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SUPPORT THEOREMS FOR GENERALIZED MONOTONE FUNCTIONS

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ABSTRACT. This note presents a complete solution of the support problem for functions that are generalized monotone in the sense of Beckenbach. The key tool of the proof is Tornheim's uniform convergence theorem. As applications, we improve some known support results and give an abstract version of the Hermite–Hadamard inequality.

1. INTRODUCTION

The usual definition of classical convexity is equivalent to the following property. Each interpolating line intersects the graph alternately and leaves the function under the graph. In this note, we shall study a generalized convexity notion which captures this geometric feature. To introduce it, first we replace Euclidean lines by interpolation families.

Definition. *A set of real valued continuous functions $\mathscr{B}_n(I)$ is a* Beckenbach family *if its members are defined on the interval I and, for all elements $(t_k, s_k)_{k=1}^n$ of $I \times \mathbb{R}$ with pairwise distinct first coordinates, there exists a unique member x of $\mathscr{B}_n(I)$ such that $x(t_k) = s_k$ for all $k = 1, \dots, n$.*

We shall refer to the members of Beckenbach families as *generalized lines*. Motivated by the reformulation above, we can attach a monotonicity notion to each Beckenbach family:

Definition. *A function $f\colon I \to \mathbb{R}$ is* generalized monotone *with respect to the Beckenbach family $\mathscr{B}_n(I)$ if, for all elements $t_1 < \dots < t_n$ of I, the following inequalities hold*

$$(-1)^{n-k}(f(t) - x(t)) \geq 0, \qquad t \in [t_k, t_{k+1}] \cap I, \quad k \in \{0, \dots, n\}$$

under the conventions $t_0 := \inf(I)$ and $t_{n+1} := \sup(I)$, where x denotes the unique generalized line of $\mathscr{B}_n(I)$ fulfilling the interpolation properties $x(t_1) = f(t_1), \dots, x(t_n) = f(t_n)$.

Linear Beckenbach families play a distinguished role in many fields of mathematics. Such an n-parameter family is the linear hull of continuous functions $\omega_1, \dots, \omega_n\colon I \to \mathbb{R}$ characterized by the independence property

$$\det\begin{pmatrix} \boldsymbol{\omega}_n(t_1) & \dots & \boldsymbol{\omega}_n(t_n) \end{pmatrix} > 0,$$

where $\boldsymbol{\omega}_n := (\omega_1, \dots, \omega_n)^T$ and the nodes $t_1 < \dots < t_n$ of I are arbitrary. A linear Beckenbach family is called a *Chebyshev system*. Considering a Chebyshev system as a vector instead of a linear hull is quite convenient and widely accepted in the technical literature. Following this convention, we will identify the Chebyshev system $\mathscr{L}(\boldsymbol{\omega}_n)$ by its positive base $\boldsymbol{\omega}_n$.

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We can describe the induced monotonicity in terms of an equivalent determinant inequality in case of Chebyshev systems. A function $f\colon I \to \mathbb{R}$ is monotone with respect to the Chebyshev system $\boldsymbol{\omega}_n$ if and only if, for all $t_0 \leq \dots \leq t_n$ in I, we have that

$$\det \begin{pmatrix} \boldsymbol{\omega}_n(t_0) & \dots & \boldsymbol{\omega}_n(t_n) \\ f(t_0) & \dots & f(t_n) \end{pmatrix} \geq 0.$$

Among Chebyshev systems, extended and complete ones have a particular importance. Instead of their rigorous definition we recall that they can be characterized via the positivity of their Wronskians' minors. (The presumed regularity is involved into the definition.) A typical example for an extended and complete n-parameter Chebyshev system is the vector space of polynomials $\mathscr{P}_n(I)$ of degree at most $(n-1)$. Using the vector formalism, the polynomial system is $\boldsymbol{\pi}_n(t) = (1, \dots, t^{n-1})$. In this case, we speak about *higher-order* or *polynomial monotonicity*. Observe that the higher-order monotonicity induced by $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are the notions of classical monotonicity and convexity.

The idea of extending convexity in the presented way can be traced back to Popoviciu [11]. However, he concentrated mostly on the particular polynomial case, while this case seems to appear first in the dissertation [6] by Hopf. Beckenbach studied only two-parameter families [1]. Inspired by Popoviciu and Beckenbach, Tornheim [14] was who revisited the general case and obtained fundamental results. For further interesting historical and important technical details, we refer to the work of Roberts and Varberg [12]. Our general reference on higher-order monotonicity, besides the pioneer work of Popoviciu [11], is the monograph [8] by Kuczma. Karlin and Studden [7] give an excellent overview on the theory of Chebyshev systems and their applications.

2. Motivations, aims, heuristics

Our first and most important motivating result is the *support property* stating that each convex function has a supporting line in any interior point of the domain.

Theorem. *If I is a real interval, $p \in I^\circ$, and $f\colon I \to \mathbb{R}$ is a convex function, then there exists and affine function $a\colon I \to \mathbb{R}$ such that $a(p) = f(p)$ and $a \leq f$.*

The second motivation is the inequality named after Hermite [5] and Hadamard [4]. This gives lower and upper estimations for the integral average of convex functions involving the endpoints and the midpoint of the underlying compact interval.

Theorem. *If $f\colon [a,b] \to \mathbb{R}$ is a convex function, then it is integrable and satisfies the inequality*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

We point out that these results have a tight interaction: if we regard the support and the chord properties as lower and upper support ones, then the Hermite–Hadamard inequality is a consequence of them.

Applying the methods of linear algebra and numerical analysis, the Hermite–Hadamard inequality can be extended for polynomial and Chebyshev systems [2]. The geometric approach via principal supports (discussed later) allows the same generalizations (see the papers [15] and [16] by Wąsowicz) and for Beckenbach families [3], as well. For further interesting details about the Hermite–Hadamard inequality, we refer to the survey [10] by Niculescu and Persson and to the paper [9] by Mitrinović and Lacković.

The problem of generalized supports is interesting by its own sight. Roughly speaking, we can formulate it in the following way (we give the precise details later). Let $\mathscr{B}_n(I)$ be a Beckenbach family. Fix arbitrarily k nodes in I, where $k \leq n$. In each nodes, prescribe a multiplicity that sum up to n. Now consider a generalized monotone function f. We are looking for a generalized line x which intersects or grazes f at a node according to the parity of the multiplicity: odd or even, and keeps sign with f between two consecutive nodes. The problem of generalized supports appears only in partial cases in the technical literature. Wąsowicz solved it for polynomial [15] and for extended and complete Chebyshev systems [16]. Although the paper [3] deals with Beckenbach systems, it presents results for special (principal) supports.

In this note, we give the complete solution of the support problem for arbitrary Beckenbach families. The heuristics of the approach is the following. Assume that the nodes and their multiplicities are given. Pick some extra points from sufficiently small neighborhoods of the nodes accordingly to the prescribed multiplicities. Then we can interpolate the generalized monotone function f with a unique generalized line x over the extended system of points. The generalized line x is almost appropriate: between consecutive neighborhoods, it keeps sign with f properly. Thus we expect that the support we are looking for can be obtained by shrinking the diameters of the neighborhoods to zero. The most delicate question at this point is the convergence of the method: The geometric intuitions have to be justified via the tools of analysis.

The efficient tool to handle this question is the uniform convergence theorem of Tornheim [14]. It states that similarly to the polynomial case, uniform convergence on compact subsets and over n-element discrete subsets are equivalent among sequences of n-parameter Beckenbach families. However, the direct application of this result is quite inconvenient. Thus we transform the Beckenbach $\mathscr{B}_n(I)$ into the polynomial system $\mathscr{P}_n(I)$. Tornheim's result guarantees that the transformation can be done with a homeomorphism. The convergence of the transformed process can be checked using the Newton form of the Lagrange interpolation polynomials. Finally, we transform back the limit in $\mathscr{P}_n(I)$ to the original system $\mathscr{B}_n(I)$ and obtain the desired support.

As applications, we recall and improve several known results on the topic, and present operator inequalities for generalized monotone functions. These inequalities may have further applications in numerical analysis and simultaneously provide a uniform treatment for all results presented in [2] and its corresponding references.

3. Auxiliary tools

It is well-known that in the vector space of polynomials of degree at most $(n-1)$ a sequence converges in n points if and only if it converges uniformly in compact subintervals. A highly nontrivial extension of this property for Beckenbach families is due to Tornheim [14].

Lemma 1. *Let $\mathscr{B}_n(I)$ be a Beckenbach family on a real interval I, and let $x \in \mathscr{B}_n(I)$ be a fixed generalized line. If a sequence of generalized lines (x_m) converges to x in n pairwise distinct points of I, then (x_m) converges uniformly to x on each compact subinterval of I.*

Let $\mathscr{B}_n(I)$ and $\mathscr{P}_n(I)$ be Beckenbach families on a real interval I, and let $R = \{r_1, \dots, r_n\}$ be a *reference set*, consisting pairwise distinct elements of I. Then the *exchange map* is defined by the following way:

$$\Phi \colon \mathscr{B}_n(I) \to \mathscr{P}_n(I), \qquad \Phi(x) := y \quad \text{where} \quad x{\restriction_R} = y{\restriction_R} .$$

The unique interpolation property guarantees that this definition is correct and that Φ is a bijection. The next lemma subsumes the most important analytical behavior of Beckenbach families and the exchange map.

Lemma 2. *Each Beckenbach family on $[a,b]$ is a closed metric subspace of $\mathscr{C}([a,b])$. Moreover, the exchange operator is a homeomorphism between two arbitrary n-parameter Beckenbach families.*

Proof. The first statement is an immediate consequence of Lemma 1. For the second statement, we show that Φ can be split into the composition of two homeomorphisms. Let $\mathscr{B}_n$ be an arbitrary Beckenbach family on $[a,b]$, and let $R=\{r_1,\dots,r_n\}$ be a reference set. Define $F_{\mathscr{B}_n}\colon \mathscr{B}_n\to\mathbb{R}^n$ by

$$F_{\mathscr{B}_n}(x):=\big(x(r)\big)_{r\in R}.$$

By the unique interpolation property, $F_{\mathscr{B}_n}$ is a bijection. Moreover, it is a homeomorphism by Lemma 1. If $\mathscr{P}_n$ is an other Beckenbach family on $[a,b]$, then take the homeomorphism $F_{\mathscr{P}_n}$. Then $\Phi=F_{\mathscr{B}_n}\circ F_{\mathscr{P}_n}^{-1}$ is a homeomorphism, as well. □

Assume that I is a real interval and $f\colon I\to\mathbb{R}$ is a given function. Then we can define the *divided differences* recursively in the usual way:

$$[t_0;f]:=f(t_0);\qquad [t_0,\dots,t_k;f]:=\frac{[t_0,\dots,t_{k-1};f]-[t_1,\dots,t_k;f]}{t_0-t_k}.$$

Here $\{t_0,\dots,t_k\}\subseteq I$ and the term $[t_0,\dots,t_k;f]$ containing $(k+1)$ nodes is called the *kth-order divided difference*. If $\{t_1,\dots,t_n\}$ consists of pairwise distinct elements of I, then there is a unique polynomial of degree at most $(n-1)$ that interpolates the set of graph points $\{(t_i,f(t_1))\}_{i=1}^n$. This polynomial $L_{n,f}$ is the *Lagrange interpolation polynomial*. The Newton expansion provides that we can express the Lagrange interpolation polynomial in terms of the divided differences:

$$L_{n,f}(t)=\sum_{i=1}^{n}\Big([t_1,\dots,t_i;f]\cdot\prod_{j=1}^{i-1}(t-t_j)\Big). \tag{1}$$

To consider the set of nodes $\{t_1,\dots,t_m\}$ as an ordered tuple $\tau=(t_1,\dots,t_m)$ turns out to be quite convenient. In this case, we call τ a *selection*. For $j\le m$, the selection $\tau^{(j)}=(t_1,\dots,t_j)$ is the *j-slice* of τ. The *contract* of $\tau_1=(t_{1j})_{j=1}^{m_1}$ and $\tau_2=(t_{2j})_{j=1}^{m_2}$ is the selection

$$(\tau_1,\tau_2)=(t_{11},\dots,t_{1m_1},t_{21},\dots,t_{2m_2}).$$

With these notations, we abbreviate divided differences as $[\tau;f]$. The expression $[\tau_1,\tau_2;f]$ means that the divided difference is built up on the contract (τ_1,τ_2). The next useful lemma states that divided differences on contracts can be represented in an algebra of functions with coefficients depending only on the individual parts of the contract.

Lemma 3. *Let H be a real subset of at least m elements, $f\colon H\to\mathbb{R}$, and $\tau_i=(t_{ij})_{j=1}^{m_i}$ be selections of H for all $i\in\{1,\dots,k\}$ such that $m_1+\dots+m_k=m$. Then*

$$[\tau_1,\dots,\tau_k;f]=\sum_{i=1}^{k}\sum_{j=1}^{m_i}[\tau_i^{(j)};f]\cdot L_{m-j}^{(i)}(\tau), \tag{2}$$

such that $L_{m-j}^{(i)}\in\mathscr{L}_{m-j}$, and $\mathscr{L}_j$ denotes the vector space generated by the j-term product of the members $\frac{1}{u-v}$, where u and v belong to different selections.

Proof. If $m = 1$, the statement holds evidently. Assume that, for a fixed $m \in \mathbb{N}$, the formula (2) holds for possible selections. Let H be a real subset of at least $(m+1)$ elements, and $t \in H$ be an element differing from the members of the selections τ_i. Define $t_0 = t$.

We have two possibilities: Either t represents a new individual selection $\tau_0 = (t_0)$, or we attach it to some existing selection τ_i. In the first case, define τ^* by

$$\tau^* := \begin{cases} \tau_0^* = \tau_0; \\ \tau_i^* = \tau_i \quad \text{if} \quad i \in \{1, \dots, k-1\}; \\ \tau_k^* = \tau_k^{(m_k-1)}. \end{cases}$$

By definition and by assumption,

$$[\tau_0, \tau; f] = \frac{[\tau^*; f] - [\tau; f]}{t_0 - t_{km_k}} = \sum_{i,j} [\tau_i^{*(j)}; f] \cdot \frac{L_{m-j}^{*(i)}(\tau)}{t_0 - t_{km_k}} - \sum_{i,j} [\tau_i^{(j)}; f] \cdot \frac{L_{m-j}^{(i)}(\tau)}{t_0 - t_{km_k}}. \tag{3}$$

The coefficients $[\tau_i^{(j)}; f]$ and $[\tau_i^{*(j)}; f]$ coincide in (3) for $i \in \{1, \dots, k\}$ and $j \neq m_k$. Thus if we aggregate the sums, this coinciding term multiplies the element

$$\frac{L_{m-j}^{*(i)}(\tau)}{t_0 - t_{km_k}} - \frac{L_{m-j}^{(i)}(\tau)}{t_0 - t_{km_k}}$$

which is a member of $\mathscr{L}_{m+1-j}$. The coefficients $[\tau_0; f]$ and $[\tau_k^{*(m_k)}; f]$ in the first and second summand multiply elements belonging to a proper vector space $\mathscr{L}_{m+1-j}$.

In the second case, we may assume that we insert t_0 to τ_1 by the symmetry of divided differences. Now define τ^* by

$$\tau^* := \begin{cases} \tau_1^* = (\tau_0, \tau_1) \\ \tau_i^* = \tau_i \quad \text{if} \quad i \in \{2, \dots, k-1\} \\ \tau_k^* = \tau_k^{(m_k-1)} \end{cases}$$

Then the coefficients $[\tau_i^{(j)}; f]$ and $[\tau_i^{*(j)}; f]$ coincide for $i \in \{2, \dots, k\}$ provided that $j \neq m_k$. Thus if we aggregate the sums in (3), the coinciding term multiplies an element of $\mathscr{L}_{m+1-j}$ as we have seen previously. The term $[\tau_k^{(m_k)}; f]$ is missing from first sum, and multiplies an element of $\mathscr{L}_{m+1-m_k}$, again. Finally, consider the index $k = 1$. Then by symmetry, $[\tau_1^*; f] = [\tau_1, \tau_0; f]$. Therefore $[\tau_1^{(j)}; f]$ and $[\tau_1^{*(j)}; f]$ coincide for indices $j \in \{1, \dots, m_1\}$, while the case corresponding to m_1+1 does not appear in the second sum. Thus we arrive at the desired conclusion with a similar argument again.

Under the convention $\sigma = (\tau_0, \tau_1, \dots, \tau_k)$, both cases can be written in the common form

$$[\sigma; f] = \sum_{i,j} [\sigma_i^{(j)}; f] \cdot L_{m+1-j}^{(i)}(\sigma).$$

This is exactly the formula we need and hence the proof is completed. □

4. THE MAIN RESULTS

Our main result provides the existence of generalized supports under some reasonable assumptions on the divided differences.

Theorem 1. *Let $\mathscr{B}_n(I)$ be an n-parameter Beckenbach family on an interval I. Assume that the multiplicities $m_1, \dots, m_k \in \mathbb{N}$ fulfill $m_1 + \cdots + m_k = n$, and $t_1 < \cdots < t_k$ are given points in I. Define $t_0 = \inf(I)$ and $t_{k+1} = \sup(I)$. Let further $m_0 = 0$ if $t_0 \neq t_1$. If $f\colon I \to \mathbb{R}$ is a $\mathscr{B}_n(I)$-monotone function whose divided differences are bounded at t_i up to order $(m_i - 1)$, then there exists $x \in \mathscr{B}_n(I)$ such that $x(t_i) = f(t_i)$ for all $i \in \{1, \dots, k\}$ and for all $i \in \{0, \dots, k\}$,*

$$(-1)^{n-(m_0+\cdots+m_i)}\big(f(t) - x(t)\big) \geq 0 \quad \textit{where} \quad t \in [t_i, t_{i+1}] \cap I.$$

Proof. Let $\varepsilon > 0$ be arbitrary such that the neighborhoods $U(t_i, \varepsilon)$ are pairwise disjoint for pairwise distinct indices. For each index i, fix a selection $\tau_i = (t_{ij})_{j=1}^{m_i}$ in $U(t_i, \varepsilon)$ such that $t_i \in \tau_i$ hold. Consider now the unique member $x_\varepsilon \in \mathscr{B}_n$ determined by the interpolation properties

$$x_\varepsilon(t_{ij}) = f(t_{ij}) \quad \text{for all} \quad i \in \{1, \dots, k\} \text{ and } j \in \{1, \dots, m_i\}.$$

This construction guarantees that the support property to be proved is "almost" valid in the following sense:

$$(-1)^{n-(m_0+\cdots+m_i)}\big(f(t) - x_\varepsilon(t)\big) \geq 0 \quad \text{for all} \quad t \in [t_i + \varepsilon, t_{i+1} - \varepsilon]. \tag{4}$$

Indeed, this formula holds evidently for $i = k$. Assume that we proved it for a fixed index $i + 1$ with $i < k$. Since $U(t_{i+1}, \varepsilon)$ contains exactly m_{i+1} nodes, $f - x_\varepsilon$ changes sign exactly m_{i+1} times by the monotonicity property of f. Thus the sign of $f - x_\varepsilon$ on $[t_i + \varepsilon, t_{i+1} - \varepsilon]$ is the expected one:

$$(-1)^{n-(m_0+\cdots+m_{i+1})}(-1)^{m_{i+1}} = (-1)^{n-(m_0+\cdots+m_i)}.$$

The set of polynomials $\mathscr{P}_n(I)$ of degree at most $(n-1)$ is an n-parameter Beckenbach family. Consider the exchange operator $\Phi\colon \mathscr{B}_n(I) \to \mathscr{P}_n(I)$, and define $y_\varepsilon = \Phi(x_\varepsilon)$. Since y_ε interpolates the set $\{(t_{ij}; f(t_{ij}))\}$ and belongs to $\mathscr{P}_n(I)$, it must be the Lagrange interpolation polynomial.

The Newton expansion (1) shows that the Lagrange polynomial can be expressed with divided differences supported by the slices of the selection $\tau = (\tau_1, \dots, \tau_k)$. On the other hand, formula (2) of Lemma 3 ensures that these slices can be expressed in terms of the divided differences supported by τ_i and by the elements of $\mathscr{L}_j$. These divided differences are bounded by assumption. The vector space $\mathscr{L}_j$ consists of finite products of the terms $\frac{1}{u-v}$, where u and v belong to different selections. Hence the elements of $\mathscr{L}_j$ are also bounded. Thus the Bolzano–Weierstrass Theorem allows us to choose a convergent subsequence from (y_ε) as ε shrinks to zero. For simplicity we may assume that $y_\varepsilon \to y_0$ as $\varepsilon \to 0$. Then y_0 belongs to $\mathscr{P}_n(I)$ obviously.

If $x_0 = \Phi^{-1}(y_0)$, then $x_0 \in \mathscr{B}_n(I)$ and $x_\varepsilon \to x_0$ as $\varepsilon \to 0$ by Lemma 2. Therefore we can take the limit $\varepsilon \to 0$ in (4), and arrive at the desired support property with $x = x_0$. $\square$

The element $x \in \mathscr{B}_n(I)$ in Theorem 1 is called an $(m_1, \dots, m_k)$-type *generalized support*. Let us present two straightforward consequences of Theorem 1. For the second one, a nontrivial fact is needed: If $\boldsymbol{\omega}_n$ is an extended and complete Chebyshev system, then each generalized monotone function admits finite one sided $(n-1)$ order derivatives at any interior point (see Karlin and Studden [7]). Hence the regularity assumptions of Theorem 1 are satisfied evidently.

Corollary 1. *Let $\boldsymbol{\omega}_n$ be an n-parameter Chebyshev system on an interval I. Assume that the multiplicities $m_1, \dots, m_k \in \mathbb{N}$ fulfill $m_1 + \cdots + m_k = n$, and $t_1 < \cdots < t_k$ are given points in I. Define $t_0 = \inf(I)$ and $t_{k+1} = \sup(I)$. Let further $m_0 = 0$ if $t_0 \neq t_1$. If $f\colon I \to \mathbb{R}$ is a $\boldsymbol{\omega}_n$-monotone function whose divided differences are bounded at t_i up to order $(m_i - 1)$, then there exists $\omega \in \boldsymbol{\omega}_n$ such that $\omega(t_i) = f(t_i)$ for all $i \in \{1, \dots, k\}$ and for all $i \in \{0, \dots, k\}$,*

$$(-1)^{n-(m_0+\cdots+m_i)}\big(f(t) - \omega(t)\big) \geq 0 \quad \textit{where} \quad t \in [t_i, t_{i+1}] \cap I.$$

Corollary 2. *Let $\boldsymbol{\omega}_n$ be an n-parameter extended and complete Chebyshev system on an interval I. Assume that the multiplicities $m_1, \dots, m_k \in \mathbb{N}$ fulfill $m_1 + \dots + m_k = n$, and $t_1 < \dots < t_k$ are given points in I. Define $t_0 = \inf(I)$ and $t_{k+1} = \sup(I)$. Let $m_0 = 0$ if $t_0 \neq t_1$; furthermore $m_0 = 1$ if $t_0 = t_1$ and $m_k = 1$ if $t_k = t_{k+1}$. If $f\colon I \to \mathbb{R}$ is a $\boldsymbol{\omega}_n$-monotone function, then there exists $\omega \in \boldsymbol{\omega}_n$ such that $\omega(t_i) = f(t_i)$ for all $i \in \{1, \dots, k\}$ and for all $i \in \{0, \dots, k\}$,*

$$(-1)^{n-(m_0+\dots+m_i)}\big(f(t) - \omega(t)\big) \geq 0 \quad \textit{where} \quad t \in [t_i, t_{i+1}] \cap I.$$

The same statement for the particular setting of polynomial systems:

Corollary 3. *Assume that the multiplicities $m_1, \dots, m_k \in \mathbb{N}$ fulfill $m_1 + \dots + m_k = n$, and $t_1 < \dots < t_k$ are given points in I. Define $t_0 = \inf(I)$ and $t_{k+1} = \sup(I)$. Let $m_0 = 0$ if $t_0 \neq t_1$; furthermore $m_0 = 1$ if $t_0 = t_1$ and $m_k = 1$ if $t_k = t_{k+1}$. If $f\colon I \to \mathbb{R}$ is a $\boldsymbol{\pi}_n$-monotone function, then there exists $p \in \boldsymbol{\pi}_n$ such that $p(t_i) = f(t_i)$ for all $i \in \{1, \dots, k\}$ and for all $i \in \{0, \dots, k\}$,*

$$(-1)^{n-(m_0+\dots+m_i)}\big(f(t) - p(t)\big) \geq 0 \quad \textit{where} \quad t \in [t_i, t_{i+1}] \cap I.$$

Note that Corollary 3 and Corollary 2 are the main results of the papers [15] and [16] by Wąsowicz. The approach in these papers is completely different from ours. Similarly, Theorem 1 extends the main result of [3], where purely geometric ideas were followed in order to establish the special case for multiplicities $m_i = 2$.

As it is well-known, the classical support property characterizes classical convexity. Thus the evident question arises: What is the situation in the extended case? It turns out that certain generalized supports characterize while others do not characterize the underlying generalized monotonicity notion even in the Chebyshev setting. For the precise details, we refer to Wąsowicz [16].

5. Applications

Those supports which keep the pointwise ordering with the supported function have an important role in numerical analysis. They can be used in numerical quadratures or estimating the integral. In particular, they provide a convenient tool to derive Hermite–Hadamard-type inequalities. The forthcoming applications are the abstract forms of such inequalities.

Theorem 2. *Let $\mathscr{B}_{2n+1}$ be a Beckenbach family on $[a, b]$, let $A\colon \mathscr{C}[a, b] \to \mathbb{R}$ be a monotone operator, and $k \leq n$ be a fixed natural. Assume that $\{a\} \subseteq \boldsymbol{\xi} \subseteq [a, b[$ and $\{b\} \subseteq \boldsymbol{\eta} \subseteq]a, b]$ are $(k+1)$-element sets, respectively, and $F_{\boldsymbol{\xi}}, F_{\boldsymbol{\eta}}\colon \mathscr{C}[a, b] \to \mathbb{R}$ are operators such that*

$$F_{\boldsymbol{\xi}}\restriction_{\mathscr{B}_{2n+1}} \leq A\restriction_{\mathscr{B}_{2n+1}} \leq F_{\boldsymbol{\eta}}\restriction_{\mathscr{B}_{2n+1}} \qquad \textit{and} \qquad \operatorname{supp} F_{\boldsymbol{\xi}} = \boldsymbol{\xi}, \quad \operatorname{supp} F_{\boldsymbol{\eta}} = \boldsymbol{\eta}.$$

If $f\colon [a, b] \to \mathbb{R}$ is a $\mathscr{B}_{2n+1}$-monotone function whose divided differences up to order $2\left\lceil \frac{n}{k} \right\rceil - 1$ are bounded on $]a, b[$, then

$$F_{\boldsymbol{\xi}}(f) \leq A(f) \leq F_{\boldsymbol{\eta}}(f).$$

Proof. Let $u = \left\lceil \frac{n}{k} \right\rceil$ and $v = ku - n$. Then $v \in \{0, \dots, k-1\}$. Now we specify $(k - v)$ and v points of $\boldsymbol{\xi} \cap]a, b[$ with multiplicities $m = 2u$ and $2u - 2$, respectively. Let the multiplicity of a be equal to 1. Then the sum of the total multiplicities is exactly $(2n + 1)$. By Theorem 1, there exist a generalized support $x \in \mathscr{B}_{2n+1}$ such that x and f coincide on $\boldsymbol{\xi}$ and $x \leq f$ on the entire interval $[a, b]$. Applying the support condition, the order property on $\mathscr{B}_{2n+1}$, and the monotonicity of A,

$$F_{\boldsymbol{\xi}}(f) = F_{\boldsymbol{\xi}}(x) \leq A(x) \leq A(f)$$

follows. The upper estimation can be proved in the same method. ∎

Theorem 3. *Let $\mathscr{B}_{2n}$ be a Beckenbach family on $[a,b]$, let $A\colon \mathscr{C}[a,b]\to\mathbb{R}$ be a monotone operator, and $k\le n$ be a fixed natural. Assume that $\boldsymbol{\xi}\subseteq]a,b[$ and $\{a,b\}\subseteq\boldsymbol{\eta}\subseteq[a,b]$ are k- and $(k+1)$-element sets, respectively, and $F_{\boldsymbol{\xi}}, F_{\boldsymbol{\eta}}\colon \mathscr{C}[a,b]\to\mathbb{R}$ are operators such that*

$$F_{\boldsymbol{\xi}}\restriction_{\mathscr{B}_{2n}}\le A\restriction_{\mathscr{B}_{2n}}\le F_{\boldsymbol{\eta}}\restriction_{\mathscr{B}_{2n}} \qquad \text{and} \qquad \operatorname{supp} F_{\boldsymbol{\xi}}=\boldsymbol{\xi},\quad \operatorname{supp} F_{\boldsymbol{\eta}}=\boldsymbol{\eta}.$$

If $f\colon[a,b]\to\mathbb{R}$ is a $\mathscr{B}_{2n}$-monotone function whose divided differences up to order $2\left\lceil\frac{n}{k}\right\rceil-1$ are bounded on $]a,b[$, then

$$F_{\boldsymbol{\xi}}(f)\le A(f)\le F_{\boldsymbol{\eta}}(f).$$

The proof of this result is similar to the previous one therefore we omit it. Although both theorems contains lower and upper estimations, one may formulate them as one-sided inequalities keeping only the relevant assumptions.

Now we present the special cases of Theorem 2 and Theorem 3 for *principal supports*. These supports contain as much multiplicities $m_i=2$ as possible. We shall need a fundamental result of Markov and Krein about the principal representation of moment spaces induced by Chebyshev systems (for precise details we refer to Karlin and Studden again [7]). We give a hint only to the left-hand side inequality of the odd order case.

Corollary 4. *Let $\boldsymbol{\omega}_{2n+1}$ be a Chebyshev system on $[a,b]$ and let $\rho\colon[a,b]\to\mathbb{R}$ be a positive integrable function. There exist uniquely determined base points $\xi_1,\dots,\xi_n$ and $\eta_1,\dots,\eta_n$ of $]a,b[$ furthermore positive coefficients $\alpha_0,\dots,\alpha_n$ and $\beta_1,\dots,\beta_{n+1}$ such that, for any $\boldsymbol{\omega}$-convex function $f\colon[a,b]\to\mathbb{R}$ whose first order divided differences are bounded on $]a,b[$,*

$$\alpha_0 f(a)+\sum_{j=1}^{n}\alpha_j f(\xi_j)\le\int_a^b f\rho\le\sum_{j=1}^{n}\beta_j f(\eta_j)+\beta_{n+1}f(b).$$

Hint. By the Krein–Markov Theorem, there exist uniquely determined base points $\xi_1,\dots,\xi_n$ in $]a,b[$ and positive coefficients $\alpha_0,\dots,\alpha_n$ such that

$$F_{\boldsymbol{\xi}}(\omega_j):=\alpha_0\omega_j(a)+\sum_{i=1}^{n}\alpha_i\omega_j(\xi_i)=\int_a^b\omega_j\rho=:A(\omega_j)$$

holds for all $\omega_j\in\boldsymbol{\omega}_{2n+1}$. Thus $F_{\boldsymbol{\xi}}\restriction_{\boldsymbol{\omega}_{2n+1}}=A\restriction_{\boldsymbol{\omega}_{2n+1}}$ by linearity of $F_{\boldsymbol{\xi}}$ and A. If $\boldsymbol{\xi}=\{a,\xi_1,\dots,\xi_n\}$ then $\operatorname{supp}\mathscr{F}_{\boldsymbol{\xi}}=\boldsymbol{\xi}$. Therefore the special case $k=n$ of Theorem 2 reduces to the statement. □

Corollary 5. *Let $\boldsymbol{\omega}_{2n}$ be a Chebyshev system on $[a,b]$ and $\rho\colon[a,b]\to\mathbb{R}$ be a positive integrable function. There exist uniquely determined base points $\xi_1,\dots,\xi_n$ and $\eta_1,\dots,\eta_{n-1}$ of $]a,b[$ furthermore positive coefficients $\alpha_1,\dots,\alpha_n$ and $\beta_0,\dots,\beta_n$ such that, for any $\boldsymbol{\omega}$-convex function $f\colon[a,b]\to\mathbb{R}$ whose first order divided differences are bounded on $]a,b[$,*

$$\sum_{j=1}^{n}\alpha_j f(\xi_j)\le\int_a^b f\rho\le\beta_0 f(a)+\sum_{j=1}^{n-1}\beta_j f(\eta_j)+\beta_n f(b).$$

The regularity theorem of Tornheim guarantees that a $\mathscr{B}_n$-monotone function is differentiable provided that the members are differentiable and $n\ge 3$ (see Tornheim's paper [14]). Under these extra assumptions on Chebyshev systems, the boundedness of divided differences in Corollary 4 and in Corollary 5 is automatically fulfilled. In fact, the algebraic approach presented in [2] ensures that boundedness can completely be canceled. The recent forms of the mentioned corollaries can be relaxed.

If the Chebyshev system is polynomial, then we can express the nodes of the corresponding Markov–Krein principal representations as the zeros of orthogonal polynomials. Similarly, the coefficients involved can be obtained via integral expressions containing the zeros and the polynomials. The principal representations are left- and right-hand side Radau quadratures $\mathcal{R}_{n;l}$ and $\mathcal{R}_{n;r}$ for the odd case and the Gauss and Lobatto quadratures $\mathcal{G}_n$ and $\mathcal{L}_{n-1}$ for the even case. For their exact form, we refer to [2]. Then Corollary 4 and Corollary 5 reduce to the next inequalities.

Corollary 6. *Any $\boldsymbol{\pi}_{2n+1}$-monotone function $f\colon [a,b] \to \mathbb{R}$ fulfills the inequalities*

$$\mathcal{R}_{n;l}(f) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \mathcal{R}_{n;r}(f).$$

Corollary 7. *Any $\boldsymbol{\pi}_{2n}$-monotone function $f\colon [a,b] \to \mathbb{R}$ fulfills the inequalities*

$$\mathcal{G}_n(f) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \mathcal{L}_{n-1}(f).$$

As one can easily check, the classical inequality of Hermite [5] and Hadamard [4] is the particular case of Corollary 7 for $n = 1$.

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