On the Number of Hyperedges in the Hypergraph of Lines and Pseudo-discs

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Abstract

Consider a hypergraph whose vertex set is a family of n lines in general position in the plane, and whose hyperedges are induced by intersections with a family of pseudo-discs. We prove that the number of t-hyperedges is bounded by $O_t(n^2)$ and that the total number of hyperedges is bounded by $O(n^3)$. Both bounds are tight.

1 Introduction

A family $\mathcal{F}$ of simple Jordan regions in $\mathbb{R}^2$ is called *a family of pseudo-discs* if for any $c_1, c_2 \in \mathcal{F}$, $|\partial(c_1) \cap \partial(c_2)| \leq 2$, where $\partial(c)$ is the boundary of c. Given a set P of points in $\mathbb{R}^2$ and a family $\mathcal{F}$ of pseudo-discs, define the geometric hypergraph $H(P, \mathcal{F})$ whose vertices are the points of P, and any pseudo-disc $c \in \mathcal{F}$ defines a hyperedge of all points contained in c.

The family of hypergraphs $H(P, \mathcal{F})$ – for a general $\mathcal{F}$ and in the special case where all elements of $\mathcal{F}$ are convex – have been studied extensively (see, e.g., [1, 3, 6, 9, 13]). In particular, it was proved in [7] that for any $P, \mathcal{F}$, the Delaunay graph of $H(P, \mathcal{F})$ (namely, the restriction of H to hyperedges of size 2) is planar, and that for any fixed t, the number of hyperedges of $H(P, \mathcal{F})$ of size t is bounded by $O(t^2|P|)$. This result was generalized in [11] (see also [4]) to the case where P is a family of pseudo-discs instead of points, and the hyperedges are defined by non-empty intersections of any element in $\mathcal{F}$ with the elements of P.

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In this note we consider hypergraphs $H = H(\mathcal{L}, \mathcal{F})$ whose vertex set $\mathcal{V}(H) = \mathcal{L}$ is a family of lines in the plane, and whose hyperedges are induced by intersections with a family $\mathcal{F}$ of pseudo-discs. Namely, any $c \in \mathcal{F}$ defines the hyperedge

$$e_c = \{\ell \in \mathcal{L} : \ell \cap c \neq \emptyset\} \in \mathcal{E}(H).$$

We assume that the geometric objects are in general position, in the sense that no 3 lines pass through a common point, no line passes through an intersection point of two boundaries of pseudo-discs.

Unlike the hypergraphs of points w.r.t. pseudo-discs, $H(P, \mathcal{F})$, the number of hyperedges in a hypergraph $H(\mathcal{L}, \mathcal{F})$, of lines w.r.t. pseudo-discs, of any fixed size, may be quadratic in the number of vertices. Such a hypergraph was demonstrated in a beautiful paper of Aronov et al. [5]. They showed that for any family $\mathcal{L}$ of lines, if $\mathcal{F}$ consists of the inscribed circles of the triangles formed by any triple of lines, then for any $t \geq 3$, the number of t-hyperedges (i.e., hyperedges of size t) in $H(\mathcal{L}, \mathcal{F})$ is exactly $\binom{n-t+2}{2}$.

For any fixed t, there exist hypergraphs $H(\mathcal{L}, \mathcal{F})$ in which the number of t-hyperedges is larger than in the construction of Aronov et al. [5], even when $\mathcal{F}$ is allowed to contain only discs (as some of those discs might not be inscribed in a triangle formed by the lines). We prove that the number of t-hyperedges cannot be significantly larger for any hypergraph $H(\mathcal{L}, \mathcal{F})$ of lines with respect to pseudo-discs.[1] Specifically, we prove:

Theorem 1.1. *Let $\mathcal{L}$ be a family of n lines in the plane, let $\mathcal{F}$ be a family of pseudo-discs, and assume both families are in general position. Then*

$$|\{e \in \mathcal{E}(H(\mathcal{L}, \mathcal{F})) : |e| = t\}| = O_t(n^2).$$

Our techniques combine probabilistic and planarity arguments, together with exploiting properties of arrangements of lines, in particular the *zone theorem.*

In addition, we show that for any choice of $\mathcal{L}$ and $\mathcal{F}$, the total number of hyperedges in $H(\mathcal{L}, \mathcal{F})$ does not exceed $O(n^3)$. This upper bound is tight, since the total number of hyperedges in the hypergraph presented by Aronov et al. [5] is $\binom{n}{3}$.

Proposition 1.2. *Let $\mathcal{L}$ be a family of n lines in the plane, let $\mathcal{F}$ be a family of pseudo-discs, and assume both families are in general position. Then $|\mathcal{E}(H(\mathcal{L}, \mathcal{F}))| = O(n^3)$.*

2 Preliminaries

In this section we present previous results and simple lemmata that will be used in our proofs.

2.1 Pseudo-discs

The two following lemmata are standard useful tools when handling families of pseudo-discs:

[1] For the difference between hypergraphs induced by pseudo-discs and hypergraphs induced by discs, see [10] and the references therein.

Lemma 2.1 (Lemma 1 in [15], based on [16]). *Let $\mathcal{F}$ be a family of pseudo-discs, $D \in \mathcal{F}, x \in D$. Then D can be continuously shrunk to the point x, such that at each moment during the shrinking process, the family obtained from $\mathcal{F}$ remains a family of pseudo-discs.*

Lemma 2.2 (Lemma 2 in [15]). *Let $\mathcal{B}$ be a family of pairwise disjoint closed connected sets in $\mathbb{R}^2$. Let $\mathcal{F}$ be a family of pseudo-discs. Define a graph G whose vertices correspond to the sets in B and connect two sets $B, B' \in \mathcal{B}$ if there is a set $D \in \mathcal{F}$ such that D intersects B and B' but not any other set from B. Then G is planar, hence $|E(G)| < 3|V(G)|$.*

2.2 Arrangements and zones

A finite set $\mathcal{L}$ of lines in $\mathbb{R}^2$ determines an *arrangement* $\mathcal{A}$. The 0-dimensional faces of $\mathcal{A}$ (namely, the intersections of two distinct lines from $\mathcal{L}$), are called *the vertices of* $\mathcal{A}$, the 1-dimensional faces are called *the edges of* $\mathcal{A}$, and the 2-dimensional faces are *the cells of* $\mathcal{A}$. Clearly, all cells are convex. The *cell complexity* of a cell f in $\mathcal{A}$, denoted by *comp*(f), is the number of lines incident with the cell. The *zone* of an additional line ℓ, is the set of faces of $\mathcal{A}$ intersected by ℓ. The *complexity of a zone* is the sum of the cell complexities of the faces in the zone of ℓ, i.e., total number of edges of these faces, counted with multiplicities.

Theorem 2.3 (Zone Theorem [8]). *In an arrangement of n lines, the complexity of the zone of a line is $O(n)$.*

The best possible upper bound in the theorem is $\lfloor 9.5(n-1) \rfloor - 3$, obtained by Pinchasi [14].

We shall use a generalization of the theorem, for which an extra definition is needed. Given an arrangement $\mathcal{A}$ and a line ℓ, the 1-zone of ℓ is defined as the zone of ℓ, and for $t > 1$ the t-zone of ℓ is defined as the set of all faces adjacent to the $(t-1)$-zone, that do not belong to any i-zone for $i < t$. The $(\leq t)$-*zone* of ℓ is the union of the i-zones of ℓ for all $1 \leq i \leq t$.

The following generalization of the zone theorem was given as Exercise 6.4.2 in [12]. Its proof can be found in [17, Prop. 1].

Lemma 2.4 ([17]). *Let $\mathcal{A}$ be an arrangement of n lines. Then for any t, the $\leq t$-zone of any additional line ℓ contains at most $O(tn)$ vertices.*

By planarity, this implies:

Corollary 2.5. *Let $\mathcal{A}$ be an arrangement of n lines. Then for any t, the $\leq t$-zone of any additional line ℓ has complexity $C_{\leq t}(\ell) = O(tn)$.*

2.3 Leveraging from 2-hyperedges to t-hyperedges

The following lemma allows bounding the number of t-hyperedges in a hypergraph $H = (\mathcal{V}, \mathcal{E})$ in terms of the number of its 2-hyperedges (i.e., the size of its Delaunay sub-hypergraph) and its *VC-dimension.*

Let us recall the classical definition of VC-dimension. A subset $\mathcal{V}' \subseteq \mathcal{V}$ is *shattered* if all its subsets are realized by hyperedges, meaning $\{\mathcal{V}' \cap e \colon e \in \mathcal{E}\} = 2^{\mathcal{V}'}$. The *VC-dimension* of H, denoted by $VC(H)$, is the cardinality of a largest shattered subset of $\mathcal{V}$, or $+\infty$ if arbitrarily large subsets are shattered.

Lemma 2.6 (Theorem 6 (ii),(iii) in [2]). *Let $H = (\mathcal{V}, \mathcal{E})$ be an n-vertex hypergraph. Suppose that there exists an absolute constant c such that for every $\mathcal{V}' \subset \mathcal{V}$, the Delaunay graph of the sub-hypergraph induced by $\mathcal{V}'$ has at most $c|\mathcal{V}'|$ edges. Then the VC-dimension d of H is at most $2c+1$, and the number of hyperedges of size at most t in H is $O(t^{d-1}n)$.*

The lemma generalizes similar results proved in [4, 7] for hypergraphs of pseudo-discs with respect to pseudo-discs. The assertion regarding the VC-dimension is a simple observation. (Indeed, if a set of d vertices is shattered, then we have $\binom{d}{2} \leq cd$, and thus, $d-1 \leq 2c$, or equivalently, $d \leq 2c+1$.) The assertion regarding the number of hyperedges is more involved.

3 The number of t-hyperedges in $H(\mathcal{L}, \mathcal{F})$

In this section we prove Theorem 1.1. We prove the following stronger statement:

Proposition 3.1. *Let $\mathcal{L}$ be a family of n lines in the plane, let $\mathcal{F}$ be a family of pseudo-discs, and assume both families are in general position. Then for each $\ell \in \mathcal{L}$,*

$$|\{e \in \mathcal{E}(H(\mathcal{L}, \mathcal{F})) : |e| = t, \ell \in e\}| = O_t(n).$$

Consequently, $|\{e \in \mathcal{E}(H(\mathcal{L}, \mathcal{F})) : |e| = t\}| = O_t(n^2)$.

Proof of Proposition 3.1. First we prove the statement for hyperedges of size 3, and then we leverage the result to general hyperedges.

3-hyperedges. Fix a line ℓ. We observe that for a pseudo-disc c that defines a 3-hyperedge $\{\ell, \ell', \ell''\}$ there exists a cell of $\mathcal{A}(\mathcal{L} \setminus \{\ell\})$ which is in the ≤ 2-zone of ℓ in $\mathcal{A}(\mathcal{L} \setminus \{\ell\})$ such that c intersects two edges of this cell where one of these edges is on ℓ' and the second is on ℓ''. With every such pseudo-disk c we associate one such cell f_c and one such pair of edges of this cell, and denote this pair by e_c.

Define a graph $G = (V, E)$ whose vertices are all edges in the (≤ 2)-zone of ℓ in $\mathcal{A}(\mathcal{L} \setminus \{\ell\})$, and whose edges are the pairs e_c associated with the pseudo-disks that define a 3-hyperedge. Note that for any hyperedge $e = \{\ell, \ell', \ell''\}$ we choose exactly one pair of edges of $\mathcal{A}(\mathcal{L} \setminus \{\ell\})$ - one is on ℓ' and one is on ℓ'' - that form a corresponding edge of G. Thus by construction, $|E|$ is equal to the number of 3-hyperedges containing ℓ, and so, we want to prove that $|E| = O(n)$.

Consider a single cell f of $\mathcal{A}(\mathcal{L} \setminus \{\ell\})$. For each pseudo-disk c that defines a 3-hyperedge containing l and has $f_c = f$, c does not intersect any other edge of f besides the two edges in e_c (as otherwise, c would intersect at least 4 lines of $\mathcal{L}$). Hence, the restriction of G to the edges of the cell f (after removing their endpoints), satisfies the assumptions of Lemma 2.2. Thus, by Lemma 2.2, the subgraph of G induced by the edges of f is planar, and hence, its number of edges is at most 3 times the complexity of f. Summing over all cells in the (≤ 2)-zone of ℓ, we obtain $|E| \leq 3 \sum_f comp(f) = O(n)$ by Corollary 2.5, and therefore, $|E| = O(n)$, as asserted.

t-hyperedges. Fix a line ℓ, and consider the hypergraph H' whose vertex set is $\mathcal{L} \setminus \{\ell\}$ and whose edge set is $\{e \setminus \{\ell\} : e \in \mathcal{E}(H), \ell \in e\}$. The 2-hyperedges of H' correspond to 3-hyperedges of H containing ℓ, and thus, by the first step, their number is $O(n)$. Furthermore, for any $\mathcal{L}' \subset \mathcal{L} \setminus \{\ell\}$, the number of 2-hyperedges in the restriction of H' to $\mathcal{L}'$ is $O(|\mathcal{L}'|)$, by the same argument. Therefore, H' satisfies the assumptions of Lemma 2.6, which implies that the VC-dimension d of H' is constant, and that the number C_{t-1} of $(t-1)$-hyperedges of H' is $O(t^{d-1}n)$.

Finally, the number of t-hyperedges of H that contain ℓ is equal to C_{t-1}. This completes the proof. □

4 The total number of hyperedges in $H(\mathcal{L}, \mathcal{F})$

In this section we prove Proposition 1.2.

Proof of Proposition 1.2. By Lemma 2.1 we can shrink the pseudo-discs one by one, such that the shrinking of each pseudo-disc $c \in \mathcal{F}$ is stopped when it becomes tangent to two lines. (Formally, first c is shrunk until the first time it is tangent to some line in $\mathcal{L}$, and then it is shrunk towards the tangency point until the next time it is tangent to some line in $\mathcal{L}$.) By the general position assumption, we can perform the shrinking process in such a way that the obtained geometric objects (i.e., lines and shrinked pseudo-discs) are also in general position. We replace each $c \in \mathcal{F}$ by its shrunk copy. Let $\mathcal{F}'$ be the obtained family. Then $H(\mathcal{L}, \mathcal{F}) = H(\mathcal{L}, \mathcal{F}')$, and by a tiny perturbation we can assume that all tangencies are in a point.

For any two lines $\ell_1, \ell_2 \in \mathcal{L}$, denote by $\mathcal{F}'(\ell_1, \ell_2)$ the set of all pseudo-discs in $\mathcal{F}'$ that are tangent to both ℓ_1 and ℓ_2. We claim that for any $\ell_1, \ell_2 \in \mathcal{L}$, $|\mathcal{E}(H(\mathcal{L}, \mathcal{F}'(\ell_1, \ell_2)))| = O(n)$, and this implies $|\mathcal{E}(H)| = O(n^3)$, the assertion of Proposition 1.2.

To show this, for any $c \in \mathcal{F}'(\ell_1, \ell_2)$, we define $x_{\ell_1,\ell_2}(c) = c \cap \ell_1 \in \mathbb{R}^2$ and $y_{\ell_1,\ell_2}(c) = c \cap \ell_2 \in \mathbb{R}^2$ (see Figure 1). In each of the four wedges that ℓ_1, ℓ_2 form, we define a linear order relation on the elements of $\mathcal{F}'(\ell_1, \ell_2)$: $c \prec c'$ if the segment $[x_{\ell_1,\ell_2}(c), y_{\ell_1,\ell_2}(c)]$ is completely above the segment $[x_{\ell_1,\ell_2}(c'), y_{\ell_1,\ell_2}(c')]$ (that is, if the points $x_{\ell_1,\ell_2}(c), y_{\ell_1,\ell_2}(c)$ are closer to the intersection point within the wedge than the points $x_{\ell_1,\ell_2}(c'), y_{\ell_1,\ell_2}(c')$, respectively).

First, we claim that this relation is well defined, since for $c \neq c'$ two such segments never intersect. Indeed, assume to the contrary they intersect, so that $y_{\ell_1,\ell_2}(c')$ is above $y_{\ell_1,\ell_2}(c)$, while $x_{\ell_1,\ell_2}(c')$ is below $x_{\ell_1,\ell_2}(c)$. The pseudo-disc c divides the remainder of the wedge into two connected components – the part 'above' it and the part 'below' it. Now, consider the points $x_{\ell_1,\ell_2}(c'), y_{\ell_1,\ell_2}(c')$. In the boundary of c', these points are connected by two curves. As these points are in different connected components w.r.t. c, each of these curves intersects c at least twice, which means that c, c' intersect at least 4 times, a contradiction.

Second, we claim that in each wedge, every line in $\mathcal{L}$ intersects a subset of consecutive elements of $\mathcal{F}'(\ell_1, \ell_2)$ under the order $\prec$. Indeed, assume that some line ℓ intersects two pseudo-discs c_1, c_3, as depicted in Figure 1. We want to show it must intersect c_2 as well. Like above, c_2 divides the wedge (without it) into two connected components. By the same argument as above, c_1 cannot intersect the component below c_2 (as otherwise, it would cross c_2 four times). Similarly, c_3 cannot intersect the component above c_2. Thus, either ℓ intersects

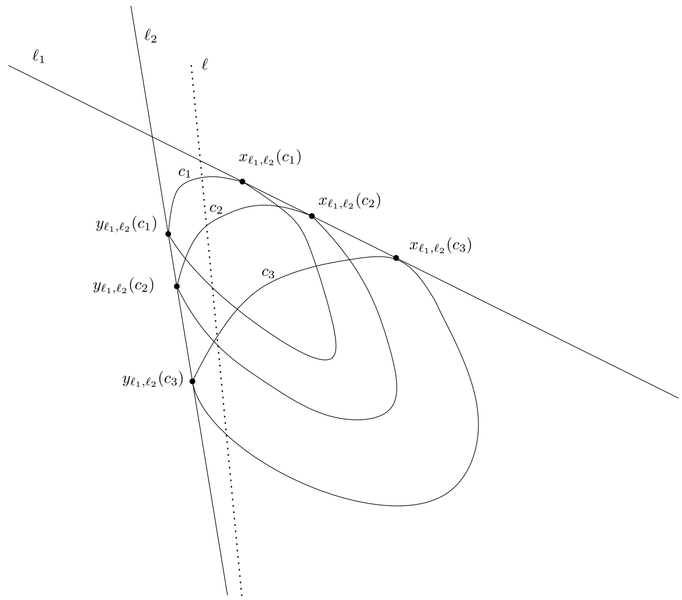

Figure 1: Illustration for the proof of Proposition 1.2 - c_1, c_2, c_3 are tangent to the lines ℓ_1, ℓ_2, and $c_1 \prec c_2 \prec c_3$.

at least one of c_1, c_3 inside c_2, or ℓ contains a point above c_2 and a point below c_2. In both cases, ℓ must intersect c_2.

Finally, by passing over all elements of $\mathcal{F}'(\ell_1, \ell_2)$ in each wedge, from the smallest to the largest, according to the order $\prec$, the number of times that the hyperedge defined by the current pseudo-disc is changed is linear in $|\mathcal{L}|$. Indeed, any such change is caused by appearance or disappearance of some line, and each line in $\mathcal{L}$ appears at most once and disappears at most once, along the proccess. Therefore, in each wedge, $|\mathcal{E}(H(\mathcal{L}, \mathcal{F}'(\ell_1, \ell_2)))| = O(n)$, and summing over all pairs $\{\ell_1, \ell_2\} \in \mathcal{L}$, we get $|\mathcal{E}(H)| = O(n^3)$. $\square$

5 Open Problems

We conclude this note with a few open problems.

Hypergraph of lines and inscribed pseudo-discs. A natural question is whether the arguments of Aronov et al. [5] can be extended from discs to pseudo-discs. We have found that all their arguments would go through if we knew that every triangle has an inscribed pseudo-disc. More precisely, we would need that for any triangle formed by three sides a, b, c, there is a pseudo-disc $d \in \mathcal{F}$, contained in the closed triangle, that intersects every side in exactly one point, or if there is no such $d \in \mathcal{F}$, then we can add such a new pseudo-disc d to $\mathcal{F}$ such that $\mathcal{F} \cup \{d\}$ still forms a pseudo-disc family. Unfortunately, it seems that such a theory has not been developed yet, not even for $\mathcal{F}$ all whose elements are convex.

We note that for the related problem regarding circumscribed pseudo-discs, even a stronger result is known. Specifically, it was shown in [16, Thm. 5.1] that for any three points a, b, c, there is a pseudo-disc $d \in \mathcal{F}$ such that $a, b, c \in \partial d$, or if there is no such $d \in \mathcal{F}$, then we can add such a new pseudo-disc d to $\mathcal{F}$ such that $\mathcal{F} \cup \{d\}$ still forms a pseudo-disc family.

Dependence on t in Theorem 1.1. While we showed the quadratic dependence on n in Theorem 1.1 to be tight, the dependence on t is not clear. It seems plausible that

$$|\{e \in \mathcal{E}(H(\mathcal{L}, \mathcal{F})) : |e| = t\}| = O(tn^2),$$

but we have not been able to prove this. On the other hand, even the stronger upper bound $O(n^2)$ for any fixed t, that would immediately imply Proposition 1.2 might hold.

Analogue of Lemma 2.6 for 3-sized hyperedges. It seems plausible that one can prove the following analogue of Lemma 2.6 for 3-sized hyperedges: If in some hypergraph on n vertices, for any induced hypergraph, the number of 3-sized hyperedges is quadratic in the number of vertices, then for any fixed t, the number of t-sized hyperedges is $O_t(n^2)$. Such a strong leveraging lemma would allow an easier proof of Theorem 1.1.

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