dx} - \frac{dX_2(a)}{dx} = a_1\cos(\beta a) - a_2\sin(\beta a) + a_3\cosh(\beta a) + a_4\sinh(\beta a) - b_1\cos(\beta a) - b_2\sin(\beta a) + b_3\cosh(\beta a) - b_4\sinh(\beta a) = 0. \quad (21)$$

Substituting the second derivatives of Eqs (4) and (5) to Eq. (13) gives:

$$\frac{d^2X_1(a)}{dx^2} - \frac{d^2X_2(a)}{dx^2} = -a_1\sin(\beta a) - a_2\cos(\beta a) + a_3\sinh(\beta a) + a_4\cosh(\beta a) + b_1\sin(\beta a) + b_2\cos(\beta a) - b_3\sinh(\beta a) - b_4\cosh(\beta a) = 0. \quad (22)$$

2.3. Matrix form and eigenvalues

Equations (14) to (22), excluding Eq. (17), are to be arranged in a matrix (8 × 8) in order to obtain the determinant in order to obtain the transcendental equation as follows:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin(\beta L) & -\cos(\beta L) & \sinh(\beta L) & \cosh(\beta L) \\ 0 & 0 & 0 & 0 & -\cos(\beta L) + \frac{\mu}{\rho A}\beta\sin(\beta L) & \sin(\beta L) + \frac{\mu}{\rho A}\beta\cos(\beta L) & \cosh(\beta L) + \frac{\mu}{\rho A}\beta\sinh(\beta L) & \sinh(\beta L) + \frac{\mu}{\rho A}\beta\cos h(\beta L) \\ \sin(\beta a) & \cos(\beta a) & \sin(\beta a) & \cosh(\beta a) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin(\beta a) & \cos(\beta a) & \sinh(\beta a) & \cosh(\beta a) \\ \cos(\beta a) & -\sin(\beta a) & \cosh(\beta a) & \sinh(\beta a) & -\cos(\beta a) & \sin(\beta a) & -\cosh(\beta a) & -\sinh(\beta a) \\ -\sin(\beta a) & -\cos(\beta a) & \sinh(\beta a) & \cosh(\beta a) & \sin(\beta a) & \cos(\beta a) & -\sinh(\beta a) & -\cosh(\beta a) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (23)$$

Solving for the determinant, leads to the transcendental equation as follows:

$$\frac{1\rho A}{7}\Big(\big(-28A\rho\cosh(\beta a)^2 - 28A\rho\cos(\beta a)^2 + 56\mu\beta\sinh(\beta a)\cosh(\beta a) - 56\mu\beta\sin(\beta a)\cosh(\beta a) + 56\rho A\big)\sinh(\beta L) + 24\mu\beta\cosh(\beta L)\sinh(\beta a)\sin(\beta a) - 28\cos(\beta L)\big(A\rho\cosh(\beta a)\sinh(\beta a) - A\rho\cos(\beta a)\sin(\beta a) + 2\mu\beta\cos(\beta a)^2 - 2\mu\beta)\big)\sinh(\beta L) + 28\left(A\rho\cosh(\beta a)\sinh(\beta a) - A\rho\cos(\beta a)\sin(\beta a) - \frac{8\mu\beta\cosh(\beta a)^2}{7} + \frac{8\mu\beta}{7}\right)\cosh(\beta L)\sin(\beta L) + 28A\big(\big(\cosh(\beta a)^2\cos(\beta L) - \cos(\beta a)^2\cos(\beta L)\big)\cosh(\beta L) + 2\sinh(\beta a)\sin(\beta a)\rho\big) = 0. \quad (24)$$

The natural frequencies are given by Eq. (25) [8]:

$$f_n = \frac{\beta_n^2}{2\pi}\sqrt{\frac{EI}{\rho A}}, \text{ [Hz]}. \quad (25)$$

3. VALIDATION OF THE MODEL

A beam model with specific properties is analyzed using the analysis method in the previous sections to obtain the values of the natural frequencies for the first four eigenvalues and the first four mode shapes, then the same beam is analyzed using the FEA in order to compare the results and calculate the error percentage between both methods. Table 1 lists the parameters of the validation model.

Since Eq. (24) has an infinite number of zeros and that the interest of this research is the first four modes, the parameters in Table 1 are substituted to Eq. (24) and then it is plotted for a domain of 4 π is illustrated in Fig. 2.

Then the zeros of the plot are extracted and tabulated in Table 2 as follows:

Substituting β values in Eq. (25), gives the natural frequencies values listed in Table 3.

In order to obtain the β values for different second pin location, instead of repeating the previous step, Fig. 3 is used for calculating β values can be obtained from Eq. (24). Two variables are defined as follows, $\delta = \beta.l$ and $\lambda = a/l$. After

Table 1. The parameters of the validation model

Parameter	Value
L	1 m
ρ	7,850 kg m^{-3}
E	2.05·e^{+11} N m^{-2}
μ	0.628 kg
I	1.333·e^{-8} m^{4}
A	4·e^{-4} m^{2}
a	0.3 m

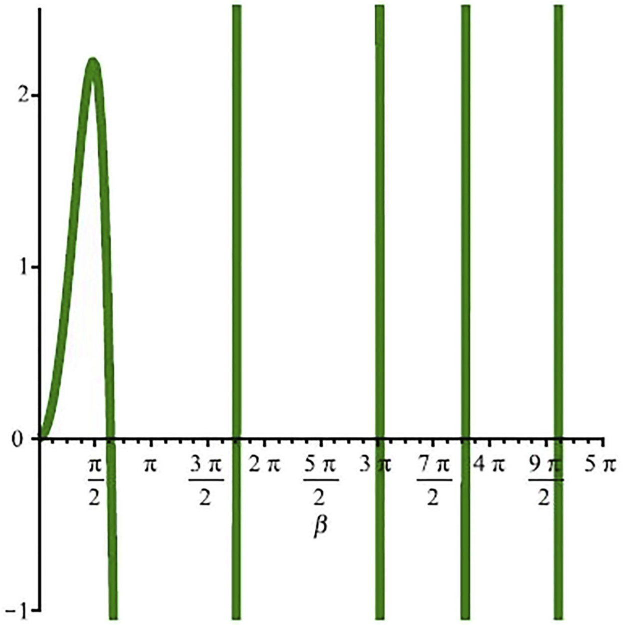

Fig. 2. Beta values plot

Table 2. Extracted beta values

β	Value
1	1.99
2	5.52
3	9.53
4	11.85

Table 3. Natural frequencies values

β	f_n [Hz]
1	18.72
2	143.25
3	426.33
4	659.5

rearranging and substituting them into Eq. (32) then plotting the new expression for δ and λ it yields to Fig. 3. The curves in the plot correspond to the modes one to four from bottom to up respectively.

For the mode shapes, the boundary condition equations are used in order to obtain relationships between the coefficients $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4. Since there are no enough equations to obtain the values of the coefficients, the

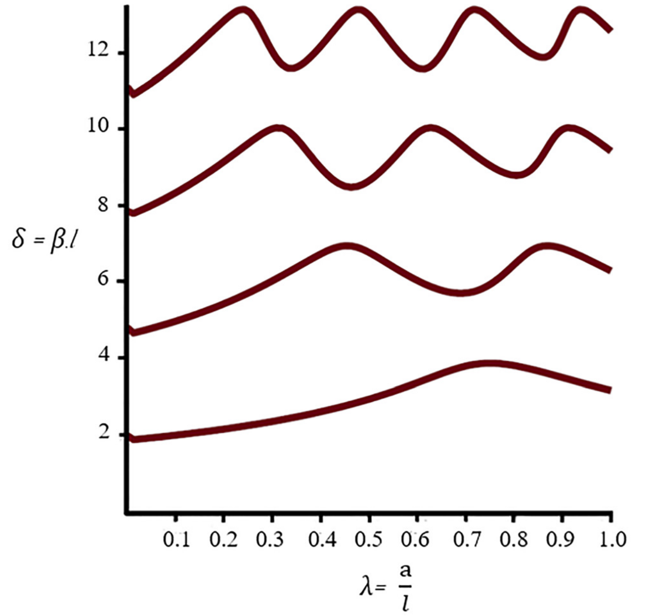

Fig. 3. Eigenvalues for the first 4 mode

obtained values will be normalized by dividing all the coefficients by b_4. Then the boundary condition equations can be rewritten as:

$$\frac{-a_1\beta^2\sin(\beta a)}{b_4}-\frac{a_2\beta^2\cos(\beta a)}{b_4}+\frac{a_3\beta^2\sinh(\beta a)}{b_4}+\frac{a_4\beta^2\cosh(\beta a)}{b_4}+\frac{b_1\beta^2\sin(\beta a)}{b_4}+\frac{b_2\beta^2\cos(\beta a)}{b_4}-\frac{b_3\beta^2\sinh(\beta a)}{b_4}=\beta^2\cosh(\beta a), \tag{26}$$

$$\frac{a_1\beta\cos(\beta a)}{b_4}-\frac{a_2\beta\sin(\beta a)}{b_4}+\frac{a_3\beta\cosh(\beta a)}{b_4}+\frac{a_4\beta\sinh(\beta a)}{b_4}-\frac{b_1\beta\cos(\beta a)}{b_4}-\frac{b_2\beta\sin(\beta a)}{b_4}-\frac{b_3\beta\cosh(\beta a)}{b_4}=\beta\sinh(\beta a), \tag{27}$$

$$\frac{b_1\sin(\beta a)}{b_4}+\frac{b_2\cos(\beta a)}{b_4}+\frac{b_3\sinh(\beta a)}{b_4}=-\cosh(\beta a), \tag{28}$$

$$\frac{a_1\sin(\beta a)}{b_4}+\frac{a_2\cos(\beta a)}{b_4}+\frac{a_3\sinh(\beta a)}{b_4}+\frac{a_4\cosh(\beta a)}{b_4}=0, \tag{29}$$

$$\frac{b_1\left(-\beta^3\cos(\beta L)+\frac{\mu}{\rho A}\beta^4\sin(\beta L)\right)}{b_4}+\frac{b_2\left(\beta^3\sin(\beta L)+\frac{\mu}{\rho A}\beta^4\cos(\beta L)\right)}{b_4}+\frac{b_3\left(\beta^3\cosh(\beta L)+\frac{\mu}{\rho A}\beta^4\sinh(\beta L)\right)}{b_4}=-\left(\beta^3\sinh(\beta L)+\frac{\mu}{\rho A}\beta^4\cosh(\beta L)\right), \tag{30}$$

$$\frac{-b_1\beta^2\sin(\beta L)}{b_4} - \frac{b_2\beta^2\cos(\beta L)}{b_4} + \frac{b_3\beta^2\sinh(\beta L)}{b_4} = -\beta^2\cosh(\beta L), \quad (31)$$

$$-\frac{a_2\beta^2}{b_4} + \frac{a_4\beta^2}{b_4} = 0. \quad (32)$$

The values of the coefficients from a_1 to b_3 divided by b_4 can be obtained. Substituting the obtained coefficients values to Eq. (3) and then plotting for each β value, the mode shapes are as it is illustrated in Fig. 4.

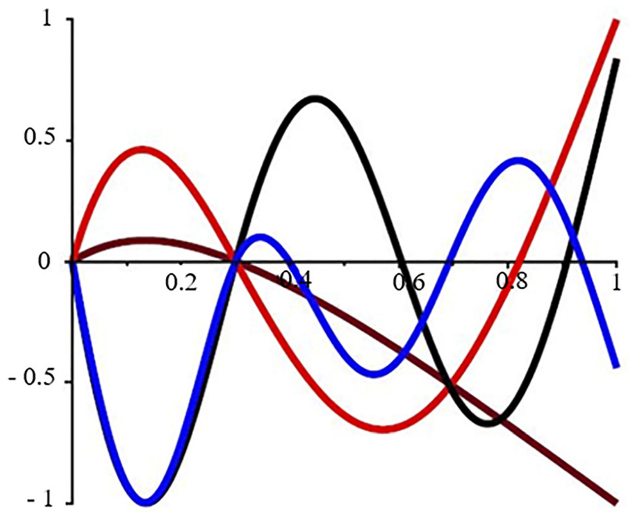

Fig. 4. Proposed model mode shapes

Table 4. Theory and FEA method natural frequencies values with the error percentage

Mode	Theory [Hz]	FEA [Hz]	Error
1	18.72	18.79	0.37%
2	143.25	143.77	0.36%
3	426.33	426.20	0.03%
4	659.50	658.13	0.21%

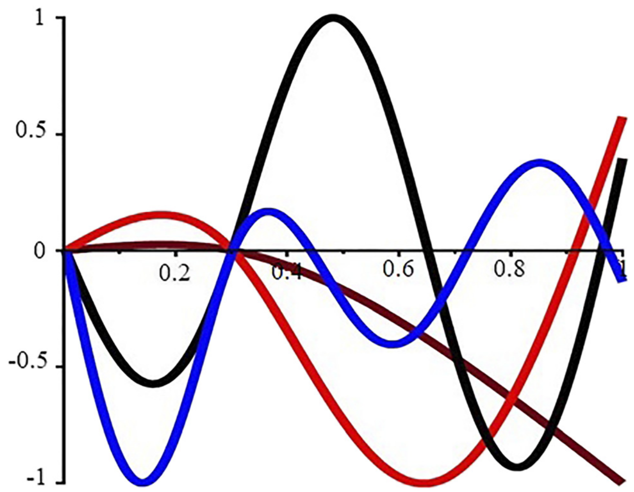

Fig. 5. The normalized FEA mode shapes

4. FEA AND COMPARISON

The same model introduced in the last section is analyzed using the FEA method to find the values of the first four natural frequencies and for plotting the mode shapes. Table 4 lists the values of the natural frequencies of the FEA method and the proposed model in addition to the error percentage.

Figure 5 shows the normalized mode shapes obtained by the FEA.

5. CONCLUSIONS

A pinned-pinned-free beam with a mass attached to the free end is studied. Euler-Bernoulli beam theory is used to derive the transcendental equation, which can be applied to different second-pin location and different attached mass values. The eigenvalues for calculating the natural frequencies and the mode shapes were obtained for a validation model. FEA of the same validation model were carried out and the first four natural frequencies were compared to the proposed method. The comparison of the results shows a very good match in the natural frequency values. However, the mode shapes for both methods take the same form with some differences in the peak's values. The difference can be attributed to the normalizing process of the mode shapes.

The developed analytical model can be helpful in the design of many engineering applications like crane arms, machine-tool spindles or even machine-tool boring bars. Future work will include investigating different attached mass to beam-mass ratio in addition to the effect of the second pin location on the accuracy of the developed model, as well as analyzing the model under the application of force.

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