TRIANGLES IN INTERSECTING FAMILIES

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ABSTRACT. We prove the following the generalized Turán type result. A collection $\mathcal{T}$ of r sets is an r-triangle if for every $T_1, T_2, \ldots, T_{r-1} \in \mathcal{T}$ we have $\cap_{i=1}^{r-1} T_i \neq \emptyset$, but $\cap_{T \in \mathcal{T}} T$ is empty. A family $\mathcal{F}$ of sets is r-wise intersecting if for any $F_1, F_2, \ldots, F_r \in \mathcal{F}$ we have $\cap_{i=1}^{r} F_i \neq \emptyset$ or equivalently if $\mathcal{F}$ does not contain any m-triangle for $m = 2, 3, \ldots, r$. We prove that if $n \geq n_0(r,k)$, then the r-wise intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ containing the most number of $(r+1)$-triangles is isomorphic to $\{F \in \binom{[n]}{k} : |F \cap [r+1]| \geq r\}$.

1. Introduction

Turán type problems form one of the most studied areas in extremal combinatorics. They ask for the maximum size of a combinatorial structure that avoids some forbidden substructure. Most common examples are the following: the maximum number of edges in an H-free graph on n vertices, the maximum number of sets in $2^{[n]}$ that avoids some inclusion pattern P, the maximum number of sets in in $2^{[n]}$ satisfying some intersection property. Recently, so called *generalized* Turán problems have attracted the attention of researchers, especially in the domain of extremal graph theory. Given a forbidden graph H, what is the maximum number of copies of a fixed graph F among n-vertex H-free graphs G? A rapidly growing literature addresses problems of this type.

There exist some results of similar flavor concerning uniform intersecting families [5, 6, 7] or the union of two intersecting families [12]. In this short note, we consider k-uniform intersecting families $\mathcal{F} \subseteq \binom{[n]}{k} := \{F \subseteq \{1, 2, \ldots, n\} : |F| = k\}$. By definition, if $\mathcal{F}$ is intersecting, then it cannot contain a pair of disjoint sets. But what is the maximum number of triples F, F', F'' that a family $\mathcal{F} \subseteq \binom{[n]}{k}$ can contain with $F \cap F' \cap F'' = \emptyset$? Such triples are called *triangles*. A natural and well-known candidate of an intersecting family with many triangles is the following. For any 3-subset X of $[n]$, we define $\mathcal{F}_{X,k} = \{F \in \binom{[n]}{k} : |F \cap X| \geq 2\}$ and $\mathcal{F}'_{X,k} = \{F \in \binom{[n]}{k} : |F \cap X| = 2\}$. We write $n_{3,k}$ to denote the number of triangles in $\mathcal{F}_{X,k}$.

Theorem 1.1. *For every $k \geq 2$ if $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting with $n \geq 4k^6$, then the number of triangles in $\mathcal{F}$ is at most $n_{3,k}$ and equality holds if and only if $\mathcal{F}'_{X,k} \subseteq \mathcal{F} \subseteq \mathcal{F}_{X,k}$ for some 3-subset X of $[n]$.*

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More generally, one can consider r*-wise intersecting families.* $\mathcal{F}$ has this property if for any $F_1, F_2, \dots, F_r \in \mathcal{F}$, we have $\cap_{i=1}^r F_i \neq \emptyset$. As proved by Frankl [3], the maximum size of an r-wise intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ is $\binom{n-1}{k-1}$ whenever $\frac{r-1}{r}n \geq k$ holds. The extremal family is unique unless $r = 2$ and $n = 2k$: all k-sets containing a fixed element of the ground set. The r-wise intersecting property can be formulated via forbidden configurations. A family $\mathcal{T}$ of k-sets with $\cap_{T\in\mathcal{T}} T = \emptyset$ is an r-triangle if $|\mathcal{T}| = r$ and for any $T_1, T_2, \dots, T_{r-1} \in \mathcal{T}$ we have $\cap_{i=1}^{r-1} T_i \neq \emptyset$. Let $\Delta_{r,k}$ denote the set of all k-uniform r-triangles. Then $\mathcal{F} \subseteq \binom{[n]}{k}$ is r-wise intersecting if and only if it is $\Delta_{m,k}$-free for all $2 \leq m \leq r$. Therefore, a natural generalization of Theorem 1.1 would be to maximize the number of $(r+1)$-triangles in r-wise intersecting families.

For any $X \subset [n]$ and integer k we define $\mathcal{F}_{X,k} = \{F \in \binom{[n]}{k} : |F \cap X| \geq |X| - 1\}$ and $\mathcal{F}'_{X,k} = \{F \in \binom{[n]}{k} : |F \cap X| = |X| - 1\}$. Finally, we write $n_{r+1,k}$ to denote the number of $(r+1)$-triangles in $\mathcal{F}_{X,k}$ with $|X| = r + 1$.

Theorem 1.2. *For every $k \geq r \geq 3$ if $\mathcal{F} \subseteq \binom{[n]}{k}$ is r-wise intersecting with $n \geq 4k^{r(r+1)}$, then the number of $(r+1)$-triangles in $\mathcal{F}$ is at most $n_{r+1,k}$ and equality holds if and only if $\mathcal{F}'_{X,k} \subseteq \mathcal{F} \subseteq \mathcal{F}_{X,k}$ for some $(r+1)$-subset X of $[n]$.*

It is not hard to see and we will show after Proposition 2.2, if $r > k$, then every r-wise intersecting k-uniform family $\mathcal{F}$ is trivial, i.e. all sets in $\mathcal{F}$ share a common element and thus $\mathcal{F}$ does not contain any ℓ-triangles for any $\ell \geq 2$. Therefore Theorem 1.2 covers all interesting cases.

2. Preliminaries

In this section, we gather some easy statements that we will need during the proofs.

Proposition 2.1. *For any fixed $2 \leq r \leq k$ and $n \geq k^2$ we have*

$$n_{r+1,k} \geq \frac{1}{2}\binom{n-r-1}{k-r}^{r+1} \geq \frac{(n-1)^{r+1}}{2k^{r+1}}.$$

Proof. As any $(r+1)$-triangle in $\mathcal{F}_{X,k}$ must contain one member of $\mathcal{F}_x = \{F \in \mathcal{F}_{X,k} : x \notin F\}$ for each $x \in X$ and these sets should not have a common element outside X, by exclusion-inclusion, we obtain the following lower bound on $n_{r+1,k}$:

$$\binom{n-r-1}{k-r}^{r+1} - (n-r-1)\binom{n-r-2}{k-r-1}^{r+1} \geq \binom{n-r-1}{k-r}^{r+1}\left[1 - \frac{k^{r+1}}{n^r}\right] \geq \frac{1}{2}\binom{n-r-1}{k-r}^{r+1}.$$

□

The *covering number* $\tau(\mathcal{F})$ of a family $\mathcal{F}$ of sets is the smallest size of a *cover* set X with $X \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. The family of covers of $\mathcal{F}$ is denoted by $\mathcal{C}_{\mathcal{F}}$.

Proposition 2.2. *If $\mathcal{F}$ is r-wise intersecting with $\tau(\mathcal{F}) = t$, then $\mathcal{F}$ is $[(r-2)(t-1)+1]$-intersecting.*

Proof. Suppose not and let $F, F' \in \mathcal{F}$ be two sets with $X = F \cap F'$ of size at most $(r-2)(t-1)$. Then one can partition X to $X_1, X_2, \ldots, X_{r-2}$ with $|X_i| \leq t-1$ for i. As $\tau(\mathcal{F}) = t$, none of the X_i's is a cover of $\mathcal{F}$, so for all i there exists $F_i \in \mathcal{F}$ disjoint with X_i. But then $F \cap F' \cap \bigcap_{i=1}^{r-2} F_i = \emptyset$ which contradicts the r-wise intersecting property of $\mathcal{F}$. □

Proposition 2.2 implies that if $r > k$, then we have $\tau(\mathcal{F}) = 1$. Indeed, if $\tau(\mathcal{F}) \geq 2$, then a k-uniform r-wise intersecting family $\mathcal{F}$ must be k-intersecting, which means that $\mathcal{F}$ must consist of a single k-set contradicting $\tau(\mathcal{F}) \geq 2$.

The next two propositions provide some very weak bounds on the size of intersecting families with respect to their covering number. Many stronger results are known (see e.g. [4, 5, 8, 9, 10, 11]), but even those would not yield linear thresholds for Theorems 1.1 and 1.2. On the other hand, the proofs of the propositions are so easy and short that we can include them for sake of self-containedness.

Proposition 2.3. *If $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family with $\tau(\mathcal{F}) \geq t$, then $|\mathcal{F}| \leq k^t \binom{n-t}{k-t}$.*

Proof. For any subset $S \subset [n]$ let the degree of S be $d_{\mathcal{F}}(S) := |\{F \in \mathcal{F} : S \subseteq F\}|$ and we write D_m for the maximum degree over all sets of size m. Observe that as long as $m < t$, we have $D_m \leq k \cdot D_{m+1}$. Indeed, for any m-subset S there exists $F \in \mathcal{F}$ with $S \cap F = \emptyset$, so every $S \subseteq F' \in \mathcal{F}$ must meet F and thus $d_{\mathcal{F}}(S) \leq \sum_{x \in F} d_{\mathcal{F}}(S \cup \{x\})$. Clearly, we have $D_t \leq \binom{n-t}{k-t}$ and $D_0 = |\mathcal{F}|$. This finishes the proof. □

Not necessarily distinct families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ of sets are said to be *cross-t-intersecting* if for any $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, \ldots, F_m \in \mathcal{F}_m$ we have $| \cap_{i=1}^m F_i| \geq t$.

Proposition 2.4. *If $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m \subseteq \binom{[n]}{k}$ are cross-t-intersecting with $\tau(\mathcal{F}_i) \geq 2$ for all $i = 1, 2, \ldots, m$, then $|\mathcal{F}_i| \leq \frac{k}{t}\binom{k}{t}\binom{n-t-1}{k-t-1}$ for all $i = 1, 2, \ldots, m$.*

Proof. Observe first that $\tau(\mathcal{F}_i) \geq 2$ implies that all $\mathcal{F}_i$s are non-empty. Without loss of generality, we bound the size of $\mathcal{F}_1$. Fixing $G \in \mathcal{F}_2$ we have $|\mathcal{F}_1| \leq \frac{1}{t} \sum_{x \in G} |\{F \in \mathcal{F}_1 : x \in F\}|$. As $\tau(\mathcal{F}_2) \geq 2$, for every $x \in G$ there exists $G_x \in \mathcal{F}_2$ with $x \notin G_x$. So any $F \in \{F \in \mathcal{F}_1 : x \in F\}$ must meet G_x in at least t elements, thus $|\{F \in \mathcal{F}_1 : x \in F\}| \leq \binom{k}{t}\binom{n-t-1}{k-t-1}$. □

3. Proofs

For any family $\mathcal{F}$ we will write $\mathcal{N}(\Delta_{r+1}, \mathcal{F})$ to denote the number of $(r+1)$-triangles in $\mathcal{F}$.

Proof of Theorem 1.1. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family. We can assume that $\mathcal{F}$ is maximal as adding sets to $\mathcal{F}$ can only increase the number of triangles. Therefore we can assume that any k-set F that contains a cover $C \in \mathcal{C}_{\mathcal{F}}$ belongs to $\mathcal{F}$. (Note that this is true only for maximal intersecting families, but not for maximal r-wise intersecting families with $r \geq 3$.)

Claim 3.1. If $n \geq 2k$ and $C, C' \in \mathcal{C}_{\mathcal{F}}$, then $C \cap C' \neq \emptyset$.

Proof of Claim. Suppose not. Then, as $n \geq 2k$, there exist k-sets F, F' with $C \subseteq F, C' \subseteq F'$ and $F \cap F' = \emptyset$. This contradicts the intersecting property of $\mathcal{F}$. $\square$

We consider several cases according to $\tau(\mathcal{F})$.

Case I $\tau(\mathcal{F}) = 1$.
Then the unique element of a singleton cover belongs to all sets of $\mathcal{F}$, so $\mathcal{F}$ does not contain any triangles.

Case II $\tau(\mathcal{F}) \geq 3$.
Then by Proposition 2.3, we have $|\mathcal{F}| \leq k^3\binom{n-3}{k-3}$, and thus $\mathcal{N}(\Delta_3, \mathcal{F}) \leq \binom{|F|}{3} \leq k^9\binom{n-3}{k-3}^3$ which, if $n \geq 2k^4$, is smaller than $\frac{1}{2}\binom{n-3}{k-2}^3$, and thus than the value of $n_{3,k}$ by Proposition 2.1.

Case III $\tau(\mathcal{F}) = 2$
By Claim 3.1, $\mathcal{C}_\mathcal{F} \cap \binom{[n]}{2}$ is either a graph star S_ℓ with ℓ leaves or a graph triangle. We consider first the case when we have a star with center c.

Suppose first $\ell \geq 3$. Then any triangle must contain a set F that does not contain c. As $\ell \geq 3$ and F must contain all leaves, there are at most $\binom{n-\ell-1}{k-\ell} \leq \binom{n-4}{k-3}$ such sets. Thus the number of triangles is at most $\binom{n-4}{k-3} \cdot \binom{|\mathcal{F}|}{2} \leq k^4\binom{n-4}{k-3}\binom{n-2}{k-2}^2$ by Proposition 2.3. If $n \geq 4k^5$, then this is smaller than $\frac{1}{2}\binom{n-3}{k-2}^3$, and thus than $n_{3,k}$.

Suppose next $\ell = 2$ with $\mathcal{C}_\mathcal{F} \cap \binom{[n]}{2} = \{\{c,x\},\{c,y\}\}$. As $\{x,y\}$ is not a cover, there must be a set $F' \in \mathcal{F}$ with $x, y \notin F'$. On the other hand every $F \in \mathcal{F}$ with $c \notin F$ must contain both x and y (as $\{c,x\}$ and $\{c,y\}$ are covers) and meet F', so their number is not more than $k\binom{n-3}{k-3}$. So the number of triangles in $\mathcal{F}$ is at most $k\binom{n-3}{k-3}\binom{|\mathcal{F}|}{2} \leq k^5\binom{n-3}{k-3}\binom{n-2}{k-2}^2$, which is smaller than $\frac{1}{2}\binom{n-3}{k-2}^3$ if $n \geq 4k^6$.

Finally, suppose that $\ell = 1$, i.e. $\{x,y\}$ is a unique cover set of $\mathcal{F}$ of size 2. We try to bound the size of $\mathcal{F}_x = \{F \in \mathcal{F} : x \in F, y \notin F\}$ and $\mathcal{F}_y = \{F \in \mathcal{F} : x \notin F, y \in F\}$. As $\{x\}$ and $\{y\}$ are not covers but $\{x,y\}$ is, they are not empty. Fix $F_y \in \mathcal{F}_y$ and consider $F \in \mathcal{F}_x$. All such F must meet F_y in some $y' \neq y$. There are $k-1$ choices for y' and for each y', as $\{x,y'\}$ is not a cover, there exists $F_{y'} \in \mathcal{F}$ with $x, y' \notin F_{y'}$. So every $F \ni x, y'$ must meet $F_{y'}$, so there are at most $k\binom{n-3}{k-3}$ such sets. Summing over all y', we obtain $|\mathcal{F}_x| \leq (k-1)k\binom{n-3}{k-3}$. An identical argument yields $|\mathcal{F}_y| \leq (k-1)k\binom{n-3}{k-3}$. Observe that every triangle must contain a set both from $\mathcal{F}_x$ and $\mathcal{F}_y$ - otherwise its three sets would have a common element as $\{x,y\}$ is a cover. Therefore the number of triangles in $\mathcal{F}$ is at most $|\mathcal{F}_x| \cdot |\mathcal{F}_y| \cdot |\mathcal{F}| \leq k^6\binom{n-3}{k-3}^2\binom{n-2}{k-2}$ which is smaller than $\frac{1}{2}\binom{n-3}{k-2}^3 \leq n_{3,k}$ if $n \geq 2k^4$.

In all cases, we obtained that $\mathcal{N}(\Delta_3, \mathcal{F}) < n_{3,k}$, so we must have that $\mathcal{C}_\mathcal{F} \cap \binom{[n]}{2}$ is a graph triangle $\{x,y,z\}$. So every $F \in \mathcal{F}$ must contain at least 2 of x, y, z and thus $\mathcal{F} \subseteq \mathcal{F}_{\{x,y,z\},k}$. In order to have as many triangles as possible, $\mathcal{F}$ must contain all sets in $\mathcal{F}'_{x,y,z}$. This finishes the proof of Theorem 1.1. $\square$

We continue with the proof of the general result.

Proof of Theorem 1.2. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an r-wise intersecting family. We start with a lemma on the covering number of $\mathcal{F}$.

Lemma 3.2. *If $\mathcal{F} \subseteq \binom{[n]}{k}$ is r-wise intersecting with $\tau(\mathcal{F}) \geq 3$, then $\mathcal{N}(\Delta_{r+1}, \mathcal{F}) < n_{r+1,k}$.*

Proof. By Proposition 2.2, $\mathcal{F}$ is $(2r-3)$-intersecting with $\tau(\mathcal{F}) > 1$, so applying Proposition 2.4 with $\mathcal{F}_i = \mathcal{F}$ we obtain $|\mathcal{F}| \leq k\binom{k}{2r-3}\binom{n-2r+2}{k-2r+2}$. Clearly, we have

$$\mathcal{N}(\Delta_{r+1}, \mathcal{F}) \leq \binom{|\mathcal{F}|}{r+1} \leq \left[k^{2r-2}\binom{n-2r+2}{k-2r+2}\right]^{r+1} < \frac{1}{2}\binom{n-r-1}{k-r}^{r+1} \leq n_{r+1,k},$$

where the strict inequality uses $r \geq 3$ and $n \geq 2k^{2r-1}$, and the last inequality is Proposition 2.1. □

If $\tau(\mathcal{F}) = 1$, then $\mathcal{F}$ does not contain any $(r+1)$-triangles, so by Lemma 3.2, we can assume $\tau(\mathcal{F}) = 2$. Let G be the graph with vertex set $[n]$ and edge set $\mathcal{C}_{\mathcal{F}} \cap \binom{[n]}{2}$. By Proposition 2.2, we know that $\mathcal{F}$ is $(r-1)$-intersecting with $\tau(\mathcal{F}) = 2$, therefore, by Proposition 2.4 with $\mathcal{F}_i = \mathcal{F}$, we obtain $|\mathcal{F}| \leq k^r\binom{n-r}{k-r}$. We will use this bound several times below.

Lemma 3.3. *If G contains an induced path on 3 vertices, then $\mathcal{N}(\Delta_{r+1}, \mathcal{F}) < n_{r+1,k}$.*

Proof. Suppose xc, yc are edges of G but xy is not. Then any $(r+1)$-triangle must contain a set $F \in \mathcal{F}$ with $c \notin F$. As xc and yc are covers of $\mathcal{F}$, all such sets must contain both x and y. Furthermore, as xy is not a cover of $\mathcal{F}$, there must exist $F' \in F$ with $x, y \notin F'$. By Proposition 2.2, we have $|F \cap F'| \geq r-1$, so the number of possible Fs is at most $\binom{k}{r-1}\binom{n-r-1}{k-r-1}$. Therefore, the number of $(r+1)$-triangles in $\mathcal{F}$ is at most

$$\binom{k}{r-1}\binom{n-r-1}{k-r-1}|\mathcal{F}|^r \leq k^{r^2+r-1}\binom{n-r-1}{k-r-1}\binom{n-r}{k-r}^r < \frac{1}{2}\binom{n-r-1}{k-r}^{r+1} \leq n_{r+1,k},$$

where we used $n \geq 4k^{r^2+r}$ for the strict inequality. □

Lemma 3.4. *If G has maximum degree larger than r, then $\mathcal{N}(\Delta_{r+1}, \mathcal{F}) < n_{r+1,k}$.*

Proof. Suppose c has degree at least $r+1$ in G. Any $(r+1)$-triangle must contain a set $F \in \mathcal{F}$ with $c \notin F$. All such F must contain all neighbors of c as any neighbor together with c form a cover of $\mathcal{F}$. Therefore, the number of such sets is at most $\binom{n-r-2}{k-r-1}$. Thus the number of $(r+1)$-triangles in $\mathcal{F}$ is at most

$$\binom{n-r-2}{k-r-1}|\mathcal{F}|^r \leq k^{r^2}\binom{n-r-2}{k-r-1}\binom{n-r}{k-r}^r < \frac{1}{2}\binom{n-r-1}{k-r}^{r+1} \leq n_{r+1,k},$$

where we used $n \geq 4k^{r^2+1}$ for the strict inequality. □

Lemma 3.3 yields that all components of G are cliques, and Lemma 3.4 shows that no component has size more than $r+1$.

Lemma 3.5. *If G contains a component of size smaller than $r+1$ that is not an isolated vertex, then $\mathcal{N}(\Delta_{r+1},\mathcal{F}) < n_{r+1,k}$.*

Proof. Without loss of generality we can assume that $1,2,\dots,m$ form a clique in G with $m \le r$. Therefore every set $F \in \mathcal{F}$ meets $[m]$ in at least $m-1$ vertices. Also, every $(r+1)$-triangle must contain a set F_i with $i \notin F_i$ for all $i=1,2,\dots,m$. Let $\mathcal{F}_i$ denote $\{F \in \mathcal{F} : i \notin F\}$ and $\mathcal{F}'_i = \{F \setminus [m] : F \in \mathcal{F}_i\}$, $\mathcal{F}' = \cup_{i=1}^m \mathcal{F}'_i$. Then $|\mathcal{F}_i| = |\mathcal{F}'_i|$ and $\mathcal{F}'_i$, $\mathcal{F}'$ are $(k-m+1)$-uniform.

Observe first that we have $\tau(\mathcal{F}'_i) \ge 2$ for all i. Indeed, if $\{x\}$ was a cover of $\mathcal{F}'_i$, then $x \notin [m]$ and $\{i,x\}$ would be a cover of $\mathcal{F}$, so an edge in G, but x and i are in different components of G.

Next we claim that the $\mathcal{F}'_i$s are cross-$(r-m+1)$-intersecting (as $m \le r$, this a positive number). Indeed, if for some $F'_1 \in \mathcal{F}'_1, F'_2 \in \mathcal{F}'_2, \dots, F'_m \in \mathcal{F}'_m$ we had $|\cap_{i=1}^m F'_i| \le r-m$, then for any $x \in \cap_{i=1}^m F'_i =: X$ there exists $F_x \in \mathcal{F}$ with $x \notin F_x$, because $\tau(\mathcal{F}) \ge 2$. Then $\cap_{i=1}^m F_i \cap_{x\in X} F_x = \emptyset$, where $F_i \in \mathcal{F}_i$ with $F_i \setminus [m] = F'_i$. This contradicts the r-wise intersecting property of $\mathcal{F}$ and thus proves our claim.

Applying Proposition 2.4, we obtain $|\mathcal{F}_i| = |\mathcal{F}'_i| \le \frac{k-m+1}{r-m+1}\binom{k-m+1}{r-m+1}\binom{n-r+m-2}{k-r-1} \le k^{r-m+2}\binom{n-r+m-2}{k-r-1}$. Observe that

$$\mathcal{N}(\Delta_{r+1},\mathcal{F}) \le |\mathcal{F}|^{r+1-m} \cdot \prod_{i=1}^m |\mathcal{F}_i| \le \left(k^r\binom{n-r}{k-r}\right)^{r+1-m} \cdot \left[k^{r-m+2}\binom{n-r+m-2}{k-r-1}\right]^m$$

This last expression decreases with m, so takes its maximum at $m=2$. So the number of $(r+1)$-triangles in $\mathcal{F}$ is at most

$$k^{r(r+1)}\binom{n-r}{k-r}^{r-1}\binom{n-r}{k-r-1}^2 < \frac{1}{2}\binom{n-r-1}{k-r}^{r+1} \le n_{r+1,k}$$

if $n \ge 4k^{\frac{r(r+1)+2}{2}}$ holds. □

We obtained that unless G consists only of cliques of size $r+1$, the number of $(r+1)$-triangles in $\mathcal{F}$ is strictly smaller than $n_{r+1,k}$. On the other hand, if X is the vertex set of a clique of size $r+1$ in G, then $\mathcal{F} \subseteq \mathcal{F}_{X,k}$ as every $F \in \mathcal{F}$ must meet X in at least $|X|-1$ elements to intersect all edges of the clique. To contain as many $(r+1)$-triangles as possible, we must have $\mathcal{F}'_{X,k} \subseteq \mathcal{F}$. This finishes the proof of Theorem 1.2. □

It would be interesting to obtain a different proof of Theorems 1.1 and 1.2 that results in a smaller threshold on n with respect to k and r. Also, the non-uniform version of these theorems are of interest. We conjecture that an analogous statement should hold.

Conjecture 3.6. *For any $r \ge 2$ there exists $n_0 = n_0(r)$ such that if $\mathcal{F} \subseteq 2^{[n]}$ is r-wise intersecting with $n \ge n_0$, then the number of $(r+1)$-triangles in $\mathcal{F}$ is at most $\mathcal{N}(\Delta_{r+1},\mathcal{F}_X)$, where $\mathcal{F}_X = \{F \subseteq [n] : |F \cap X| \ge |X|-1\}$ for some $(r+1)$-set X.*

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