Nonlinear Analysis

GENERALIZED FRACTALS IN SEMIMETRIC SPACES

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ABSTRACT. The aim of the present paper is to extend the classical fractal theory using condensing maps and generalized contractions of semimetric spaces. Our method is independent from the approach of Hutchinson: It is based on the Kuratowski measure of noncompactness and avoids completely the Blaschke Completeness Theorem.

1. INTRODUCTION

The fundamental paper of Hutchinson [14] considers fractals as self-similar objects, gives an existence and uniqueness theorem for them, and enlightens the deep connection between the self-similarity and the Hausdorff-dimension. Adopting the philosophy of Hutchinson, we shall define fractals as special invariant sets. Let $\mathscr{F}$ be a family of self-maps of a nonempty set X. We say that $H \subseteq X$ is an *$\mathscr{F}$-invariant set* if it fulfills the *invariance equation*

$$H = \bigcup \mathscr{F}(H). \tag{1}$$

Assume additionally that (X, d) is a metric space. Then under an *$\mathscr{F}$-fractal* we mean a nonempty, compact, and $\mathscr{F}$-invariant subset of X. Hutchinson's result states that if $\mathscr{F}$ is a finite family of contractions of a complete metric space, then there exists precisely one $\mathscr{F}$-fractal.

The original approach in [14] briefly is the following. If we consider the right-hand side of the invariance equation as a set-valued map, the resolvability of (1) turns into a fixed point problem. By the Blaschke Theorem [5], the family of nonempty and compact subsets is complete with respect to the metric of Pompeiu [22] and Hausdorff [13]. On the other hand, the right-hand side of (1) is a contraction in this metric space. Thus it has a unique fixed point by the Banach Contraction Principle [1]. This unique fixed point is a nonempty, compact, and $\mathscr{F}$-invariant subset, in other words: an $\mathscr{F}$-fractal.

The other impact of Hutchinson's paper [14] is the fractal dimension formula: If a finite family of contractive similarities $\mathscr{F}$ of a complete metric space fulfills the open set condition and the magnitudes of the members are $q_1, \dots, q_n$, then the Hausdorff dimension of the induced fractal is obtained as the unique solution of the exponential equation

$$q_1^r + \dots + q_n^r = 1. \tag{2}$$

Our aim is to extend classical fractal theory in double sense. Firstly, we shall suppose that the embedding space is a *semimetric space*. These spaces are obtained by dropping the third axiom, the triangle inequality from the usual definition of metric spaces due to Fréchet [10]. Secondly,

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we shall use *condensing maps* or *generalized contractions* instead of contractions. Generalized contractions are self-maps $f\colon X \to X$ of the underlying semimetric space satisfying the inequality

$$d\big(f(x), f(y)\big) \leq \varphi\big(d(x,y)\big). \tag{3}$$

The precise details about semimetric spaces and generalized contractions are detailed and discussed in the next section. To achieve our aim, we follow a totally different way then Hutchinson [14]. This alternative approach was initiated in [3] and [4]. The key tools are the Kuratowski noncompactness measure [18], the fixed point theorem of Bessenyei and Páles [2], and the fundamental results of Chrząszcz, Jachymski, and Turoboś on semimetric spaces [8]. These tools guarantee the *existence* of generalized fractals. The Hausdorff–Pompeiu (semi)metric is needed only for the *uniqueness* part.

We shall present a variant of the fractal dimension formula, as well: In the exponential expression (2), the factors of contractions are replaced with the right-upper Dini-derivatives of the comparison functions related to the generalized contractions.

2. Semimetric spaces

Throughout in this paper, $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ stand for the sets of nonnegative and extended nonnegative reals, respectively. The set of positive integers is denoted by $\mathbb{N}$. We define the *composite iterates* of a map $F\colon X \to X$ inductively in the usual way: $F^1 := F$ and $F^{n+1} := F \circ F^n$ whenever $n \in \mathbb{N}$.

Under a *semimetric space* we mean a pair (X,d) where X is a nonempty set, and $d\colon X^2 \to \mathbb{R}_+$ is a positive definite, symmetric function, that is, fulfills the first two axioms of metric spaces by Fréchet [10]. In a semimetric space, the notions of *open ball*, and *Cauchy sequence* or *convergent sequence*, finally *continuity* of functions between semimetric spaces can be introduced similarly as in metric spaces. A semimetric space is *complete* if any Cauchy sequence is convergent. For open balls in a semimetric space (X,d), we use the usual notation

$$U(x_0, r) = \{x \in X \mid d(x_0, x) < r\}.$$

Having these notions, we have essentially two possibilities to introduce topology. The *neighborhood topology* is generated by the *open sets*. A set is open, if each point belongs to the set with an open ball. The *sequential topology* is defined in the following way. The *closure point* of a set is a point which can be obtained as the limit of a sequence whose members belong to the set. A set is *closed* if it agrees with its closure points. In metric spaces, the complement of sequentially closed sets are topologically open. However, in semimetric spaces this property does not hold in general: The neighborhood topology and the sequential topology may differ from each other. Semimetric spaces can have further pathological properties. For example, the neighborhood topology is not necessarily Hausdorff; the limit of a sequence is not necessarily unique; a convergent sequence is not necessarily Cauchy. The paper [8] by Chrząszcz, Jachymski, and Turoboś and the references therein give a nice and detailed overview of this topic.

In spite of these deviations, semimetric spaces have tight relation with metric spaces as the next construction illustrates. Let (X,d) a be semimetric space. We say that $\Phi\colon \overline{\mathbb{R}}_+^2 \to \overline{\mathbb{R}}_+$ is a *triangle function* for d, if Φ is symmetric and monotone increasing in both of its arguments, satisfies $\Phi(0,0) = 0$ and for all $x, y, z \in X$,

$$d(x,y) \leq \Phi\big(d(x,z), d(y,z)\big).$$

Obviously, metric spaces are semimetric spaces with $\Phi(u,v) := u+v$. Ultrametric spaces are also semimetric spaces if we choose $\Phi(u,v) := \max\{u,v\}$. Not claiming completeness, we present here some intensively studied cases that can be interpreted in this framework:

- $\Phi(u,v) = c(u+v)$ induces the *c-relaxed triangle inequality*, where $c \geq 1$;
- $\Phi(u,v) = c\max\{u,v\}$ induces the *c-inframetric inequality*, where $c \geq 1$;
- $\Phi(u,v) = (u^p + v^p)^{1/p}$ induces the *pth-order triangle inequality*, where $p > 0$.

It turns out that each semimetric space can be equipped with a triangle function which is the smallest one among triangle functions with respect to the pointwise ordering; for details, see [2]. This optimal triangle function is called the *basic triangle function*. Having the basic triangle function, we can introduce two further important properties. A semimetric space is

- *normal* if its basic triangle function is real-valued;
- *regular* if its basic triangle function is continuous at the origin.

Clearly, the examples above are both normal and regular semimetric spaces. Throughout this paper, we will focus on regular semimetrics. A relevant study of these spaces is the paper [8]. One of its main results characterizes regular semimetric spaces in terms of uniform equivalence. We say that the semimetrics d and ρ on X are *uniformly equivalent*, if the identity map $\mathrm{id}\colon (X,d) \to (X,\rho)$ is uniformly bi-continuous. That is,

(i) for all $\varepsilon > 0$ there exits $\delta > 0$ such that $\rho(x,y) < \varepsilon$ whenever $d(x,y) < \delta$; and
(ii) for all $\varepsilon > 0$ there exits $\delta > 0$ such that $d(x,y) < \varepsilon$ whenever $\rho(x,y) < \delta$.

Now the fundamental result of Chrząszcz, Jachymski, and Turoboś [8] reads as follows.

Proposition 1. *A semimetric is regular if and only if it is uniformly equivalent to a metric.*

Uniform equivalence in itself guarantees that neighborhood topology, sequential topology, continuity, and any notions related to sequences are invariants of such semimetrics. Moreover, in view of this theorem the pathologies mentioned earlier disappear. In regular semimetric spaces, the neighborhood and sequential topologies coincide and the Hausdorff property holds. In other words, the topological behavior of regular semimetric spaces and metric spaces cannot be distinguished.

Besides the above facts on semimetric spaces, we shall need an extension of the Banach Contraction Principle [1]. In order to formulate it, the next notions are needed. Under a *comparison function* we mean an increasing, right-continuous function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ fulfilling $\varphi(t) < t$ for $t > 0$. Clearly, any comparison function vanishes at zero: $\varphi(0) = 0$. Since the composition of nondecreasing, right-continuous functions remains nondecreasing and right-continuous, the iterates of comparison functions are comparison functions, as well.

Let (X,d) be an arbitrary semimetric space. We say that the map $f\colon X \to X$ is a *generalized contraction* with comparison function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ if it satisfies the inequality (3) for all elements $x, y \in X$. To the best of our knowledge, the first fixed point results for maps satisfying (3) were obtained by Browder [7], by Boyd and Wong [6] and by Matkowski [21]. A variant for certain kind of semimetric spaces appears in the paper by Jachymski, Matkowski, and Świątkowski [16]. The key tool of our approach is based on the result of Bessenyei and Páles [2]:

Proposition 2. *The composite iterates of a generalized contraction in a regular semimetric space generates Cauchy sequences pointwise. Each generalized contraction of a complete and regular semimetric space has a unique fixed point.*

We close this section by turning back to normal semimetric spaces. In any semimetric space (X, d), a set H is *bounded* by definition if there exists $x_0 \in X$ and $r > 0$ such that $H \subseteq U(x_0, r)$. The *diameter* of H is $\operatorname{diam}(H) := \sup\{d(x, y) \mid x, y \in H\}$. In normal semimetric spaces, boundedness can be characterized in the same way as in metric spaces:

Proposition 3. *If (X, d) is a normal semimetric space and $H \subseteq X$, then the following statements are equivalent:*

(i) for all $x_0 \in X$ there exists $r > 0$ such that $H \subseteq U(x_0, r)$;
(ii) there exists $x_0 \in X$ and $r > 0$ such that $H \subseteq U(x_0, r)$;
(iii) $\operatorname{diam}(H) < +\infty$.

Proof. Denote the basic triangle function of (X, d) by Φ. Implication $(i) \Rightarrow (ii)$ is trivial. To prove $(ii) \Rightarrow (iii)$, let $a, b \in H$ be arbitrary. Then, by partial monotonicity of Φ,

$$d(a, b) \leq \Phi\big(d(x_0, a), d(x_0, b)\big) \leq \Phi(r, r) =: r_0.$$

Thus $\operatorname{diam}(H) \leq r_0 < +\infty$ since d is normal. For the implication $(iii) \Rightarrow (i)$, assume that $r_0 := \operatorname{diam}(H) < +\infty$ and let $x_0 \in X$ be arbitrary. Fix $y \in H$, and let $x \in H$. Then $d(x, y) \leq r_0$ and hence

$$d(x_0, x) \leq \Phi\big(d(x_0, y), d(y, x)\big) \leq \Phi\big(d(x_0, y), r_0\big) =: r.$$

Normality ensures that $r < +\infty$ and the inequality above shows that $H \subseteq U(x_0, r)$, and the proof is completed. □

3. EXISTENCE AND UNIQUENESS OF GENERALIZES FRACTALS

The aim of this section is to develop a general method that guarantees the resolvability of (1) among proper compact sets. The extension concerns both the underlying space and the mapping: Our main result is an existence theorem for those fractals that are generated by condensing maps of complete and regular semimetric spaces. The proof completely different from the standard ones since it avoids the hyperspace of fractals and hence the Blaschke Completeness Theorem [5]. The main idea is to combine the arguments of our recent papers [3] and [4] with the results of Chrząszcz, Jachymski, and Turoboś [8]. Finally we present a Hutchinson-type result for generalized contractions of a complete and regular semimetric space. The construction of the Hausdorff–Pompeiu metric plays role only in its uniqueness issue.

Our key tool is the next construction. Let (X, d) be a semimetric space and $H \subseteq X$. Then the (extended) nonnegative real number

$$\chi(H) = \inf\big\{\varepsilon > 0 \mid \exists x_1, \dots, x_n \in X :\ H \subseteq U(x_1, \varepsilon) \cup \dots \cup U(x_n, \varepsilon)\big\}$$

is the *noncompactness measure* of H, while the finite set $E = \{x_1, \dots, x_n\}$ is an *ε-net* for H. In metric setting, these notion was invented by Kuratowski [18], and its properties with nice applications can be found in the monograph of Granas and Dugundji [11]. The next lemma subsumes its properties in semimetric context.

Lemma 1. *If (X, d) is a regular semimetric space, then*

(i) $\chi(A) \leq \chi(B)$ whenever $A \subseteq B \subseteq X$;
(ii) $\chi(A \cup B) \leq \max\{\chi(A), \chi(B)\}$ for all $A, B \subseteq X$;
(iii) $\chi(A) = 0$ if and only if $\chi(\overline{A}) = 0$.

Proof. Statement (i) is obvious, since each finite ε-net of B is a finite ε-net for A provided that $A \subseteq B$. To prove (ii), let $\varepsilon > \max\{\chi(A), \chi(B)\}$. Then there exist E_A and E_B finite ε-nets for A and B, respectively. Then $E_A \cup E_B$ is a finite ε-net for $A \cup B$, resulting in

$$\chi(A \cup B) \leq \varepsilon.$$

Taking the limit $\varepsilon \to \max\{\chi(A), \chi(B)\}$, we arrive at the desired inequality. Thus to verify statement (iii), we have to show only that $\chi(A) = 0$ implies $\chi(\overline{A}) = 0$. Let $\varepsilon > 0$ be arbitrary. By regularity, the basic triangle function Φ is continuous at the origin. Thus there exists $\delta > 0$ such that $\Phi(\delta, \delta) < \varepsilon$. Since $\chi(A) = 0$, there exists a finite δ-net for A:

$$A \subseteq \bigcup_{x \in E} U(x, \delta).$$

Regularity guarantees that the topological and analytical closure of A coincide by Proposition 1. Therefore

$$\overline{A} \subseteq \bigcup_{a \in A} U(a, \delta).$$

If $a^* \in \overline{A}$ is arbitrary, then there exists $a \in A$ such that $a^* \in U(a, \delta)$ and there exists $x \in E$ such that $a \in U(x, \delta)$. Thus

$$d(x, a^*) \leq \Phi\big(d(x, a), d(a^*, a)\big) \leq \Phi(\delta, \delta) < \varepsilon.$$

This shows that E is a finite ε-net for $\overline{A}$. Since $\varepsilon > 0$ was arbitrary, $\chi(\overline{A}) = 0$ follows. □

The third property can be improved in metric spaces so that $\chi(\overline{A}) = \chi(A)$. However, the present relaxed form, combining with Proposition 1 is still strong enough to prove the semimetric version of the Hausdorff Compactness Theorem. We note here that a subset of a semimetric space is *total bounded* if its noncompactness measure is zero.

Lemma 2. *In a regular semimetric space, a subset is compact if and only if it is complete and total bounded. In a complete and regular semimetric space, a subset is relatively compact if and only if its Kuratowski measure of noncompactness is zero.*

Proof. Let (X, d) be a regular semimetric space. By Proposition 1, there exists a metric ρ uniformly equivalent to d. We note first that compactness, completeness, and total boundedness are invariants of uniform equivalence. That is, a set $H \subseteq X$ is

- (i) compact in d if and only if it is compact in ρ;
- (ii) complete in d if and only if it is complete in ρ;
- (iii) total bounded in d if and only if it is total bounded in ρ.

The first and second properties are immediate consequences of Proposition 1. Now assume that H is total bounded in ρ and let $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $d(x, y) < \varepsilon$ whenever $\rho(x, y) < \delta$. Since H is total bounded, there exists a finite δ-net E for H: If $x \in H$ is arbitrary, we can find $y \in E$ such that $\rho(x, y) < \delta$. In this case, $d(x, y) < \varepsilon$, showing that E is a finite ε-net for H in d. The converse statement can be proved similarly. Thus the first part of the lemma is a consequence of the Hausdorff Compactness Theorem.

For the second part, assume that H is relatively compact. Then $\overline{H}$ is total bounded by the first part. That is, $\chi(\overline{H}) = 0$; in particular, $\chi(H) = 0$ by the first property of Lemma 1. Conversely, assume that $\chi(H) = 0$. Then $\chi(\overline{H}) = 0$ by the third property of Lemma 1. Thus $\overline{H}$ is a total bounded. On the other hand, it is a closed subset in a complete and regular semimetric space and hence $\overline{H}$ is complete. By the first part again, $\overline{H}$ must be compact. □

Each family $\mathscr{F}$ of self-maps of a set X induces a set-valued operator via the right-hand side of the invariance equation (1). From now on, we shall refer it as the *invariance operator*:

$$F\colon \mathscr{P}(X) \to \mathscr{P}(X), \qquad F(H) = \bigcup \mathscr{F}(H) \tag{4}$$

Evidently, a subset of X is $\mathscr{F}$-invariant if and only if it is a fixed point of the invariance operator. This easy observation enables us to use the standard methods of Fixed Point Theory for fractals. First we recall a structure-free, set theoretical one. The *Kantorovich iteration* determined by the invariance operator (4) and starting from a set $H \in \mathscr{P}(X)$ is the sequnece

$$\big(F^n(H)\big)_{n\in\mathbb{N}}.$$

We say that the set $H \in \mathscr{P}(X)$ is *$\mathscr{F}$-subinvariant*, if the invariance property holds only with inclusion: $H \subseteq F(H)$. The next result is a version of the Kantorovich Fixed Point Theorem [17]. For convenience, we present its proof.

Lemma 3. *If $\mathscr{F}$ is a family of self-maps of a set X, then the Kantorovich iteration extends each $\mathscr{F}$-subinvariant set to a smallest $\mathscr{F}$-invarint set.*

Proof. Using induction and the $\mathscr{F}$-subinvariant property, we can prove that the sequence $(F^n(H))$ of sets is an ascending chain. Denote the union of the chain by H_0. It is immediate to see that the invariance operator is interchangeable with the union. Thus

$$F(H_0) = F\Big(\bigcup_{n\in\mathbb{N}} F^n(H)\Big) = \bigcup_{n\in\mathbb{N}} F^{n+1}(H) = \bigcup_{n\in\mathbb{N}} F^n(H) = H_0,$$

showing that H_0 is an $\mathscr{F}$-invariant set. Now assume that $H \subseteq K$ and K is an $\mathscr{F}$-invariant set. Since the invariance operator is monotone with respect to inclusion, $F(H) \subseteq F(K) = K$ follows. Iterating F and applying induction one can easily obtain that $F^n(H) \subseteq K$ holds for all $n \in \mathbb{N}$. Thus the definition of H_0 results in $H_0 \subseteq K$. ☐

Certain kind of maps are strongly connected with the Kuratowski measure of noncompactness (see Granas and Dugundji [11] for the metric case). A map $f\colon X \to X$ of a semimetric space (X, d) is called *condensing* if it is continuous and decreases the noncompactness measure of bounded, not relatively compact sets. That is, $\chi(f(H)) < \chi(H)$ provided that $H \subseteq X$ is bounded and not relatively compact. The role of the two defining properties of condensing maps is enlightened in the next lemmas.

Lemma 4. *If $\mathscr{F}$ is a finite family of self-maps of a complete and regular semimetric space whose members decrease the Kuratowski noncompactness measure of bounded, not relatively compact sets, then any bounded, $\mathscr{F}$-invariant set is relatively compact.*

Proof. Let $\mathscr{F} = \{f_1, \dots, f_n\}$. Assume to the contrary, that an $\mathscr{F}$-invariant set H is bounded, but not relatively compact. Then $\chi(H)$ is finite and positive by (iii) of Lemma 1. By (ii) of Lemma 1 again,

$$\begin{aligned}\chi(H) = \chi(F(H)) &= \chi(f_1(H) \cup \cdots \cup f_n(H)) \\ &= \max\{\chi(f_1(H)), \dots, \chi(f_n(H))\} \\ &< \max\{\chi(H), \dots, \chi(H)\} = \chi(H)\end{aligned}$$

follows, which is a contradiction. ☐

Lemma 5. *If $\mathscr{F}$ is a finite family of continuous self-maps of a regular semimetric space, then the closure of any relatively compact $\mathscr{F}$-invariant set is $\mathscr{F}$-invariant, as well.*

Proof. Let (X,d) be a regular semimetric space and let $H \subseteq X$ be $\mathscr{F}$-invariant. By Proposition 1, we can apply topological arguments in the standard way. Let $y \in F(\overline{H})$. Then $y \in f(\overline{H})$ for some $f \in \mathscr{F}$. That is, $y = f(x)$ where $x \in \overline{H}$. Take a sequence (x_k) from H tending to x. Then $f(x_k) \in H \subseteq \overline{H}$ since H is $\mathscr{F}$-invariant. Continuity guarantees that $y = f(x) \in \overline{H}$, resulting in the inclusion $F(\overline{H}) \subseteq \overline{H}$. By continuity again and by the compactness of $\overline{H}$, the set $F(\overline{H})$ is compact, as well. In particular, it is closed. Thus $H = F(H) \subseteq F(\overline{H})$ shows that H is a subset of the closed set $F(\overline{H})$. Therefore $\overline{H} \subseteq F(\overline{H})$ which is exactly the reversed inclusion. □

The essence of the previous lemmas is our first main result on the existence of generalized fractals. We may regard it as the topological correspondence of Lemma 3. The proof gives a method to approximate fractals: The Kantorovich iteration plays the role of the Banach–Picard one. Let us emphasize that the method avoids completely the hyperspace of fractals.

Theorem 1. *If $\mathscr{F}$ is a finite family of condensing self-maps of a normal, complete and regular semimetric space such that the invariance operator generates a bounded Kantorovich iteration on a nonempty $\mathscr{F}$-subinvariant set H, then there exists a smallest $\mathscr{F}$-fractal containing H.*

Proof. Consider the invariant set H_0 generated by H and guaranteed by Lemma 3. Then H_0 is bounded and hence $\overline{H_0}$ is compact by Lemma 4. Moreover, it is also invariant by Lemma 5. Thus $\overline{H_0}$ is a fractal since $H \subseteq \overline{H_0}$ and H is nonempty. Finally, $H_0 \subseteq K$ whenever K is a fractal containing H by Lemma 3. Therefore $\overline{H_0} \subseteq \overline{K} = K$ and the proof is completed. □

Fixed point theorems help to create subinvariant sets in an evident way. Indeed, if x_0 is a fixed point of some map in $\mathscr{F}$ then the singleton $\{x_0\}$ serves as an $\mathscr{F}$-subinvariant set. This fact results in the existence part of the Hutchinson Theorem with the help of the Banach Contraction Principle; we shall revisit this issue in details later. Similarly, the main result of [4] can easily be deduced from Theorem 1 using the fixed point theorem of Darbo [9] and Sadovskiĭ [25]. It states that a finite family of condensing self maps of a bounded, closed subset in a Banach space always generates a fractal.

Moreover, Theorem 1 may work even in very simple cases when the Hutchinson Theorem does not. We illustrate it with the examples below by modifying the invariance equation of the Cantor set. In each cases, the semimetric space is the set of nonnegative reals $\mathbb{R}_+$ equipped with the Euclidean metric. Let $\mathscr{F} = \{f_1, f_2\}$ where $f_2(x) = \frac{1}{3}x + \frac{2}{3}$. If $f_1(x) = x$, then the smallest $\mathscr{F}$-fractal containing zero is the set

$$\left\{\frac{2}{3} + \cdots + \frac{2}{3^n} \mid n \in \mathbb{N}\right\} \cup \{0, 1\}.$$

The existence of a smallest $\mathscr{F}$-fractal containing zero also follows if $f_1(x) = \arctan(x)$. Finally, we recall an example of Rhoades [23]. If $f_1(x) = 1/(1+x)$, then it is a generalized contraction with comparison function $\varphi(t) = t/(1+t)$. Its unique fixed point is $x_0 = (\sqrt{5}-1)/2$ and there exits a smallest $\mathscr{F}$-fractal containing x_0 by Theorem 1. In each cases, f_1 is not a Banach contraction thus Hutchinson's result cannot be applied.

Unfortunately we cannot state uniqueness in Theorem 1. For example, the interval $[0,1]$ is another $\mathscr{F}$-invariant set in the first example above. It turns out that we can adopt completely Hutchinson's result for families of generalized contraction. In what follows, we shall focus on this kind of families. First we show that the class of generalized contractions is one of the most important subclasses of condensing maps.

Lemma 6. *Each generalized contraction of a complete and regular semimetric space is a condensing map.*

Proof. Let (X,d) be a complete, regular semimetric space and let $f\colon X\to X$ be a generalized contraction with comparison function φ. If $H\subseteq X$ is a bounded, not relatively compact set and $\varepsilon>\chi(H)$ then there exists a finite ε-net E for H. For arbitrary $x\in H$, choose $h\in E$ such that $d(x,h)<\varepsilon$. Then,

$$d(f(x),f(h))\le\varphi(d(x,h))\le\varphi(\varepsilon).$$

Therefore $f(E)$ is a finite $\varphi(\varepsilon)$-net for $f(H)$. Thus $\chi(f(H))\le\varphi(\varepsilon)$. Taking the limit $\varepsilon\to\chi(H)$ and applying the right continuity of φ, we arrive at

$$\chi(f(H))\le\varphi(\chi(H)).$$

By Lemma 2, $\chi(H)>0$ whenever H is not relatively compact. Thus, by the properties of comparison functions, $\varphi(\chi(H))<\chi(H)$. That is, f decreases the Kuratowski noncompactness measure of bounded, not relatively compact sets. On the other hand, generalized contractions are continuous. In other words, generalized contractions are condensing maps. □

Given a semimetric space (X,d), denote the family of nonempty and compact subsets by $\mathscr{K}(X)$. For $A,B\in\mathscr{K}(X)$, define the (extended) real number

$$d_{HP}(A,B):=\inf\Big\{r>0\mid A\subseteq\bigcup_{b\in B}U(b,r),\ B\subseteq\bigcup_{a\in A}U(a,r)\Big\}.$$

The next lemma shows that $(\mathscr{K}(X),d_{HP})$ is a semimetric space under suitable assumptions. This space is the *fractal space* over (X,d).

Lemma 7. *The fractal space over a normal and regular semimetric space is a regular semimetric space.*

Proof. Let (X,d) be a normal and regular semimetric space with basic triangle function Φ, and consider its fractal space $(\mathscr{K}(X),d_{HP})$. Normality and regularity ensure that compact sets, in particular: fractals are bounded and closed. Observe first, that d_{HP} has finite values. Indeed, for arbitrary $A,B\in\mathscr{K}(X)$, there exist positive numbers α,β and there exist $x,y\in X$ such that

$$A\subseteq U(x,\alpha)\quad\text{and}\quad B\subseteq U(y,\beta)$$

by boundedness of compact sets. Thus, for all $a\in A$ and for all $b\in B$,

$$\begin{aligned}d(a,b)&\le\Phi\big(d(x,a),d(x,b)\big)\\&\le\Phi\big(d(x,a),\Phi\big(d(x,y),d(y,b)\big)\big)\\&\le\Phi\big(\alpha,\Phi\big(d(x,y),\beta\big)\big)=:r\end{aligned}$$

by the monotonicity properties of Φ. Moreover, $r<+\infty$ since Φ is real valued. Therefore,

$$a\in U(b,r)\subseteq\bigcup_{b\in B}U(b,r)\quad\text{and}\quad b\in U(a,r)\subseteq\bigcup_{a\in A}U(a,r).$$

Hence $d_{HP}(A,B)\le r<+\infty$. If $A=B$, then $d_{HP}(A,B)=0$ obviously holds. Conversely, assume that $d_{HP}(A,B)=0$ for some $A,B\in\mathscr{K}(X)$. Let $a\in A$ be fixed. Then, for all $n\in\mathbb{N}$, there exists $b_n\in B$, such that $d(a,b_n)<1/n$. Thus (b_n) tends to $a\in A$. Since B is closed and the space is regular, $a\in B$. However, $a\in A$ is arbitrary, consequently $A\subseteq B$. The other inclusion can be proved similarly, resulting in $A = B$. The symmetry follows directly from the definition. These arguments proves that the fractal space is a semimetric space, indeed.

Finally, for fixed $A, B, C \in \mathscr{K}(X)$, let $d_{HP}(A, B) < r_1$ and let $d_{HP}(B, C) < r_2$. If $a \in A$ is arbitrary, then there exists $b \in B$ and $c \in C$, such that $d(a, b) < r_1$ and $d(b, c) < r_2$. Thus $d(a, c) < \Phi(r_1, r_2)$. Interchanging the role of A and C, we can conclude that

$$A \subseteq \bigcup_{c\in C} U(c, \Phi(r_1, r_2)) \quad \text{and} \quad C \subseteq \bigcup_{a\in A} U(a, \Phi(r_1, r_2)).$$

In other words, $d_{HP}(A, C) \le \Phi(r_1, r_2)$ whenever $d_{HP}(A, B) < r_1$ and $d_{HP}(B, C) < r_2$. Consider now the function $\Phi^*\colon \mathbb{R}_+^2 \to \overline{\mathbb{R}}$ defined by

$$\Phi^*(u, v) = \lim_{n\to\infty} \Phi\Big(u + \frac{1}{n}, v + \frac{1}{n}\Big).$$

The partial monotonicity of Φ ensures that Φ^* is well-defined, while the previous observation guarantees that Φ^* is a triangle function for d_{HP}. Thus $\Psi \le \Phi^*$ where Ψ stands for the basic triangle function of the fractal space. Now we prove that Φ^* is continuous at the origin. Let $\varepsilon > 0$ arbitrary. By the regularity of the underlying space, Φ is continuous at the origin thus there exists $\delta > 0$ such that $\Phi(u, v) < \varepsilon$ if $0 \le u, v < \delta$. If $n \in \mathbb{N}$ is sufficiently large,

$$0 \le u + \frac{1}{n}, v + \frac{1}{n} < \delta \quad \text{hence} \quad \Phi\Big(u + \frac{1}{n}, v + \frac{1}{n}\Big) < \varepsilon.$$

Taking the limit $n \to \infty$, we arrive at $\Phi^*(u, v) \le \varepsilon$. The inequality $0 \le \Psi \le \Phi^*$ implies that Ψ is continuous at the origin, as well. □

Hutchinson's original idea is to handle the invariance equation (1) as a fixed point problem of the invariance operator (4) in the fractal space. His key observation is that invariance operator inherits the contraction properties of the generating family $\mathscr{F}$. Our last auxiliary result corresponds to this idea. However, we shall need it only for proving the uniqueness of generalized fractals.

Lemma 8. *If $\mathscr{F}$ is a finite family of generalized contractions of a normal and regular semimetric space, then the invariance operator* (4) *is a generalized contraction in the fractal space, and the sequence of its composite iterates generates Cauchy sequences pointwise.*

Proof. Let $f_1, \dots, f_n\colon X \to X$ be generalized contractions of a normal and regular semimetric space X with comparison functions $\varphi_1, \dots, \varphi_n$. First we show, that $\varphi := \max\{\varphi_1, \dots, \varphi_n\}$ is a comparison function, as well. Clearly, φ is right-continuous, increasing, and

$$\varphi(t) = \max\{\varphi_1(t), \dots, \varphi_n(t)\} < \max\{t, \dots, t\} = t.$$

In the second step, we show that the invariance operator F is a generalized contraction with comparison function φ. Note that $F(A)$ is compact if A is compact since the members of $\mathscr{F}$ are continuous. Thus $F\colon \mathscr{K}(X) \to \mathscr{K}(X)$. Now let $A, B \in \mathscr{K}(X)$ and choose $\varepsilon > 0$ such that $d_{HP}(A, B) < \varepsilon$. Then, for any $a \in A$ there exists $b \in B$ such that $a \in U(b, \varepsilon)$. Hence

$$d(f_k(a), f_k(b)) \le \varphi_k\big(d(a, b)\big) \le \varphi\big(d(a, b)\big) \le \varphi(\varepsilon),$$

yielding

$$f_k(a) \in U(f_k(b), \varphi(\varepsilon)) \subseteq \bigcup_{y\in F(B)} U(y, \varphi(\varepsilon)).$$

Consequently,

$$F(A) \subseteq \bigcup_{y \in F(B)} U(y, \varphi(\varepsilon)).$$

Similar arguments result in

$$F(B) \subseteq \bigcup_{y \in F(A)} U(y, \varphi(\varepsilon)).$$

Thus $d_{HP}(F(A), F(B)) \le \varphi(\varepsilon)$. Taking the limit $\varepsilon \to d_{HP}(A, B)$ and applying the right-continuity of φ, we arrive at the desired contraction property of F. The second statement follows from this fact and from Proposition 2. □

To end this section, we present a complete extension of Hutchinson's classical result when the invariance operator is induced by a finite family of generalized contractions of a complete, normal, and regular semimetric space.

Theorem 2. *If $\mathscr{F}$ is a finite family of generalized contractions of a complete, normal and regular semimetric space, then there exists a unique $\mathscr{F}$-fractal.*

Proof. Clearly, Proposition 2 provides a nonempty $\mathscr{F}$-subinvariant set using the fixed point of the generalized contraction. (The singleton containing the fixed point is $\mathscr{F}$-subinvariant.) Thus there exists a nonempty and smallest $\mathscr{F}$-invariant set by Lemma 3. Furthermore, the Kantorovich iteration produces a Cauchy sequence in the fractal space by Lemma 8 and hence this $\mathscr{F}$-invariant set is bounded. Applying Theorem 1 we arrive at the existence part. In other words, we found a fixed point of the invariance operator. However, the invariance operator is a generalized contraction in the fractal space by Lemma 7, and a generalized contraction has at most one fixed point. This yields uniqueness. □

A complete metric space is a complete, normal and regular semimetric space. Thus Hutchinson's result [14] is a direct consequence of the theorem above. Moreover, relaxed metric spaces, inframetric spaces, and pth order spaces mentioned in the previous section are normal and regular semimetric spaces, as well. Therefore Theorem 2 applies for these settings. We omit the explicit formalizations of these corollaries. It is also worth mentioning that the third example after Theorem 1 is also in the scope of Theorem 2 and hence we can state uniqueness in that case.

We call the attention that a wide range of extensions in *metric* spaces via *metric* fixed point theorems can also be obtained if we follow the original ideas of Hutchinson [14]. An excellent summary of the topic (with further important developments) is the survey by Leśniak, Snigireva, and Strobin [20]. The precise connections between metric fixed point theorems can be found in the papers by Rhoades [23] and by Jachymski and Jóźwik [15].

4. The Hausdorff dimension of generalized fractals

The aim of this section is to give a correspondence of Hutchinson's fractal dimension formula involving the exponential equation (2). The Hausdorff measure on a semimetric space can be introduced formally as in metric spaces. For details, we refer to Rogers [24], but for convenience we sketch it briefly. The book of Halmos [12] serves as our measure theoretic reference. Having a set function $\mu\colon \mathscr{P}(X) \to [0, +\infty]$ on a nonempty set X, the definition

$$\mu^*(A) := \inf \left\{ \sum_{C \in \mathscr{C}} \mu(C) \mid \mathrm{card}(\mathscr{C}) \le \mathbb{N},\, C \subseteq X,\, A \subseteq \bigcup \mathscr{C} \right\}$$

results in a nonnegative valued, σ-subadditive set function $\mu^*\colon \mathscr{P}(X) \to [0,+\infty]$ or briefly: in an *outer measure* on X. We say that $A \subseteq X$ is μ^*-measurable if, for all $T \subseteq X$,

$$\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \setminus A).$$

The Carathéodory Theorem states that the μ^*-measurable subsets of X form a σ-algebra on which μ^* is a measure. To construct the Hausdorff measure in semimetric spaces, we apply the Carathéodory construction using a suitable set function. If $\Gamma\colon \mathbb{R}_+ \to \mathbb{R}$ is the gamma function and $r \geq 0$, then define

$$\alpha(r) = \frac{\Gamma^r(\frac{1}{2})}{\Gamma(1+\frac{r}{2})}.$$

Let (X,d) be a semimetric space. From now on, we are going to use the notion of diameter as it is introduced right before Proposition 3. Denote the family of subsets with diameter at most δ by $\mathscr{B}_\delta(X)$ and define ν_δ^r by

$$\nu_\delta^r(A) := \begin{cases} \dfrac{\alpha(r)}{2^r}\operatorname{diam}^r(A) & \text{if } A \in \mathscr{B}_\delta(X), \\ +\infty & \text{if } A \notin \mathscr{B}_\delta(X). \end{cases}$$

Now we can introduce a set function $\chi^r\colon \mathscr{P}(X) \to [0,+\infty]$ with the outer measure $\chi_\delta^r = \big(\nu_\delta^r\big)^*$ via the formula

$$\chi^r(A) = \lim_{\delta\to 0+0} \chi_\delta^r(A).$$

Using the same arguments as in the metric case, one can prove that this function is well-defined and it is an outer measure on the underlying semimetric space. In the forthcomings, we refer it as the *r-dimensional outer Hausdorff measure*. We denote the σ-algebra of the χ^r-measurable sets by $\mathscr{H}^r$. The *r-dimensional Hausdorff measure* is the restriction of χ^r to this σ-algebra, and since it will not be confusing, we denote it by χ^r, as well. Finally, the *Hausdorff dimension* of a set $A \subseteq X$ is the (extended) real number

$$\dim_{\mathscr{H}}(A) := \sup\{r \geq 0 \mid \chi^r(A) > 0\}.$$

The most important properties of the Hausdorff dimension are still valid if we use semimetrics:

- If $\chi^r(A) < +\infty$ and $t > r$, then $\chi^t(A) = 0$.
- If $r > \dim_{\mathscr{H}}(A)$, then $\chi^r(A) = 0$.
- If $r < \dim_{\mathscr{H}}(A)$, then $\chi^r(A) = +\infty$.

In the first auxiliary result we show that, in certain cases, outer measures inherit the relation of the generating set functions.

Lemma 9. *If $\mu\colon \mathscr{P}(X) \to \overline{\mathbb{R}}_+$ and $\nu\colon \mathscr{P}(Y) \to \overline{\mathbb{R}}_+$ are given and $f\colon X \to Y$ satisfies $\nu \circ f \leq \mu$ on $\mathscr{P}(X)$, then $\nu^* \circ f \leq \mu^*$. If f is a bijection and $\nu \circ f = \mu$ on $\mathscr{P}(X)$, then $\nu^* \circ f = \mu^*$.*

Proof. We may assume that $\mu^*(A)$ is finite. Let $c > \mu^*(A)$ and let $\mathscr{C}$ be a countable cover for A such that $\sum_{C\in\mathscr{C}} \mu(C) < c$. Obviously $\{f(C) \mid C \in \mathscr{C}\}$ is a countable cover for $f(A)$. Thus

$$\nu^*(f(A)) \leq \sum_{C\in\mathscr{C}} \nu(f(C)) \leq \sum_{C\in\mathscr{C}} \mu(C) < c.$$

Taking the limit $c \to \mu^*(A)$, we arrive at the first statement. The second statement follows from the first one applying the just have proved inequality for f and its inverse. $\square$

In what follows, we shall need the next concepts. Let $H \subseteq \mathbb{R}$ be a nonempty set and let $t_0 \in H$ be a right limit point of H. The *right-upper Dini-derivative* of a function $\varphi\colon H \to \mathbb{R}$ at t_0 is the (extended) real number

$$D\varphi(t_0) := \limsup_{t \to t_0+0} \frac{\varphi(t) - \varphi(t_0)}{t - t_0}.$$

Let (X, d) be a semimetric space. We say that a function $f\colon X \to X$ is *φ-Lipshitz* if it fulfills (3) with $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(0) = 0$ and $D\varphi(0) < +\infty$. We point out that each generalized contraction is a φ-Lipschitz map. Additionally, if φ is strictly monotone increasing, differentiable at zero, $\varphi'(0) > 0$, and (3) holds with equality, then we speak about a *φ-similarity*. The next lemma clarifies the connection of the Hausdorff measure and φ-Lipschitz maps.

Lemma 10. *If f is a φ-Lipschitz self-map of a semimetric space (X, d) and $A \subseteq X$, then*

$$\chi^r\big(f(A)\big) \le \big(D\varphi(0)\big)^r \cdot \chi^r(A).$$

Moreover, if f is a surjective φ-similarity, then this inequality is valid with equality, and $A \subseteq X$ is Hausdorff-measurable if and only if $f(A)$ is Hausdorff-measurable.

Proof. If $\varepsilon > D\varphi(0)$, then there exists a right neighborhood I_ε of zero such that $\varphi(t) \le \varepsilon t$ whenever $t \in I_\varepsilon$ since $\varphi(0) = 0$. Let $\delta \in I_\varepsilon$ be fixed and let A be such that $\operatorname{diam}(A) < \delta$. If $u, v \in f(A)$, then $u = f(x)$ and $v = f(y)$ where $x, y \in A$. Furthermore

$$d(u, v) = d(f(x), f(y)) \le \varphi(d(x, y)) \le \varepsilon \operatorname{diam}(A).$$

Hence the diameter of $f(A)$ is at most $\varepsilon \dim(A)$. This property ensures that

$$\nu^r_{\varepsilon\delta}(f(A)) = \frac{\alpha(r)}{2^r} \operatorname{diam}^r(f(A)) \le \frac{\alpha(r)}{2^r} \varepsilon^r \operatorname{diam}^r(A) = \varepsilon^r \nu^r_\delta(A)$$

holds for all $\delta \in I_\varepsilon$ and for all $A \in \mathscr{B}_\delta(X)$. Thus by Lemma 9,

$$\chi^r_{\varepsilon\delta}(f(A)) = (\nu^r_{\varepsilon\delta})^*(f(A)) \le (\varepsilon^r \nu^r_\delta)^*(A) = \varepsilon^r \chi^r_\delta(A).$$

Taking first the limit $\delta \to 0$ and then $\varepsilon \to D\varphi(0)$, we arrive at the first statement.

Now assume that f is a surjective φ-similarity. Since $\varphi^{-1}(\{0\}) = \{0\}$, it must be a bijection. Then f^{-1} is a ψ-similarity with $\psi := \varphi^{-1}$. Therefore the inequality of the first part results in

$$\chi^r(A) = \chi^r(f^{-1}(f(A))) \le \big(D\psi(0)\big)^r \cdot \chi^r(f(A)).$$

Observe that $\varphi'(0) > 0$ implies $D\psi(0) = 1/\varphi'(0) > 0$. Rearranging the inequality above, we get the reverse one.

Finally assume that $A \subseteq X$ is Hausdorff-measurable and let $q = \varphi'(0)$. Then, since f is a bijection, for all $T \subseteq X$,

$$\begin{aligned}
\chi^r\big(f(T)\big) &= q^r \chi^r(T) = q^r\big(\chi^r(T \cap A) + \chi^r(T \setminus A)\big) \\
&= \chi^r\big(f(T \cap A)\big) + \chi^r\big(f(T \setminus A)\big) \\
&= \chi^r\big(f(T) \cap f(A)\big) + \chi^r\big(f(T) \setminus f(A)\big).
\end{aligned}$$

Since f is a bijection, each subset S of X can be obtained as $S = f(T)$ with suitable $T \subseteq X$. Thus the calculation above ensures that $f(A)$ is Hausdorff-measurable. The converse statement follows similarly. □

It turns out that the fractals of a regular semimetric space are Hausdorff measurable. We shall prove a more general result by showing that even Borel sets have this feature. As in the standard setting, first we characterize the so-called (semi)metric outer measures and then conclude that the Hausdorff outer measure possesses this property. We adopt the basic idea from [12]. However, the proof contains several independent movements, thus we share its details. We define the *distance* among subsets of a semimetric space by $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}$.

Lemma 11. *Let μ be an outer measure on a regular semimetric space (X,d). The Borel sets of X are μ-measurable if and only if all nonempty sets $A, B \subseteq X$ of positive distance satisfy*

$$\mu(A \cup B) = \mu(A) + \mu(B). \tag{5}$$

In particular, the Borel sets of a regular semimetric space are Hausdorff-measurable.

Proof. For necessity of (5), assume that $d(A,B) > 0$. We show that $\overline{A} \cap B = \emptyset$. Indeed, assume to the contrary that this is not the case. Then there exits $b \in B$ and $a_n \in A$ such that $a_n \to b$ by Proposition 1. Thus $0 < d(A,B) \leq d(a_n, b) \to 0$ as $n \to \infty$, which is a contradiction. Now let $T = A \cup B$. Since $\overline{A}$ is μ-measurable,

$$\begin{aligned} \mu(A \cup B) = \mu(T) &= \mu(\overline{A} \cap T) + \mu(T \setminus \overline{A}) \\ &= \mu(A \cup (\overline{A} \cap B)) + \mu(B \setminus \overline{A}) \\ &= \mu(A) + \mu(B). \end{aligned}$$

For sufficiency of (5), denote the basic triangle function of the semimetric space by Φ, and define α^* attached to $\alpha > 0$ by

$$\alpha^* := \lim_{\delta \to 0+0} \Phi(\alpha, \delta).$$

The partial monotonicity of Φ ensures that the definition is correct. Furthermore, the continuity of Φ at the origin implies that, for all $\varepsilon > 0$ there exists α such that $\alpha^* < \varepsilon$. This property ensures that we can construct a strictly monotone decreasing null sequence (α_n) satisfying $\alpha^*_{n+1} < \alpha_n$.

Let $U \subseteq X$ be an open set and let $T \subseteq X$ be such that $\mu(T) < +\infty$. Consider the sequences of sets (A_n) and (B_n) defined by

$$A_n = \{x \in T \cap U \mid d(x, X \setminus U) \geq \alpha_n\} \qquad \text{and} \qquad B_1 = A_1, \quad B_{n+1} = A_{n+1} \setminus A_n.$$

Clearly, (A_n) is an ascending chain. Moreover, the odd subsequence of (B_n) consists of sets with positive distance. To verify this property, we shall check it only for consecutive members. Assume to the contrary that $d(B_{2n-1}, B_{2n+1}) = 0$. Then, for all $\delta > 0$ we can find elements $b_{2n-1} \in B_{2n-1}$ and $b_{2n+1} \in B_{2n+1}$ such that $d(b_{2n-1}, b_{2n+1}) < \delta$. If $y \in X \setminus U$,

$$\alpha_{2n-1} \leq d(y, b_{2n-1}) \leq \Phi\big(d(y, b_{2n+1}), d(b_{2n-1}, b_{2n+1})\big) \leq \Phi(\alpha_{2n}, \delta).$$

Taking the limit $\delta \to 0$, we arrive at the contradiction $\alpha_{2n-1} \leq \alpha^*_{2n} < \alpha_{2n-1}$.

Thus, applying induction and (5), we arrive at

$$\mu\Big(\bigcup_{k=1}^{n} B_{2k-1}\Big) = \sum_{k=1}^{n} \mu(B_{2k-1}); \quad \text{hence} \quad \mu\Big(\bigcup_{k=1}^{\infty} B_{2k-1}\Big) \geq \sum_{k=1}^{\infty} \mu(B_{2k-1}).$$

The same argument results in the corresponding estimation for even indices. Therefore,

$$+\infty > 2\mu(T) \geq \mu\Big(\bigcup_{k=1}^{\infty} B_{2k-1}\Big) + \mu\Big(\bigcup_{k=1}^{\infty} B_{2k}\Big) \geq \sum_{k=1}^{\infty} \mu(B_k).$$

Let $\varepsilon > 0$ be arbitrary. Since $T \cup U = A_n \cup B_{n+1} \cup B_{n+2} \cup \dots$ and $\sum \mu(B_k)$ is convergent, there exists $n \in \mathbb{N}$ such that

$$\varepsilon > \sum_{k=n}^{\infty} \mu(B_k) \geq \mu\big((T \cap U) \setminus A_n\big).$$

Thus, by (5) again,

$$\begin{aligned}\mu(T \cap U) + \mu(T \setminus U) &\leq \mu\big((T \cap U) \setminus A_n\big) + \mu(A_n) + \mu(T \setminus U)\\ &\leq \varepsilon + \mu\big((T \setminus U) \cup A_n\big)\\ &\leq \varepsilon + \mu(T).\end{aligned}$$

Since $\varepsilon > 0$ has been chosen arbitrarily, this means that U is μ-measurable.

Finally, let A and B be subsets of positive distance. By the continuity of Φ, we can choose $\delta > 0$ such that $\Phi(\delta, \delta) < d(A, B)$. Let $\mathscr{C}$ be a countable cover for $A \cup B$ which consists of sets belonging to $\mathscr{B}_\delta(X)$. Define

$$\mathscr{C}_A = \{C \in \mathscr{C} \mid C \cap A \neq \emptyset\} \quad \text{and} \quad \mathscr{C}_B = \{C \in \mathscr{C} \mid C \cap B \neq \emptyset\}.$$

Then $\mathscr{C}_A$ and $\mathscr{C}_B$ are countable covers for A and B, respectively. Moreover, these covers do not overlap. Indeed, assume to the contrary that this is not the case. Then there exist $C_A \in \mathscr{C}_A$ and $C_B \in \mathscr{C}_B$ such that $x \in C_A \cap C_B$. If $a \in A$ and $b \in B$, we arrive at the contradiction

$$d(A, B) \leq d(a, b) \leq \Phi\big(d(a, x), d(b, x)\big) \leq \Phi(\delta, \delta) < d(A, B).$$

These properties of $\mathscr{C}_A$ and $\mathscr{C}_B$ result in

$$\chi^r_\delta(A \cup B) \geq \chi^r_\delta(A) + \chi^r_\delta(B).$$

However, this inequality implies that $\mu = \chi^r$ fulfills (5). Thus Borel sets are Hausdorff-measurable, indeed. □

The main result of this section gives a condition for the Hausdorff dimension of generalized fractals of a semimetric space in terms of an exponential expression. In the second part of the statement, we use *strong* comparison functions, that is, comparison functions which are strictly monotone increasing, differentiable at zero, and their derivative at zero is positive.

Theorem 3. *If $f_1, \dots, f_n$ are generalized contractions of a complete, normal and regular semimetric space with comparison functions $\varphi_1, \dots, \varphi_n$ and $0 < \chi^r(H) < +\infty$ holds for the induced fractal H with some $r \geq 0$, then $\dim_{\mathscr{H}}(H) = r$ and*

$$1 \leq \big(D\varphi_1(0)\big)^r + \dots + \big(D\varphi_n(0)\big)^r.$$

Moreover, if $\chi^r\big(f_k(H) \cap f_l(H)\big) = 0$ whenever $k \neq l$ and each generalized contraction is a similarity with strong comparison function, then this inequality is valid with equality.

Proof. Note first that the induced fractal H exists and it is unique by Theorem 2. The assumption $0 < \chi^r(H) < +\infty$ implies immediately that $\dim_{\mathscr{H}}(H) = r$. The compactness of the sets H and $f_1(H), \dots, f_n(H)$ provides their Hausdorff-measurability by Lemma 11. Applying (1), the properties of the measure, and Lemma 10,

$$\chi^r(H) = \chi^r\Big(\bigcup_{k=1}^{n} f_k(H)\Big) \leq \sum_{k=1}^{n} \chi^r\big(f_k(H)\big) \leq \sum_{k=1}^{n} \big(D\varphi_k(0)\big)^r \chi^r(H).$$

Since $\chi^r(H)$ is positive and finite, this estimation reduces to the inequality of the theorem after simplifying. The second statement follows from the same calculation, since the assumptions provide that both inequality hold with equality. □

The right-upper Dini-derivative of the comparison function of a classical contraction coincides to the factor of the contraction. Thus if the underlying family in Theorem 3 consists of contractions, the involved inequality reduces to (2). Therefore our theorem just a weak dimension formula for generalized fractals: While Hutchinson's result [14] is a sufficient condition, Theorem 3 is a necessary one. Thus we pose the next question:

Open problem. *Does the complete extension of Hutchinson's fractal dimensional formula valid for generalized fractals of regular semimetric spaces?*

We recommend the papers [19] by Leśniak and [20] by Leśniak, Snigireva, and Strobin for further study. Although they elaborate the methods in the hyperspace, but their aims are the same as ours: To cover in a unified way the existence of fractals for a wide range of multifunctions.

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