Order
https://doi.org/10.1007/s11083-021-09581-4

Rainbow Ramsey Problems for the Boolean Lattice

Fei-Huang Chang[1] · Dániel Gerbner[2] · Wei-Tian Li[3] · Abhishek Methuku[4] · Dániel T. Nagy[2] · Balázs Patkós[2,5] · Máté Vizer[2,6]

Received: 16 July 2020 / Accepted: 30 September 2021 / Published online: 11 December 2021

Abstract
We address the following rainbow Ramsey problem: For posets P, Q what is the smallest number n such that any coloring of the elements of the Boolean lattice B_n either admits a monochromatic copy of P or a rainbow copy of Q. We consider both weak and strong (non-induced and induced) versions of this problem.

Keywords Extremal set systems · Forbidden subposet problem · Ramsey theory

✉ Balázs Patkós
patkosb@gmail.com

Fei-Huang Chang
cfh@ntnu.edu.tw

Dániel Gerbner
gerbner@renyi.hu

Wei-Tian Li
weitianli@nchu.edu.tw

Abhishek Methuku
abhishekmethuku@gmail.com

Dániel T. Nagy
nagydani@renyi.hu

Máté Vizer
vizermate@gmail.com

[1] Division of Preparatory Programs for Overseas Chinese Students, National Taiwan Normal University New Taipei City, Taipei, Taiwan

[2] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary

[3] Department of Applied Mathematics, National Chung Hsing University, Taichung, 40227, Taiwan

[4] University of Birmingham, Edgbaston, Birmingham, United Kingdom

[5] Moscow Institute of Physics and Technology, Dolgoprudny, Russia

[6] Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Budapest, Hungary

1 Introduction

In this paper we consider rainbow Ramsey-type problems for posets. Given posets P and Q, we say that $X \subseteq Q$ is a *weak copy* of P, if there is a bijection $\alpha\colon P \to X$ such that $p \leq_P p'$ implies $\alpha(p) \leq_Q \alpha(p')$. If α has the stronger property that $p \leq_P p'$ holds if and only if $\alpha(p) \leq_Q \alpha(p')$, then X is a *strong* or *induced copy of* P. A copy X of P is *monochromatic* with respect to a coloring $\phi\colon Q \to \mathbb{Z}^+$, if $\phi(q) = \phi(q')$ for all $q, q' \in X$ and *rainbow* if $\phi(q) \neq \phi(q')$ for all $q \neq q' \in X$. We will be looking for monochromatic and/or rainbow copies of some posets in the Boolean lattice B_n, the subsets of an n-element set ordered by inclusion. The set of elements of B_n corresponding to sets of the same size is called a *level* of B_n.

Definition 1.1 The *weak Ramsey number* $R(P_1, P_2, \dots, P_k)$ is the smallest number n such that for any coloring of the elements of B_n with k colors, say $1, 2, \dots, k$ there is a monochromatic copy of the poset P_i in color i for some $1 \leq i \leq k$. We simply write $R_k(P)$ for $R(P_1, P_2, \dots, P_k)$, if $P_1 = \dots = P_k = P$. We define the *strong Ramsey number* $R^*(P_1, P_2, \dots, P_k)$ and $R^*_k(P)$ for strong copies of posets analogously.

Ramsey theory of posets is an old and well investigated topic, see e.g., [11, 15]. However, the study of Ramsey problems in the Boolean lattice was initiated only recently: weak Ramsey numbers were studied by Cox and Stolee [3] and strong Ramsey numbers were investigated by Axenovich and Walzer [1]. In addition, some results in the latter one were improved by Lu and Thompson [12].

In this article, we study rainbow Ramsey numbers for the Boolean lattice.

Definition 1.2 For two posets P, Q the *weak (or not necessarily induced) rainbow Ramsey number* $RR(P, Q)$ is the minimum number n such that any coloring (using an arbitrary number of colors) of B_n admits either a monochromatic weak copy of P or a rainbow weak copy of Q. The *strong (or induced) rainbow Ramsey number* can be defined analogously and is denoted by $RR^*(P, Q)$.

Rainbow Ramsey numbers for graphs have been intensively studied (they are sometimes called constrained Ramsey numbers or Gallai–Ramsey numbers), for a recent survey see [4]. The results on the rainbow Ramsey number for Boolean posets are sporadic [2, 10]. Nevertheless, the following easy observation connects (usual) Ramsey numbers to rainbow Ramsey numbers.

Proposition 1.3 *For any pair P and Q of posets we have*
(i) $RR(P, Q) \geq R_{|Q|-1}(P)$*, and*
(ii) $RR^*(P, Q) \geq R^*_{|Q|-1}(P)$.

Proof To see **(i)** observe that if a coloring ϕ uses at most $|Q|-1$ colors, then clearly it cannot contain a rainbow weak copy of Q. Therefore any such coloring showing $R_{|Q|-1}(P) > n$ also shows $RR(P, Q) > n$. An identical proof with strong copies implies **(ii)**. □

In this paper, we show many examples of posets P, Q for which the inequality in **(i)** of Proposition 1.3 holds with equality, while in Section 3, we show another example of posets P, Q for which **(ii)** of Proposition 1.3 holds with strict inequality. Unfortunately, we do not know whether there exist posets P, Q for which **(i)** holds with strict inequality.

Many of the tools used in [1, 3] come from the related Turán-type problem, the so-called forbidden subposet problem. Let us introduce some terminology. For a poset P, a family $\mathcal{F} \subseteq B_n$ of sets is called (induced) P-*free* if $\mathcal{F}$ does not contain a weak (strong) copy of P. The size of the largest (induced) P-free family in B_n is denoted by $La(n, P)$ (resp. $La^*(n, P)$). For a poset P, we denote by $e(P)$ the maximum number m such that for any n the union of any consecutive m levels of B_n is P-free. The analogous strong parameter is denoted by $e^*(P)$. The most widely believed conjecture [5] in the area of forbidden subposet problems states that for any poset P we have

$$\lim_{n\to\infty} \frac{La(n, P)}{\binom{n}{\lfloor n/2 \rfloor}} = e(P) \text{ and } \lim_{n\to\infty} \frac{La^*(n, P)}{\binom{n}{\lfloor n/2 \rfloor}} = e^*(P).$$

It is worth noting that this conjecture is already wide open for a very simple poset called the diamond poset D_2 (defined on four elements a, b, c, and d with relations $a < b, c$ and $b, c < d$). See [9] for the best known bounds in this direction.

For a family $\mathcal{F} \subseteq B_n$ of sets, its *Lubell-mass* is $\lambda_n(\mathcal{F}) = \sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}}$. For a poset P, we define $\lambda_n(P)$ to be the maximum value of $\lambda_n(\mathcal{F})$ over all P-free families $\mathcal{F} \subseteq B_n$ and $\lambda_{max}(P)$ is defined to be $\sup_n \lambda_n(P)$. Its finiteness follows from the fact that every poset P is a weak subposet of $C_{|P|}$ (where C_l denotes the l-chain, the totally ordered set of size l) and the k-LYM-inequality stating that $\lambda_n(\mathcal{F}) \le k$ for any C_{k+1}-free family $\mathcal{F} \subseteq B_n$. Analogously, $\lambda_n^*(P)$ is the maximum value of $\lambda_n(\mathcal{F})$ over all induced P-free families $\mathcal{F} \subseteq B_n$ and $\lambda_{max}^*(P)$ is defined to be $\sup_n \lambda_n^*(P)$. It was proved to be finite by Méroueh [13].

Observe that, by definition of $e(P)$ and $e^*(P)$, we have $e(P) \le \lambda_n(P)$ and $e^*(P) \le \lambda_n^*(P)$ for every poset P and integer $n \ge e(P)$ or $n \ge e^*(P)$. We say that a poset is *uniformly Lubell-bounded* if $e(P) \ge \lambda_n(P)$ holds for all positive integers n. Similarly, a poset is *uniformly induced Lubell-bounded* if $e^*(P) \ge \lambda_n^*(P)$ holds for all positive integers n. An instance of posets eqipped with this property is the class of chain posets C_l. For $k \ge 2$ the *generalized diamond* poset D_k consists of $k+2$ elements $a, b_1, b_2, \ldots, b_k, c$ with relations $a < b_i < c$ for $1 \le i \le k$. Griggs, Li and Lu [6] proved that infinitely many of the D_k's are uniformly Lubell-bounded and Patkós [14] proved that an overlapping but distinct and infinite subset of the D_k's is uniformly induced Lubell-bounded. For more uniformly Lubell-bounded posets, see [8].

In [1] and [3], it was observed that if P is uniformly Lubell-bounded or uniformly induced Lubell-bounded, then $R_k(P) = k \cdot e(P)$ or $R_k^*(P) = k \cdot e^*(P)$ holds, respectively.

Our main result concerning weak rainbow Ramsey numbers extends the above observation.

Theorem 1.4 *Let P be a uniformly Lubell-bounded poset and $\mathcal{F} \subseteq B_n$ be a family of sets with $\lambda_n(\mathcal{F}) > e(P)(k-1)$. Then any coloring of $\phi\colon \mathcal{F} \to \mathbb{Z}^+$ admits either a monochromatic weak copy of P or a rainbow copy of C_k.*

Corollary 1.5 *If P is uniformly Lubell-bounded, then $RR(P, Q) = e(P)(|Q| - 1)$ holds for any poset Q.*

Proof As $\lambda_n(B_n) = n + 1$, the inequality $RR(P, Q) \le e(P)(|Q| - 1)$ is a direct consequence of Theorem 1.4 as any poset Q is a weak subposet of $C_{|Q|}$.

Let $n = (|Q| - 1)e(P) - 1$. The lower bound $RR(P, Q) > n$ follows from coloring B_n so that the color classes form a partition of the levels of B_n into $|Q| - 1$ intervals, each of size $e(P)$. As we use only $|Q| - 1$ colors, we avoid rainbow copies of Q and by definition of $e(P)$ we avoid monochromatic copies of P. □

For strong copies of posets, the coloring from the proof of Corollary 1.5 yields the same lower bound $RR^*(P, Q) \geq e^*(P)(|Q| - 1)$, but one can easily observe that in most cases this trivial lower bound can be improved by slightly modifying the above coloring: If Q does not have a unique smallest element, then one can color $\emptyset$ with an otherwise unused color i. Since no other sets are colored i, it does not help to create a strong monochromatic copy of P, and since Q does not have a unique smallest element, it does not help to create a strong rainbow copy of Q. Therefore one can introduce the following function. For any poset Q, let $f(Q) = 0$, if Q has both a unique largest and a unique smallest element, let $f(Q) = 2$, if Q has neither largest nor smallest element, and define $f(Q) = 1$ otherwise. One obtains $RR^*(P, Q) \geq e^*(P)(|Q| - 1) + f(Q)$ for all posets P and Q. For this lower bound, the strong version of Corollary 1.5 would be expected for P being uniformly induced Lubell-bounded. Nonetheless, we will show the above inequality is strict when $P = C_2$, the chain of two elements, and $Q = A_k$, the antichain of size k in Section 3. So we ask the following question.

Question 1.6 For which uniformly induced Lubell-bounded posets P, does one have

$$RR^*(P, Q) = e^*(P)(|Q| - 1) + f(Q) \tag{1}$$

for every poset Q?

Despite the above counterexample to Eq. 1, we prove that it holds for most uniformly induced Lubell-bounded posets P and $Q = A_3$. Indeed, we have a general upper bound for $RR^*(P, A_k)$ for any poset P and $k \geq 2$.

Theorem 1.7 *Given an integer $k \geq 2$, let $m_k = \min\{m : \binom{m}{\lfloor m/2 \rfloor} \geq k\}$. For any poset P we have*

$$RR^*(P, A_k) \leq \lfloor (k-1)\lambda^*_{max}(P) \rfloor + m_k.$$

Moreover, if P is not C_1 or C_2, then we have

$$RR^*(P, A_3) \leq \lfloor 2\lambda^*_{max}(P) \rfloor + 2.$$

Since $\lambda^*_{max}(P) = e^*(P)$ for every uniformly induced Lubell-bounded poset P, we have the next corollary immediately from the latter part of Theorem 1.7.

Corollary 1.8 *For every uniformly induced Lubell-bounded poset P other than C_1 or C_2 we have*

$$RR^*(P, A_3) = 2 + 2e^*(P).$$

Structure of the paper The remainder of the paper is organized as follows: Theorem 1.4 and other results on weak copies are proved in Section 2. Section 3 contains the proofs of the counterexample to Eq. 1 and Theorem 1.7.

Notation For $n \in \mathbb{Z}^+$ we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. For a set F, we write $\mathcal{U}_F = \mathcal{U}_{n,F} = \{G \subseteq [n] : F \subseteq G\}$, $\mathcal{D}_F = \mathcal{D}_{n,F} = \{G \subseteq [n] : G \subseteq F\}$, and $\mathcal{I}_F = \mathcal{I}_{n,F} = \mathcal{U}_{n,F} \cup \mathcal{D}_{n,F}$. For sets $F \subseteq H$, we write $B_{F,H} = \{G : F \subseteq G \subseteq H\}$. For integers $0 \leq a \leq b \leq n$, we write $\lambda_n(B_{a,b}) = \lambda_n(B_{F,H})$ for some $F \subseteq H \subseteq [n]$ with $|F| = a$, $|H| = b$. Let B_n^- and $B_{F,H}^-$ denote the *truncated* Boolean lattices obtained by removing the smallest and the largest element of the cubes B_n and $B_{F,H}$, respectively. For a coloring $\phi\colon B_n \to \mathbb{Z}^+$, let $\|\phi\|$ denote the number of colors used by ϕ. For a coloring $\phi\colon B_n \to \mathbb{Z}^+$ and a positive integer i, let $\mathcal{H}_i = \mathcal{H}_{\phi,i} = \{F \subseteq [n] : \phi(F) = i\}$. We use $\binom{n}{\le k}$ to denote $\sum_{j=0}^{k}\binom{n}{j}$. All logarithms are of base 2 in this paper.

2 Weak Copies

In this section, we prove Theorem 1.4 and some other results on weak Ramsey and weak rainbow Ramsey numbers. We start with a couple of definitions.

We denote by $\mathbf{C}_n$ the set of all maximal chains in B_n. For a family $\mathcal{F} \subseteq B_n$ and set $F \in \mathcal{F}$, we define $\mathbf{C}_{n,F} = \mathbf{C}_{n,F,\mathcal{F}}$ to be the set of those maximal chains $\mathcal{C} \in \mathbf{C}_n$ for which the largest set of $\mathcal{F} \cap \mathcal{C}$ is F. Then the *max-partition* of $\mathbf{C}_n$ consists of the blocks $\mathbf{C}_{n,F}$ for each $F \in \mathcal{F}$ and $\mathbf{C}_{n,-}$ which contains all maximal chains $\mathcal{C}$ with $\mathcal{F} \cap \mathcal{C} = \emptyset$.

The Lubell mass $\lambda_n(\mathcal{F}) = \sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}}$ is the average number of sets of $\mathcal{F}$ in a maximal chain $\mathcal{C}$ chosen uniformly at random from $\mathbf{C}_n$. As observed by Griggs and Li [7], if we condition on the largest set F in $\mathcal{F} \cap \mathcal{C}$, then we obtain

$$\lambda_n(\mathcal{F}) = \sum_{F\in\mathcal{F}} \frac{|\mathbf{C}_{n,F}|}{n!}\lambda_{|F|}(\mathcal{D}_F \cap \mathcal{F}).$$

Proof of Theorem 1.4 We proceed by induction on k. The base case $k = 1$ is trivial as any colored set forms a “rainbow” copy of C_1. Let $k \ge 2$ and suppose the statement is proven for $k-1$ and let $\mathcal{F} \subseteq B_n$ be a family of sets with $\lambda_n(\mathcal{F}) > e(P)(k-1)$. Fix a coloring $\phi : \mathcal{F} \to \mathbb{Z}^+$ and consider the max-partition $\{\mathbf{C}_{n,F} : F \in \mathcal{F}\} \cup \{\mathbf{C}_{n,-}\}$. Using

$$\lambda_n(\mathcal{F}) = \sum_{F\in\mathcal{F}} \frac{|\mathbf{C}_{n,F}|}{n!}\lambda_{|F|}(\mathcal{D}_F \cap \mathcal{F}),$$

we obtain a set $F \in \mathcal{F}$ with $\lambda_{|F|}(\mathcal{D}_F \cap \mathcal{F}) > e(P)(k-1)$. Let $\mathcal{F}_1 = \{G \in \mathcal{D}_F : \phi(G) = \phi(F)\}$. If $\mathcal{F}_1$ contains a weak copy of P, then we are done as, by definition, $\mathcal{F}_1$ is monochromatic. Otherwise, as P is uniformly Lubell-bounded, we have $\lambda_{|F|}(\mathcal{F}_1) \le e(P)$ and thus

$$\lambda_{|F|}((\mathcal{D}_F \cap \mathcal{F}) \setminus \mathcal{F}_1) > e(P)(k-1) - e(P) = e(P)(k-2).$$

Applying our inductive hypothesis to $(\mathcal{D}_F \cap \mathcal{F}) \setminus \mathcal{F}_1$ we either obtain a monochromatic weak copy of P or a rainbow copy of C_{k-1}. As all sets in $(\mathcal{D}_F \cap \mathcal{F}) \setminus \mathcal{F}_1$ are colored differently than F, we can extend the rainbow copy of C_{k-1} to a rainbow copy of C_k by adding F. □

Remark Note that a simple modification of the above proof shows that if P is a uniformly induced Lubell-bounded poset and $\mathcal{F} \subseteq B_n$ is a family of sets with $\lambda_n(\mathcal{F}) > e^*(P)(k-1)$, then any coloring of $\phi : \mathcal{F} \to \mathbb{Z}^+$ admits either a monochromatic strong copy of P or a rainbow copy of C_k, and therefore $RR^*(P, C_k) = e^*(P)(k-1)$ holds.

The equality in Proposition 1.3 (i) holds for uniformly Lubell-bounded posets P and any posets Q. To find posets P and Q with $RR(P, Q) > R_{|Q|-1}(P)$, we have to choose a non-uniformly Lubell-bounded poset as P. However, regardless of P, Proposition 1.3 (i) still holds with equality if Q is one of the following posets: for $r \ge 2$ the *r-fork* poset V_r consists of a minimum element and r other elements that form an antichain. Similarly, for $s \ge 2$ *the s-broom* poset Λ_s consists of a maximum element and s other elements that form an antichain.

Proposition 2.1 *For any poset P, we have*
(i) $RR(P, V_r) = R_r(P)$, *and*
(ii) $RR(P, \Lambda_s) = R_s(P)$.

Proof By Proposition 1.3, $RR(P, V_r) \geq R_r(P)$. Let $n = R_r(P)$. Any coloring $\phi \colon B_n \to \mathbb{Z}^+$ with $\|\phi\| \geq r+1$ contains a rainbow weak copy of V_r: the empty set and one representative from each of any other r color classes.

The proof of (ii) is similar by taking the universal set $[n]$ and one representative from each of any s other color classes if $\|\phi\| \geq s+1$. □

If P and Q are both fork posets, then we have $RR(V_r, V_k) = R_k(V_r)$ for any $r, k \geq 1$. In our next result, we manage to determine this value asymptotically for fixed k. We write $f_k(r) = R_k(V_r)$ for simplicity. A simple way to define a k-coloring of B_n is to color sets of the same size with the same color such that color classes consist of consecutive levels. Formally, let $i_1, i_2, \ldots, i_k$ be positive integers with $\sum_{j=1}^{k} i_j = n+1$ and consider the coloring $\phi(F) = h$ if and only if $\sum_{j=1}^{h-1} i_j \leq |F| < \sum_{j=1}^{h} i_j$. (The empty sum equals 0, so $\phi(F) = 1$ if and only if $|F| < i_1$ holds.) We call such a coloring ϕ a *consecutive level k-coloring* and define $g_k(r)$ to be the smallest integer n such that any consecutive level k-coloring of B_n admits a monochromatic weak copy of V_r. By definition, we have $g_k(r) \leq f_k(r)$.

For $c \in (0,1)$ let $h(c) = -c \log c - (1-c)\log(1-c)$, the *binary entropy function*. Note that for $c \in (0,1)$ and n large enough we have

$$\frac{1}{\sqrt{n}} 2^{nh(c)} \leq \binom{n}{\lfloor cn \rfloor} \leq 2^{nh(c)}.$$

We will use the fact that for $0 < \varepsilon \leq 1/2$ and $k \leq (1/2 - \varepsilon)n$ we have $\frac{\binom{n}{k-1}}{\binom{n}{k}} = \frac{k}{n-k} \leq \frac{1/2-\varepsilon}{1/2+\varepsilon} =: c$. It implies

$$\binom{n}{\leq k} = \sum_{i=0}^{k} \binom{n}{i} \leq \binom{n}{k} \sum_{i=0}^{k} c^{k-i} \leq \frac{1}{1-c} \binom{n}{k}. \tag{2}$$

In the proof we omit floor and ceiling signs for simplicity.

Theorem 2.2 *For any positive integer k there exists a constant c_k such that*

$$\lim_{r \to \infty} \frac{g_k(r)}{\log r} = \lim_{r \to \infty} \frac{f_k(r)}{\log r} = c_k.$$

Moreover, $c_1 = 1$ and the sequence $\{c_k\}_{k=1}^{\infty}$ satisfies the equality $c_{k+1} h\left(\frac{c_{k+1} - c_k}{c_{k+1}}\right) = 1$ for any $k \geq 1$.

Proof The proof is based on the recursive inequalities contained in the following claim. In part **(i)** of Claim 2.3, the term $\min\{a : \binom{a + f_k(2r-1)}{\leq a} > r\}$ ensures that in $B_{f_k(2r-1)+a}$ the levels $0, 1, \ldots, a$ contain together more than r sets. Similarly, in part **(ii)** of Claim 2.3 the term $\max\{a : \binom{a + g_k(r)}{\leq a} \leq r\}$ ensures that in $B_{g_k(r)+a}$ the levels $0, 1, \ldots, a$ contain together at most r sets.

Claim 2.3 For any $k \geq 1$ and $r \geq 1$ we have
(i) $f_{k+1}(r) \leq f_k(2r-1) + \min\{a : \binom{a + f_k(2r-1)}{\leq a} > r\}$,

(ii) $g_{k+1}(r) \geq g_k(r) + \max\{a : \binom{a+g_k(r)}{\leq a} \leq r\} + 1$.

Proof of the claim Let $N = f_k(2r-1) + \min\{a : \binom{a+f_k(2r-1)}{\leq a} > r\}$ and let us consider a coloring $\phi : B_N \to [k+1]$. Without loss of generality we may assume $\phi(\emptyset) = k+1$ for the empty set $\emptyset$. Assume first that there exists a set $F \in B_N$ with $|F| \leq \min\{a : \binom{a+f_k(2r-1)}{\leq a} > r\}$ and $\phi(F) \neq k+1$. Then consider the k-coloring $\phi' : B_{F,[N]} \to [k]$ defined by $\phi'(G) = \phi(G)$, if $\phi(G) \in [k]$ and $\phi'(G) = \phi(F)$ otherwise. As $N - |F| \geq f_k(2r-1)$, ϕ' admits a monochromatic weak copy C of V_{2r-1} in $B_{F,[N]}$. If its color is not $\phi(F)$, then its elements have the same color in ϕ, thus C is a monochromatic weak copy of V_{2r-1} with respect to ϕ. If the color of C is $\phi(F)$ and C contains at least r sets that were colored $k+1$ in the coloring ϕ, then together with the empty set, they form a monochromatic weak copy of V_r with respect to ϕ. Otherwise C contains at least $r+1$ sets, including F, that were colored $\phi(F)$. Then F together with r other such sets form a monochromatic weak copy of V_r with respect to ϕ.

Assume next that all sets of size at most $\min\{a : \binom{a+f_k(2r-1)}{\leq a} > r\}$ are colored $k+1$. Then the empty set and r other such sets form a monochromatic weak copy of V_r. This proves **(i)**.

To prove **(ii)**, let us consider a consecutive level k-coloring $\psi : B_{g_k(r)-1} \to [k]$ defined by the positive integers $i_1, i_2, \ldots, i_k$ such that ψ does not admit a monochromatic weak copy of V_r. We "add $\max\{a : \binom{a+g_k(r)}{\leq a} \leq r\} + 1$ extra levels", i.e., we let $j_1 := \max\{a : \binom{a+g_k(r)}{\leq a} \leq r\} + 1$, and $j_{h+1} := i_h$ for all $1 \leq h \leq k$ and set $N' := \left(\sum_{h=1}^{k+1} j_h\right) - 1$. We claim that the corresponding consecutive level $(k+1)$-coloring ψ' does not admit a monochromatic weak copy of V_r, which proves **(ii)**. Indeed, by definition the union of the first j_1 layers does not contain $r+1$ sets, so no monochromatic V_r exists in this color. To see the V_r-free property of the other color classes, observe that for any set F of size j_1, the cube $B_{F,[N']}$ has dimension $g_k(r) - 1$, and the consecutive level k-coloring that we obtain by restricting ψ' to $B_{F,[N']}$ is isomorphic to ψ. If G is the set corresponding to the bottom element of a copy C of V_r, then for a j_1-subset F of G, the copy C belongs to $B_{F,[N']}$, so it cannot be monochromatic. □

To prove the theorem we proceed by induction on k. If one can use only one color, then all colorings are consecutive level 1-colorings and B_N does not admit a monochromatic V_r if and only if $2^N \leq r$, so $g_1(r) = f_1(r) = \lfloor \log r \rfloor + 1$ and $c_1 = 1$.

Assume now that the statement of the theorem is proved for some $k \geq 1$ and let us fix $\varepsilon > 0$. Observe that using Claim 2.3 **(ii)** and the inductive hypothesis we obtain that for r large enough we have

$$g_{k+1}(r) \geq g_k(r) + \max\left\{a : \binom{a + g_k(r)}{\leq a} \leq r\right\} + 1, \tag{3}$$

and $(c_k - \varepsilon)\log r \leq g_k(r) \leq (c_k + \varepsilon)\log r$. We claim that if d_k is the constant that satisfies $(d_k + c_k)h\left(\frac{d_k}{d_k+c_k}\right) = 1$, then the maximum a in Inequality Eq. 3 is at least $(d_k - \varepsilon)\log r$. Indeed, there exist positive constants c_0 and δ such that

$$\binom{(d_k-\varepsilon)\log r + g_k(r)}{\leq (d_k-\varepsilon)\log r} \leq \binom{(d_k+c_k)\log r}{\leq (d_k-\varepsilon)\log r} \leq c_0\binom{(d_k+c_k)\log r}{(d_k-\varepsilon)\log r}$$
$$\leq c_0 2^{h\left(\frac{d_k-\varepsilon}{d_k+c_k}\right)(d_k+c_k)\log r} = c_0 r^{h\left(\frac{d_k-\varepsilon}{d_k+c_k}\right)(d_k+c_k)} \leq c_0 r^{1-\delta} < r$$

holds, where for the second inequality we used $d_k < c_k$ and Inequality Eq. 2 and for the penultimate inequality we used that the entropy function is strictly increasing in (0, 1/2). Therefore, we have $g_{k+1}(r) \geq (c_k + d_k - 2\varepsilon)\log r$.

On the other hand, according to Claim 2.3 **(i)**, we have

$$f_{k+1}(r) \leq f_k(2r-1) + \min\left\{a : \binom{a + f_k(2r-1)}{\leq a} > r\right\}. \tag{4}$$

By the inductive hypothesis, for sufficiently large r we have

$$(c_k - \varepsilon)\log r \leq f_k(r) \leq f_k(2r-1) \leq (c_k + \varepsilon)\log(2r-1) \leq (c_k + 2\varepsilon)\log r.$$

We claim that the minimum a in Inequality Eq. 4 is at most $(d_k + \varepsilon)\log r$. Indeed, for some positive δ' and large enough r we have

$$\binom{(d_k+\varepsilon)\log r + f_k(2r-1)}{\leq (d_k+\varepsilon)\log r} \geq \binom{(d_k+c_k)\log r}{(d_k+\varepsilon)\log r} \geq \frac{1}{\sqrt{\log r}} 2^{h\left(\frac{d_k+\varepsilon}{d_k+c_k}\right)(d_k+c_k)\log r}$$
$$= \frac{1}{\sqrt{\log r}} r^{h\left(\frac{d_k+\varepsilon}{d_k+c_k}\right)(d_k+c_k)} \geq \frac{r^{1+\delta'}}{\sqrt{\log r}} > r.$$

Therefore, we have $f_{k+1}(r) \leq (c_k + d_k + 3\varepsilon)\log r$ and consequently

$$(c_k + d_k - 2\varepsilon)\log r \leq g_{k+1}(r) \leq f_{k+1}(r) \leq (c_k + d_k + 3\varepsilon)\log r,$$

showing $c_{k+1} = c_k + d_k$. Plugging back to the defining equation $(d_k + c_k)h\left(\frac{d_k}{d_k+c_k}\right) = 1$ we obtain $c_{k+1}h\left(\frac{c_{k+1}-c_k}{c_{k+1}}\right) = 1$ as claimed. □

Note that Cox and Steele [3] obtained general but not tight upper bounds on the Ramsey number $R(V_{r_1}, \ldots, V_{r_s}, \Lambda_{r_{s+1}}, \ldots, \Lambda_{r_t})$. Theorem 2.2 is an improvement on their result in case all target posets are the same.

3 Strong Copies

The lower bounds in most of our theorems are obtained via trivial colorings where sets of the same size receive the same color. We introduce the following parameters: let $m(P) = \max\{m : B_m$ does not contain a weak copy of $P\}$ and $m^*(P) = \max\{m : B_m$ does not contain a strong copy of $P\}$. We say that $Q \subset B_n$ is *thin* if Q contains at most one set from each level. Also, let $r^*(P) = \max\{r : B_r$ does not contain a thin, strong copy of $P\}$. Note that the corresponding weak parameter $r(P) = \max\{r : B_r$ does not contain a thin, weak copy of $P\}$ trivially equals $|P| - 2$ as $B_{|P|-1}$ contains a chain of length $|P|$ and thus a weak copy of P. Also, it is not hard to see that $r^*(P) \leq 2|P| - 2$. This is certainly true for all one and two-element posets. Then we proceed by induction on $|P|$. Fix a maximal element $p \in P$. By induction, there exists a thin, strong copy of $P \setminus \{p\}$ in B_N with $N = 2|P| - 4$. Denote the embeddig by ϕ. Set $A := \cup_{p' < p}\phi(p')$ and partition $P \setminus \{p\}$ into $R_1 = \{p' : |\phi(p')| \leq |A|\}$ and $R_2 = \{p' : |\phi(p')| > |A|\}$. Then it is easy to check that the embedding ϕ' defined as $\phi'(p') = \phi(p')$ if $p' \in R_1$, $\phi'(p') = \phi(p') \cup \{N+2\}$ if $p' \in R_2$ and $\phi'(p) = A \cup \{N+1\}$ creates a thin, strong copy of P into B_{N+2}.

In the next proposition, we prove some lower bounds using non-trivial colorings. A poset P is said to be *connected* if for any pair $p, q \in P$ there exists a sequence $r_1, r_2, \ldots, r_k$ such that $r_1 = p$, $r_k = q$ and r_i, r_{i+1} are comparable for any $i = 1, 2, \ldots, k-1$.

Proposition 3.1 *If P is a connected poset with $|P| \geq 2$ and Q is an arbitrary poset, then we have*
(i) $RR(P, Q) > m(P) + |Q| - 2$,
(ii) $RR^*(P, Q) > m^*(P) + |Q| - 2$,
(iii) $RR^*(P, Q) > r^*(Q)$.

Proof Set $N = m(P) + |Q| - 2$, $N^* = m^*(P) + |Q| - 2$ and $R = [|Q| - 2]$. Consider the colorings $\phi\colon B_N \to \{1, \ldots, |Q| - 1\}$ and $\phi^*\colon B_{N^*} \to \{1, \ldots, |Q| - 1\}$ defined by $\phi(F) = |F \cap R| + 1$ and $\phi^*(G) = |G \cap R| + 1$. Observe that ϕ and ϕ^* do not admit a rainbow copy of Q as only $|Q| - 1$ colors are used.

By definition of $m(P)$, for any set $T \subseteq R$ the family $\mathcal{F}_T = \{F \subseteq [N] : F \cap R = T\}$ cannot contain a weak copy of P. Thus a monochromatic weak copy of P (admitted by ϕ) must contain two sets F, F' with $F \in \mathcal{F}_T$ and $F' \in \mathcal{F}_{T'}$ such that $|T| = |T'|$ and $T \neq T'$. As P is connected, we can choose F, F' to be comparable. However, since each $F \in \mathcal{F}_T$ is incomparable to each $F' \in \mathcal{F}_{T'}$ as T is incomparable to T', this is a contradiction. So the coloring ϕ does not admit a monochromatic weak copy of P. This proves **(i)**, and one can prove **(ii)** in a similar way.

To see **(iii)** let us consider the trivial coloring $\phi : B_{r^*(Q)} \to \{1, \ldots, r^*(Q) + 1\}$ defined by $\phi(F) = |F| + 1$. As P is connected with $|P| \geq 2$, ϕ does not admit a monochromatic copy of P and by definition of $r^*(Q)$, ϕ does not admit a rainbow strong copy of Q. □

Proposition 3.2 *If $n \geq 4$, then $r^*(A_n) = n + 1$ holds.*

Proof Let $\mathcal{F} \subset B_n$ be a thin antichain. Then we claim $|\mathcal{F}| \leq n - 2$ holds, which shows $r^*(A_n) \geq n + 1$. Indeed, if $\emptyset \in \mathcal{F}$ or $[n] \in \mathcal{F}$, then $\mathcal{F} = \{\emptyset\}$ or $\mathcal{F} = \{[n]\}$. Also, if both a 1-element and an $(n - 1)$-element sets are in $\mathcal{F}$, they have to be complements, and then no other sets can be in $\mathcal{F}$.

For the upper bound we prove the stronger statement that B_n contains a thin antichain of size $n - 2$ with set sizes $1, 2, \ldots, n - 2$. We proceed by induction on n. The statement is trivial for $n = 4$ and $n = 5$. Assume the statement holds for some $n \geq 4$, and we prove it for $n + 2$. Hence we can find a thin antichain $\mathcal{F}$ in B_n that has cardinality $n - 2$ with set sizes $1, 2, \ldots, n - 2$. Then let $\mathcal{F}' = \{F \cup \{n + 1\} : F \in \mathcal{F}\} \cup \{[n], \{n + 2\}\}$. It is easy to see that $\mathcal{F}' \subset B_{n+2}$ is a thin antichain of size n with set sizes $1, 2, \ldots, n$. □

Propositions 3.1 and 3.2 together yield $RR^*(C_2, A_k) \geq k + 2$, which is larger than both $e^*(C_2)(|A_k| - 1) + f(A_k) = k + 1$ and $R^*_{k-1}(C_2) = k - 1$, showing that C_2 does not possess the property of Question 1.6 and that there exists a pair of posets for which Proposition 1.3 **(ii)** holds with a strict inequality.

Definition 3.3 We say that the families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_l$ are *mutually comparable* if for any $F_i \in \mathcal{F}_i$ and $F_j \in \mathcal{F}_j$ with $1 \leq i < j \leq l$ we have $F_i \subseteq F_j$ or $F_j \subseteq F_i$, and they are *mutually incomparable* if for any $F_i \in \mathcal{F}_i$ and $F_j \in \mathcal{F}_j$ with $1 \leq i < j \leq l$ we have $F_i \not\subseteq F_j$ and $F_j \not\subseteq F_i$.

Proof of Theorem 1.7 Set $N = \lfloor \lambda^*_{max}(P)(k - 1) \rfloor + m_k$ and consider a coloring $\phi\colon B_N \to \mathbb{Z}^+$. Observe that if ϕ does not admit a monochromatic induced copy of P, then for any set $S \subseteq [m_k]$, ϕ must admit at least k colors on the family $\mathcal{Q}_S = \{S \cup T : T \subseteq [N] \backslash [m_k]\}$. Indeed, if there are at most $k - 1$ colors on some $\mathcal{Q}_S$, then consider the corresponding coloring ϕ' of B_{N-m_k} such that $\phi'(\{i_1, i_2, \ldots, i_\ell\}) = \phi(S \cup \{i_1 + m_k, i_2 + m_k, \ldots, i_\ell + m_k\})$ for every set $\{i_1, i_2, \ldots, i_\ell\} \in B_{[N-m_k]}$. Then ϕ' is a $(k - 1)$-coloring of B_{N-m_k}, and one of the color classes has Lubell-mass strictly larger than $\lambda^*_{max}(P)$. So ϕ' admits a monochromatic induced copy of P in B_{N-m_k}. This implies that ϕ admits a monochromatic induced copy of P in $\mathcal{Q}_S$.

By the definition of m_k, we can pick k subsets $S_1, S_2, \dots, S_k$ of $[m_k]$ of size $\lfloor m_k/2 \rfloor$. As the S_i's form an antichain, the families $\mathcal{Q}_{S_1}, \mathcal{Q}_{S_2}, \dots, \mathcal{Q}_{S_k}$ are mutually incomparable. By the above paragraph, on each of these families ϕ admits at least k colors otherwise we find a monochromatic induced copy of P. But then we can pick a rainbow antichain from the $\mathcal{Q}_{S_i}$'s greedily: a set F_1 from $\mathcal{Q}_{S_1}$, then F_2 from $\mathcal{Q}_{S_2}$ and so on with $\phi(F_i) \neq \phi(F_j)$ for all $i < j$. This completes the proof of the first part of Theorem 1.7.

Now we prove the second part. For any P other than C_1 or C_2, $\mathcal{F} = \{\emptyset, [n]\} \subset B_n$ is induced P-free for all $n \geq 2$. Hence $\lambda^*_{max}(P) = \sup \lambda^*_n(P) \geq 2$. Let $N = \lfloor 2\lambda^*_{max}(P) \rfloor + 2$. For any coloring ψ of B^-_N, we show that it admits either a monochromatic induced copy of P or a rainbow copy of A_3. If $\|\psi\| \leq 2$, then $\lambda^*_N(B^-_N) = N - 1$ hence one of the color classes has Lubell-mass strictly larger than $\lambda^*_{max}(P)$, so by the definition of λ^*_{max}, ψ admits a monochromatic induced copy of P.

Therefore, we can assume that $\|\psi\| \geq 3$. Let $\mathcal{Q}_i = \{\{i\} \cup T : T \subseteq [N] \setminus [2]\}$ for $i = 1, 2$. Note that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are mutually incomparable. By the same reasoning as in the previous case, if ψ admits only 2 colors on some $\mathcal{Q}_i$, then we can find a corresponding 2-coloring ψ' of B_{N-2} and a monochromatic copy of P in B_{N-2} with respect to ψ'. As before, this implies that there is a monochromatic copy of P in $\mathcal{Q}_i$ with respect to ψ. Hence we consider the case that ψ admits at least three colors on each $\mathcal{Q}_i$. If there are two sets $F_1, F_2 \in \mathcal{Q}_1$ of the same size with distinct colors, then a set of third color in $\mathcal{Q}_2$ together with F_1 and F_2 form a rainbow A_3. So we may assume that all subsets of the same size in $\mathcal{Q}_1$ have the same color. Now if all sets in $\mathcal{Q}_1 \setminus \{\{1\}, ([N] \setminus [2]) \cup \{1\}\}$ are of the same color, then the corresponding coloring ψ' admits only one color on B^-_{N-2}. Since $\lambda^*_{max}(P) \geq 2$, we have $\lambda^*_{N-2}(B^-_{N-2}) = N - 3 = \lfloor 2\lambda^*_{max}(P) \rfloor - 1 > \lambda^*_{max}(P)$. Thus, ψ' admits a monochromatic P in B_{N-2} and then ψ admits a monochromatic P in $\mathcal{Q}_1$ as well. If there are at least two colors on $\mathcal{Q}_1 \setminus \{\{1\}, ([N] \setminus [2]) \cup \{1\}\}$ and sets of the same size have the same color, then we can easily find two incomparable sets from two levels of distinct colors. The two sets together with a set of third color in $\mathcal{Q}_2$ form a rainbow A_3. This completes the proof. □

Funding Open access funding provided by ELKH Alfréd Rényi Institute of Mathematics. Research supported by the National Research, Development and Innovation Office - NKFIH under the grants K 116769, K 132696, KH 130371, SNN 129364, FK 132060, and KKP-133819, by the János Bolyai Research Fellowship of the Hungarian Academy of Sciences and the Taiwanese-Hungarian Mobility Program of the Hungarian Academy of Sciences, by Ministry of Science and Technology Project-based Personnel Exchange Program 107 -2911-I-005 -505 and by the Ministry of Education and Science of the Russian Federation in the framework of MegaGrant no 075-15-2019-1926 and by the EPSRC grant no. EP/S00100X/1 (A. Methuku). Research of Vizer was supported by the New National Excellence Program under the grant number ÚNKP-20-5-BME-45.

Availability of data and material Our manuscript contains no associated data.

Code Availability Our manuscript contains no associated code.

Declarations

Conflict of Interests The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Axenovich, M., Walzer, S.: Boolean lattices: Ramsey properties and embeddings. Order **34**(2), 287–298 (2017)
2. Chen, H.-B., Cheng, Y.-J., Li, W.-T., Liu, C.-A.: The Boolean rainbow Ramsey number of antichains, Boolean Posets, and chains. Electron. J. Comb. **27**, # P34 (2020)
3. Cox, C., Stolee, D.: Ramsey numbers for partially-ordered sets. Order **35**, 557–579 (2018)
4. Fujita, S., Magnant, C., Ozeki, K.: Rainbow generalizations of Ramsey theory-a dynamic survey. Theory Appl Graphs (2014)
5. Griggs, J.R., Lu, L.: On families of subsets with a forbidden subposet. Comb. Probab. Comput. **18**(5), 731–748 (2009)
6. Griggs, J.R., Li, W.-T., Lu, L.: Diamond-free families. J. Comb. Theory Ser. A **119**(2), 310–322 (2012)
7. Griggs, J.R., Li, W.-T.: The partition method for poset-free families. J. Comb. Optim. **25**(4), 587–596 (2013)
8. Griggs, J.R., Li, W.-T.: Poset-free families and Lubell-boundedness. J. Comb. Theory Ser. A **134**, 166–157 (2015)
9. Grósz, D., Methuku, A., Tompkins, C.: An upper bound on the size of diamond-free families of sets. J. Comb. Theory Ser. A **156**, 164–194 (2018)
10. Johnston, T., Lu, L., Milans, K.G.: Boolean algebras and Lubell functions. J. Comb. Theory Ser. A **136**, 174–183 (2015)
11. Kierstead, H.A., Trotter, W.T.: A Ramsey theoretic problem for finite ordered sets. Discret. Math. **63**(2-3), 217–223 (1987)
12. Lu, L., Thompson, J.C.: Poset ramsey number for Boolean lattices. arXiv:1909.08680 (2021)
13. Méroueh, A.: A LYM inequality for induced posets. J. Comb. Theory Ser. A **155**, 398–417 (2018)
14. Patkós, B.: Induced and non-induced forbidden subposet problems. Electron. J. Comb. **22**(1), P1.30 (2015)
15. Trotter, W.T.: Ramsey theory and partially ordered sets. In: Contemporary Trends in Discrete Mathematics, Graham, R.L., et al (eds.) DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 49, pp. 337–347 (1999)