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A Convex Analysis View of the Barrier Problem

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Besides the simplex algorithm, linear programs can also be solved via interior point methods. The theoretical background of such algorithms is the classical log-barrier problem. The aim of this note is to study and generalize the barrier problem using the standard tools of Convex Analysis.

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1. Motivations

The simplex method developed by Dantzig [5, 6] approaches optimum of a given linear program on the boundary of the feasible set. However, there are alternative algorithms which find optimum through interior points. Such algorithm appears first in the paper of Fiacco and McCormick [7], while the systematic study of these methods can be traced back to Karmarkar [9]. Later Megiddo [12] proved that Karmarkar's algorithm relying on projective geometry is equivalent to an interior point method, which was named *central path method* by Huard [8]. An excellent summary on the nonlinear geometry of the central path method is the survey by Bayer and Lagarias [1, 2]. For further theoretical details and implementation issues, we refer to Roos, Terlaky, and Vial [14].

It is worth mentioning that the simplex method may require exponential many iterations as Klee and Minty [11] demonstrated. In contrary, Khachiyan [10] pointed out that any linear program can be solved in polynomial time with interior point methods. The first easily implementable such method is due to Karmarkar [9].

The idea behind the central path method briefly is the following. The original objective function is replaced by a one-parameter family of functions, while the feasible set essentially remains unchanged. If the perturbed objectives have a unique optimum for each parameter, then the optimal solutions form a parametrized curve called the *central path*. As the parameter shrinks to zero, the perturbed objective approaches to the original one. Thus the central path is expected to terminate in the optimal solution of the original linear program.

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In this note, we are going to study the *log-barrier problem*, the theoretical background of some interior point methods. The book of Borwein and Lewis [4] contains it as an exercise. The solution is based on the Fenchel conjugation. A detailed and alternative approach is presented by Vanderbei [15]. Unfortunately, the reasoning is incorrect at a point: The Heine-Borel Theorem is applied in the relative topology induced by an open subset of $\mathbb{R}^n$. (However, this fact does not affect the transparency and excellence of Vanderbei's monograph.) Our approach combines some ideas of Wright [16] and Vanderbei [15] with the standard methods of Convex Analysis. The most important tool we have to develop is the recession cone of *open* convex sets. This concept is well-known for *closed* convex sets (see Rockafellar [13]). The main result, an extension of the barrier problem provides the resolvability of concave programs in terms of two limit conditions. The most important advantage of our approach is that the second order test to verify optimality can completely be avoided by supposing only continuous differentiability on the objective.

2. Basic notions and notations

First we recall some basic facts from Linear Programming and Convex Analysis which we will use throughout the discussion. Our basic and permanent references are Vanderbei [15] and Rockafellar [13]. Let X and Y be nonempty sets, and let $D \subseteq X$ further $b \in Y$. Consider the constrained optimization problem

$$f(x) \longmapsto \max, \quad g(x) = b,$$

where the *objective function* $f\colon D \to \mathbb{R}$ and the *constraint* $g\colon X \to Y$ are given. We say that $x \in X$ is a *feasible solution* if $x \in D$ and $g(x) = b$. An element $x^* \in X$ is called a feasible *optimal* solution, if it is feasible and $f(x) \le f(x^*)$ whenever $x \in X$ is a feasible solution.

In particular, let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, furthermore $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ where $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ stands for the vector space of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then the pair of special constrained optimization problems

$$\text{(Primal)}\quad \begin{aligned} c^T x &\longmapsto \max \\ Ax &\le b, \\ x &\ge 0; \end{aligned} \qquad\qquad \text{(Dual)}\quad \begin{aligned} y^T b &\longmapsto \min \\ A^T y &\ge c, \\ y &\ge 0 \end{aligned}$$

is termed the primal-dual pair in standard form. Here (and in any further analogue situations) inequalities are meant coordinatewise. The notion of feasible (optimal) solution remains the same as previously was mentioned. To achieve equality form in the constraints, we can introduce the *slack variables* $\omega \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$:

$$\begin{aligned} c^T x &\longmapsto \max \\ Ax + \omega &= b, \\ x, \omega &\ge 0; \end{aligned} \qquad\qquad \begin{aligned} y^T b &\longmapsto \min \\ A^T y - z &= c, \\ y, z &\ge 0. \end{aligned}$$

It is well-known that the optimality of feasible primal and dual solutions can easily be checked using the Complementary Slackness Theorem:

Theorem 2.1. *Let $x = (x_1, \dots, x_n)^T$ and $y = (y_1, \dots, y_m)^T$ be feasible solutions of the primal and the dual problems, and let $\omega = (\omega_1, \dots, \omega_m)^T$ and $z = (z_1, \dots, z_n)^T$ be the attached slack variables. Then x and y are optimal for their respective problems if and only if*

$$x_1 z_1 = \dots = x_n z_n = 0 = \omega_1 y_1 = \dots = \omega_m y_m.$$

Let X be a vector space, and let $D \subseteq X$ be a convex set. We say that $f\colon D \to \mathbb{R}$ is a *concave function* if, for all $\lambda \in [0,1]$ and for all $x, y \in D$,

$$f\big(\lambda x + (1-\lambda)y\big) \geq \lambda f(x) + (1-\lambda) f(y).$$

If this inequality is strict for all $\lambda \in]0,1[$ and for all $x \neq y$, then we speak about *strictly* concave function. As it can easily be checked, the set of concave functions on D forms a convex cone with respect to the pointwise operations. We quote here the result of Bernstein and Doetch [3]: *If a concave function acting on an open, convex subset of a normed space is locally bounded from below at a point, then it is locally Lipschitz.* In the case of finite dimensional vector spaces this boundedness property holds automatically. Thus Lipschitz property (in particular: continuity) is fulfilled without any further restrictions: *Any concave function acting on an open, convex subset of a finite dimensional normed space is continuous.*

3. Convex optimization problems

We are going to study the central path in a geometrical point of view. To do this, we have to introduce first the path in an abstract setting and show its properties in two lemmas.

Let T and B be nonempty sets. We say that $F\colon T \times B \to \mathbb{R}$ fulfills the *parametrized global maximum property* if, for all $t \in T$ there exists a unique element $h(t) \in B$ such that

$$F\big(t, h(t)\big) > F(t, x)$$

holds for all $x \neq h(t)$. In this case, $h\colon T \to B$ is a function, indeed. The function h and the map F are called the *path* and its *generator*, respectively.

Now assume that T is a metric space and B is a nonempty subset of a Euclidean space E, and consider the next limit conditions for $F\colon T \times B \to \mathbb{R}$: For all $t_0 \in T$ and for all $x_0 \in \overline{B} \setminus B$,

$$\lim_{\substack{t \to t_0 \\ x \to x_0}} F(t, x) = -\infty. \tag{1}$$

For all $t_0 \in T$,

$$\lim_{\substack{t \to t_0 \\ \|x\| \to +\infty}} F(t, x) = -\infty. \tag{2}$$

These requirements on the path generator ensure that the induced central path is continuous:

Lemma 3.1. *Let T be a metric space, and B be a nonempty subset of a Euclidean space E. If $F\colon T \times B \to \mathbb{R}$ is a continuous path generator which has the limit conditions* (1) *and* (2)*, then the path $h\colon T \to B$ is continuous.*

Proof. Assume to the contrary that there exist $t_0 \in T$ and a sequence (t_k) in T such that $t_k \to t_0$ whereas $h(t_k) \nrightarrow h(t_0)$.

Firstly, consider the case when $h(t_k)$ is bounded. Without loss of generality we may assume that $h(t_k) \to x_0 \in \overline{B}$ by the Bolzano-Weierstrass Theorem. If $x_0 \in B$, then the continuity and the maximum property of F imply

$$F(t_0, x_0) = \lim_{k\to\infty} F\big(t_k, h(t_k)\big) \geq \lim_{k\to\infty} F\big(t_k, h(t_0)\big) = F\big(t_0, h(t_0)\big) > F(t_0, x_0),$$

which is a contradiction. If $x_0 \notin B$, then the properties of F and the limit condition (1) result in the contradiction

$$F\big(t_0, h(t_0)\big) = \lim_{k\to\infty} F\big(t_k, h(t_0)\big) \leq \lim_{k\to\infty} F\big(t_k, h(t_k)\big) = -\infty.$$

To complete the proof, we have to discuss the case when $h(t_k)$ is unbounded. This can be done similarly to the previous case, using the limit condition (2). □

For special generators, we have monotonicity along the central path:

Lemma 3.2. *Let $I \subseteq \mathbb{R}$ be an interval, B be a nonempty subset of a Euclidean space E, $c \in E$, and $g\colon E \to \mathbb{R}$. If the function $F\colon I \times B \to \mathbb{R}$ defined by*

$$F(t, x) = c^T x + tg(x)$$

satisfies the parametrized global maximum property, then $g \circ h$ is monotone increasing.

Proof. By the parametrized global maximum property,

$$\begin{aligned} c^T h(t) + tg\big(h(t)\big) &\geq c^T h(s) + tg\big(h(s)\big) \\ &= c^T h(s) + sg\big(h(s)\big) + (t-s)g\big(h(s)\big) \\ &\geq c^T h(t) + sg\big(h(t)\big) + (t-s)g\big(h(s)\big). \end{aligned}$$

Thus $(t-s)g\big(h(t)\big) \geq (t-s)g\big(h(s)\big)$ follows, and proof is completed. □

According to [13, p. 61], the recession cone $\operatorname{rec}(D)$ of a nonempty convex subset D of a vector space X is defined by the set of $v \in X$ such that $x + \alpha v \in D$ for every $x \in D$ and every real $\alpha \geq 0$. The recession cone of a *closed* convex set in a Euclidean space is trivial if and only if the set is bounded. As the next lemma shows, the same statements remain true if D is *open* and convex. The proof may be traced back to the facts that for open sets $\operatorname{rec}(D) = \operatorname{rec}(\overline{D})$ holds by [13, Corollary 8.3.1] and for the property of closed convex sets by [13, Theorem 8.4]. For the sake of completeness, we provide here an independent and direct approach. We note that using open sets instead of closed ones will be essential later: The objective's domain of the log-barrier problem requires it.

Lemma 3.3. *If D is a nonempty, open, convex subset of a Euclidean space X, and $v \in X$, then the next statements are equivalent:*

(i) *There exists $x \in D$, such that $x + \alpha v \in D$ for all $\alpha \geq 0$;*

(ii) *For all $x \in D$ and for all $\alpha \geq 0$, we have that $x + \alpha v \in D$.*

Moreover, D is bounded if and only if $\operatorname{rec}(D)$ *is trivial.*

Proof. Clearly, it suffices to prove the implication (i)⇒(ii). Assume that $x + \beta v \in D$ holds for some $x \in D$ and for all $\beta > 0$. Let $x_0 \in D$ be arbitrary (see Figure 1).

We may assume that x_0 does not lie on the line whose direction is v and passes trough x. By openness, x_0 belongs to D with some neighborhood U. Let $y \in U$ such that $x_0 \in [x, y]$. Consider a point $w := x_0 + \alpha v$, where $\alpha \geq 0$. Then the parallel lines (with direction v) determined by x and x_0, furthermore the point y are on the same plane. By the Axiom of Parallels, y determines a line which intersects the half-line $\{x + \beta v \mid \beta > 0\}$ at a point z. Then, $z \in D$ by assumption (i), and $w \in [y, z]$. That is, w can be represented as the convex combination of two elements of D. Since D is convex, $w \in D$ follows.

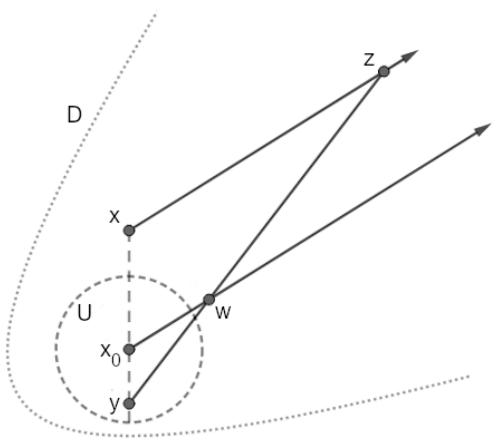

Figure 1

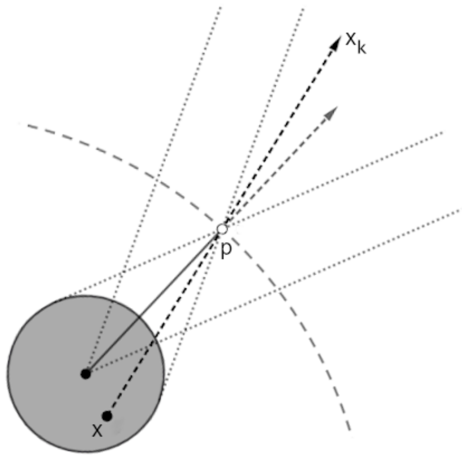

Figure 2

The necessity of the second statement is clear. Conversely, assume to the contrary that $\operatorname{rec}(D)$ is trivial while D is unbounded. Then, D contains an unbounded sequence (x_k) (see Figure 2). By (ii) and by the openness of D, we may assume that the closed unit ball with center at the origin is contained in D. Similarly, we may also assume that (x_k) does not contain the zero vector. Define the sequence (v_k) by

$$v_k := \frac{x_k}{\|x_k\|}.$$

This sequence is bounded since its members are unit vectors. The Bolzano-Weierstrass Theorem guarantees that (v_k) has a limit point v; moreover, we may assume that $v_k \to v$ as $k \to +\infty$. By the indirect assumption, $v \notin \operatorname{rec}(D)$. Thus,

$$\alpha_0 := \sup\{\alpha \geq 0 \mid \alpha v \in D\} < +\infty.$$

Furthermore, the convexity of D ensures that $\alpha v \in D$ if $\alpha < \alpha_0$ and $\alpha v \notin D$ if $\alpha > \alpha_0$. In particular, these properties show that $p := \alpha_0 v \in \partial D$. Therefore, $p \notin D$ since D is open. Consider now the set defined by

$$C := \{\alpha(p - x) \mid \alpha \geq 0, \|x\| = 1\}.$$

The convexity of closed unit ball provides that C is a convex cone. Clearly, $v \in C$, and hence $v_k \in C$ for sufficiently large indices. Thus, $x_k \in p + C$ if k is large enough. However, in such cases, p can be obtained as a convex combination of x_k and a unit vector. Both of these elements belong to D, therefore $p \in D$ by convexity. This contradiction completes the proof. □

The last auxiliary tool is an optimum property of constrained problems with strictly concave objective and linear conditions. Since the statement is well-known, we omit its proof.

Lemma 3.4. *Assume that X and Y are Euclidean spaces, $D \subseteq X$ is a nonempty, open, convex set, $A \in \mathscr{L}(X,Y)$, and $b \in Y$. If $f\colon D \to \mathbb{R}$ is a strictly concave function, and the concave program*

$$f(x) \longmapsto \max, \quad Ax = b$$

has a feasible local maximum, then the local maximum is a global one, and the maximizer is unique.

4. The main results

The main results are presented in two theorems. The first one gives a sufficient condition for a concave program to have optimal feasible solution. The limit properties involved guarantee that the "large" values of the objective are allocated in a compact subset of the domain. In the proof the recession cone plays a key role.

Theorem 4.1. *Assume that X and Y are Euclidean spaces, $D \subseteq X$ is a nonempty, open, convex set, $A \in \mathscr{L}(X,Y)$, and $b \in Y$. If a concave function $f\colon D \to \mathbb{R}$ satisfies the limit conditions*

$$\lim_{x\to x_0} f(x) = -\infty \quad \textit{and} \quad \lim_{\alpha\to+\infty} f(x+\alpha v) = -\infty \tag{3}$$

for all $x_0 \in \partial D$ and for all $v \in \operatorname{rec}(D)$, and the concave program

$$f(x) \longmapsto \max, \quad Ax = b$$

has a feasible solution, then it has a feasible optimal solution, as well.

Proof. Let $c \in \mathbb{R}$ be given, and consider the open sublevel set L_c of the objective function f:

$$L_c(f) = \{x \in D \mid f(x) > c\} = f^{-1}\big(]-\infty, c[\big).$$

We claim that $L_c(f) \subseteq X$ is open, convex and bounded for all $c \in \mathbb{R}$.

The level set is a continuous preimage of an open set, thus it is open in the relative topology of D. However, D itself is open in the Euclidean topology, therefore $L_c(f)$ is open in the Euclidean topology of X. If $x, y \in L_c(f)$ and $\lambda \in [0,1]$, then we have $\lambda x + (1-\lambda)y \in D$, and

$$f\big(\lambda x + (1-\lambda)y\big) \geq \lambda f(x) + (1-\lambda)f(y) \geq \lambda c + (1-\lambda)c = c.$$

Thus $L_c(f)$ is convex. Finally, assume to contrary that $L_c(f)$ is unbounded. Then there exists a recession direction v by Lemma 3.3. Clearly, v is a recession direction for D, as well. Let $x \in L_c(f)$ be arbitrary. By the second limit condition, there exists $\alpha > 0$ such that $f(x+\alpha v) < c$. On the other hand, the property $x + \alpha v \in L_c(f)$ implies $f(x+\alpha v) > c$, which is obviously impossible.

Now we prove that $\overline{L_c(f)} \subseteq D$. To do this, it suffices to verify that $\partial L_c(f) \subseteq D$. Assume that this is not the case and let $x_0 \in \partial L_c(f) \setminus D$. Then $x_0 \in \partial D$ since $\overline{L_c(f)} \subseteq \overline{D}$ holds evidently. Take a sequence (x_k) in $L_c(f)$ such that $x_k \to x_0$.

The inequality $f(x_k) > c$ implies $\lim_{k\to\infty} f(x_k) \geq c > -\infty$, which contradicts to the first limit property.

Closure does not effect convexity and boundedness, therefore $\overline{L_c(f)}$ is a convex, compact set by the Heine-Borel Theorem. The family $\{\overline{L_c(f)} \mid c \in \mathbb{R}\}$ covers D, and we have a feasible solution. Thus $\overline{L_{c_0}(f)}$ intersects the feasible set for some $c_0 \in \mathbb{R}$. The intersection, denoted by K, is compact, being the feasible set closed. By the continuity of f, there exists $x^* \in K$ such that

$$f(x^*) = \max_K f.$$

Now take a feasible solution $x \in D$. If $x \in K$, then $f(x^*) \geq f(x)$ by the choice of x^*. If $x \notin K$, then $x \notin L_{c_0}(f)$, and hence $f(x) \leq c_0$. On the other hand, by continuity again, $f(x^*) \geq c_0$. In other words, x^* is a feasible optimal solution of the concave program. □

The second main result is an extension of the log-barrier problem. Observe that the one-parameter family of the concave problems reduces to a linear program in standard form if we set $t = 0$. The previous results ensure that its central path exists and it is continuous. Moreover, each limit point of its graph represents an optimal solution of the original primal-dual pair. To check optimality, we do not need second order tests. Thus, we may assume that the objective is *continuously* (instead of *twice*) differentiable. This is the reason why Lagrange multipliers and complementary slackness are sufficient for us.

To formulate the statement, we shall need the concept of *standard projections*. Let X be a Euclidean space and consider its standard base. If $1 \leq k \leq \dim(X)$, then the kth standard projection $\pi_k \colon X \to \mathbb{R}$ is the function which assigns the standard coordinate x_k to all $x \in X$.

Theorem 4.2. *Let $D \subseteq \mathbb{R}_+^{n+m}$ be a nonempty, open, convex set, $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, and $b \in \mathbb{R}^m$, furthermore $c \in \mathbb{R}^n$. Assume that $g \colon D \to \mathbb{R}$ is a continuously differentiable strictly concave function such that $\partial_k g$ are nonnegative, $\pi_k \partial_k g$ are bounded, and the function*

$$(x, \omega) \longmapsto c^T x + tg(x, \omega)$$

satisfies the limit conditions (3) *for all $t > 0$. If the original primal-dual pair has a feasible solution in D, then the central path of the family of concave programs*

$$c^T x + tg(x, \omega) \longmapsto \max, \quad Ax + \omega = b$$

is well-defined and continuous. Moreover, each limit point at $t = 0$ of the central path's graph is an optimal solution of the attached primal-dual pair.

Proof. For a fixed parameter $t > 0$, the concave program above has a unique optimal solution by Lemma 3.4 and Theorem 4.1. Thus the central path of the family is well-defined, indeed. Taking into consideration the special form of the objective function, the limit conditions (3) mean that the map

$$F\big(t, (x, \omega)\big) := c^T x + tg(x, \omega)$$

fulfills the conditions (1) and (2). Thus, by Lemma 3.1, the central path is continuous. For the second statement, consider the Lagrange-functional

$$L(x,\omega,y) := c^T x + tg(x,\omega) + y^T(b - Ax - \omega).$$

Then, the unique optimal solution $x = x(t)$ and $\omega = \omega(t)$ of the perturbed problem satisfies the Lagrange system

$$\begin{aligned} \frac{\partial L}{\partial x_j} &= c_j + t\frac{\partial g}{\partial x_j} - \sum_{i=1}^{m} y_i a_{ij} &&= t\frac{\partial g}{\partial x_j} - z_j = 0, \qquad && j = 1,\dots,n \\ \frac{\partial L}{\partial \omega_i} &= &&= t\frac{\partial g}{\partial \omega_i} - y_i = 0, \qquad && i = 1,\dots,m. \end{aligned}$$

Since $\partial_k g$ are nonnegative, $y = y(t)$ and $z = z(t)$ are nonnegative, as well. Thus, y is a feasible solution of the dual problem, and z is its slack variable. Moreover, the last terms of the Lagrange system imply

$$tx_j\frac{\partial g}{\partial x_j} = x_j z_j \qquad \text{and} \qquad t\omega_i\frac{\partial g}{\partial \omega_i} = \omega_i y_i.$$

Let (x^*,ω^*,y^*,z^*) be an arbitrary limit point at $t = 0$ of the primal-dual central path's graph. Since it belongs to $\overline{D}$, it is a feasible solution of the original primal-dual pair. For simplicity, assume that $(x,\omega,y,z)(t) \to (x^*,\omega^*,y^*,z^*)$ as $t \to 0$. Passing the limit $t \to 0$ in the equations above, $0 = x_j^* z_j^*$ and $0 = \omega_i^* y_i^*$ follow for all indices i and j by the boundedness of $\pi_k \partial_k g$. Thus, by the Complementary Slackness Theorem, (x^*,ω^*,y^*,z^*) is a feasible optimal solution for the original primal-dual problem. □

Note also that the generator $F\big(t,(x,\omega)\big) = c^T x + tg(x,\omega)$ fulfills the conditions of Lemma 3.2. Thus the values of the objective increases along the central path as the parameter t approaches to zero.

Finally we revisit the classical log-barrier problem and handle it as a direct corollary of the previous theorem. Consider a linear program in standard form, of which constraints are given in equality form using slack variables. Requiring strict positivity on all the variables, for all parameters $t > 0$ perturb the original objective function in the following way:

$$\begin{gathered} c^T x + t\sum_{j=1}^{n} \log x_j + t\sum_{i=1}^{m} \log \omega_i \longmapsto \max \\ Ax + \omega = b, \quad x,\omega > 0. \end{gathered}$$

This one-parameter family of constrained programs is the classical *logarithmic barrier problem.*

Theorem 4.3. *If the original primal-dual pair has a positive feasible solution, then the central path of the barrier-problem exists and continuous. Moreover, its each limit point at zero is a feasible optimal solution of the original primal-dual pair.*

Proof. We show that the conditions of Theorem 4.2 hold. Let $D \subseteq \mathbb{R}_+^{n+m}$ be the positive orthant. Then, D is a nonempty, open and convex set. Let $\phi_k(x,\omega) = \log x_j$ if $k = j$, and $\phi_k(x,\omega) = \log \omega_i$ if $k = n+i$, and define $g\colon D \to \mathbb{R}$ by

$$g(x,\omega) = t\sum_{k=1}^{n+m} \phi_k(x,\omega).$$

Then, g is continuously differentiable and strictly concave; its partial derivatives are positive and $\pi_k \partial_k g = 1$. The objective function of the barrier-problem can be written as

$$f(x,\omega) = c^T x + tg(x,\omega).$$

Finally we prove that f satisfies the limit conditions (3). Let (x_*,ω_*) be a positive feasible solution of the primal, and (y_*,z_*) be a positive feasible solution of the dual. Using the definition of slack variables,

$$z_*^T x + y_*^T \omega = (A^T y_* - c)^T x + y_*^T (b - Ax) = y_*^T b - c^T x$$

holds for each primal feasible solution (x,ω). Thus,

$$\begin{aligned} f(x,\omega) &= t\sum_{k=1}^{n+m} \phi_k(x,\omega) + c^T x \\ &= t\sum_{k=1}^{n+m} \phi_k(x,\omega) + y_*^T b - z_*^T x - y_*^T \omega. \end{aligned}$$

If $(x_0,\omega_0) \in \partial D$ and $(x,\omega) \to (x_0,\omega_0)$, then there exists an index j such that $x_j \to 0$, or there exists an index i such that $\omega_i \to 0$. Then $\phi_k(x,\omega) \to -\infty$ where k corresponds to j or i, while $c^T x$ has a finite limit. Thus, in this case, $f(x,\omega) \to -\infty$.

If $(x,\omega) \in \operatorname{rec}(D)$, then $x,\omega > 0$. Using the convention $x_{n+i} = \omega_i$ again, the objective function can be represented as the sum of terms

$$t \log \alpha x_k - \alpha x_k + \beta.$$

Here x_k are positive therefore $f(\alpha x, \alpha\omega) \to -\infty$ as $\alpha \to +\infty$. Thus the statement follows directly from Theorem 4.2. □

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