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Centralized Clearing Mechanisms in Financial Networks: A Programming Approach*

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Abstract

We consider financial networks where agents are linked to each other with financial contracts. A centralized clearing mechanism collects the initial endowments, the liabilities and the division rules of the agents and determines the payments to be made. A division rule specifies how the assets of the agents should be rationed, the four most common ones being the proportional, the priority, the constrained equal awards, and the constrained equal losses division rules. Since payments made depend on payments received, we are looking for solutions to a system of equations. The set of solutions is known to have a lattice structure, leading to the existence of a least and a greatest clearing payment matrix. Previous research has shown how decentralized clearing selects the least clearing payment matrix. We present a centralized approach towards clearing in order to select the greatest clearing payment matrix. To do so, we formulate the determination of the greatest clearing payment matrix as a programming problem. When agents use proportional division rules, this programming problem corresponds to a linear programming problem. We show that for the other common division rules, it can be written as an integer linear programming problem.

Keywords: Financial networks, systemic risk, bankruptcy rules, clearing, integer linear programming.

JEL Classification: C71, G10.

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1 Introduction

In this paper, we consider financial networks where agents are linked to each other with financial contracts. Like the seminal paper of Eisenberg and Noe (2001), a financial network consists of agents corresponding to the financial institutions, initial endowments, and liabilities. An agent's initial endowment includes all the agent's tangible and intangible assets but excludes the claims and liabilities the agent has towards the other agents. For outstanding surveys of this literature, we refer the reader to Glasserman and Young (2016) and Jackson and Pernoud (2021).

In Eisenberg and Noe (2001), agents use proportional division rules to determine payments in case of bankruptcy, i.e., payments are proportional to liabilities. Other division rules are important too. In practice, often priority principles are invoked, where a priority order determines the seniority of the liabilities. Given a permutation determining the rank of the claims, under the priority rule (see Moulin (2000) and Chatterjee and Eyigungor (2015)) claimants are paid in a lexicographic order determined by the permutation. Other important division rules are the constrained equal awards rule and the constrained equal losses rule (for updated surveys, see Thomson (2013) and Thomson (2015)). Under the constrained equal awards division rule,[1] all claimants get the same amount, up to the value of their claim. The constrained equal losses division rule is its dual and imposes that all claimants face the same loss, up to the value of their claim. The choice of the division rule may also balance the trade-off between welfare maximization and payoff equalization (Galice, 2019). Csóka and Herings (2018) note that on top of financial networks, default contagion can also occur in other applications (i.e. supply chains, international student exchange programs, servers processing job, time banks), where again other division rules may be in place. We therefore extend the Eisenberg and Noe (2001) framework and allow for general division rules.

In claims problems, there is a single, exogenously given, bankrupt agent and a division rule is used to determine the payments to the claimants. In financial networks, there can be multiple bankrupt agents. As an agent's asset value, and therefore payments made, depends on payments received, the actual payments are endogenously determined. Like the proportional rule for claims problems can be extended to financial networks (Eisenberg and Noe, 2001), it is possible to extend any division rule for claims problems to financial networks (Groote Schaarsberg, Reijnierse, and Borm, 2013). The resulting payment matrix consist of first computing each agent's asset value as the sum of the initial endowments and the payments received and next making the payments in accordance with the given division rule.

Following Csóka and Herings (2021), a so-called clearing payment matrix satisfies three

[1] For an axiomatization of its weighted version, see Flores-Szwagrzak (2015).

conditions. First, feasibility, which states that the payments are in accordance with the given division rules. Second, limited liability, which requires that the total payments made by an agent must never exceed the asset value of the agent. Third, priority of creditors, which expresses that default is only allowed if equity, i.e., asset value minus payments made, is equal to zero. Since payments made depend on payments received, the determination of a clearing payment matrix corresponds to the solution to a fixed point problem.

In this paper, we use the system of equations as introduced in Csóka and Herings (2021) to find a clearing payment matrix. The set of solutions to this system forms a complete lattice, which implies that there is a least and a greatest clearing payment matrix. Csóka and Herings (2018) show in a decentralized set-up that a large class of decentralized clearing processes converges to the least clearing payment matrix and Ketelaars and Borm (2021) derive an analogous result in a continuous set-up. In this paper, we therefore examine how a centralized approach can be used to select the greatest clearing payment matrix. More precisely, we present a programming problem whose unique solution is the greatest clearing payment matrix. The programming problem can be written as a linear programming problem when all agents use proportional division rules. For the other common division rules, we demonstrate how the programming problem reduces to an integer linear programming problem.

The rest of the paper is organized as follows. Section 2 defines financial networks and clearing payment matrices. Section 3 illustrates the possible multiplicity of clearing payment matrices and how multiplicity may vary with the division rules that are in place. Section 4 formulates the programming problems. Section 5 makes some concluding remarks.

2 Financial Networks

A *financial network* is a quadruple $F = (N, z, L, d)$ with the following interpretation.

The finite set N consists of the *agents* in the financial network.

The vector $z \in \mathbb{R}^N_+$ represents the *endowments* of the agents, which are non-negative real numbers. The endowments of an agent include all the agent's tangible and intangible assets, but exclude the claims the agent has on the other agents.

The non-negative *liability matrix* $L \in \mathbb{R}^{N \times N}_+$ describes the mutual claims of the agents. Its entry L_{ij} is the liability of agent $i \in N$ towards agent $j \in N$ or, equivalently, the claim of agent j on agent i. It is allowed that simultaneously agent j has a claim on agent i and agent i has a claim on agent j, so $L_{ij} > 0$ and $L_{ji} > 0$ can both hold at the same time. Agent do not have claims on themselves, so we set $L_{ii} = 0$ for every $i \in N$.

The determination of the payments to be made by the agents takes place by *division* *rules* $d = (d^i)_{i \in N}$. The division rule $d^i : \mathbb{R}_+ \to \mathbb{R}^N_+$ of agent $i \in N$ describes which payments agent i makes to the agents in N as a function of agent's i estate E_i. Payments are non-negative, bounded above by the liabilities, and are such that the sum of the payments is equal to the minimum of the estate and the sum of the liabilities, so it holds that, for every $E_i \in \mathbb{R}_+$, for every $j \in N$, $d^i_j(E_i) \leq L_{ij}$, and $\sum_{j \in N} d^i_j(E_i) = \min\{E_i, \sum_{j \in N} L_{ij}\}$. Moreover, for every $j \in N$, d^i_j is required to be weakly increasing in E_i. It is well-known that the weak monotonicity of d^i implies that it is continuous, see for instance Thomson (2003). The estate of an agent in a financial network depends on the payments received on outstanding claims and is therefore determined endogenously.

Important examples of division rules are the proportional, priority, constrained equal awards, and the constrained equal losses division rules.

The division rule d^i of agent $i \in N$ is equal to the *proportional division rule* if, for every $E_i \in \mathbb{R}_+$, it assigns to claimant $j \in N$ the amount

$$d^i_j(E_i) = \begin{cases} 0, & \text{if } L_{ij} = 0, \\ \min\{\frac{L_{ij}}{\sum_{k \in N} L_{ik}} E_i, L_{ij}\}, & \text{otherwise.} \end{cases}$$

Under the proportional division rule, the estate is divided in a proportional way over the claimants, up to the value of those claims.

The division rule d^i of agent $i \in N$ is equal to a *priority division rule* if there exists a permutation $\pi : N \to \{1, \ldots, |N|\}$, determining the rank of the claims, such that, for every $E_i \in \mathbb{R}_+$,

$$d^i_j(E_i) = \max\{0, \min\{L_{ij}, E_i - \sum_{k \in N | \pi(k) < \pi(j)} L_{ik}\}\},$$

where $\{k \in N | \pi(k) < \pi(j)\}$ is the set of agents ranked before j according to π. Under a priority division rule, claims are paid sequentially to agents $\pi^{-1}(1), \pi^{-1}(2), \ldots$ as long as the estate of agent i permits this.

We next define the constrained equal awards rule. Let $i \in N$. If $E_i > \sum_{j \in N} L_{ij}$, then define the award $\lambda_i = \max_{j \in N} L_{ij}$. Otherwise, define the award $\lambda_i \in [0, \max_{j \in N} L_{ij}]$ as the unique solution to

$$\sum_{j \in N} \min\{L_{ij}, \lambda_i\} = E_i.$$

The division rule d^i of agent i is equal to the *constrained equal awards division rule* if, for every $E_i \in \mathbb{R}_+$, it assigns to claimant $j \in N$ the amount

$$d^i_j(E_i) = \min\{L_{ij}, \lambda_i\}.$$

Under the constrained equal awards division rule, all claimants get the same amount, up to the value of their claim.

The constrained equal losses rule is the dual of the constrained equal awards rule. If $E_i > \sum_{j \in N} L_{ij}$, then define the loss $\mu_i = 0$. Otherwise, define the loss $\mu_i \in [0, \max_{j \in N} L_{ij}]$ as the unique solution to

$$\sum_{j \in N} \max\{L_{ij} - \mu_i, 0\} = E_i.$$

The division rule d^i of agent $i \in N$ is equal to the *constrained equal losses division rule* if, for every $E_i \in \mathbb{R}_+$, it assigns to claimant $j \in N$ the amount

$$d^i_j(E_i) = \max\{L_{ij} - \mu_i, 0\}.$$

Under the constrained equal losses division rule, all claimants face the same loss, up to the value of their claim.

The set of all matrices in $\mathbb{R}_+^{N \times N}$ with a zero diagonal is denoted by $\mathcal{M}$. The partial order $\leq$ on $\mathcal{M}$ is defined in the usual way: For $P, P' \in \mathcal{M}$ it holds that $P \leq P'$ if and only if $P_{ij} \leq P'_{ij}$ for all $(i,j) \in N \times N$. For $P \in \mathcal{M}$ and $i \in N$, let $P_i \in \mathbb{R}^N$ denote row i of P. For $P_i, P'_i \in \mathbb{R}^N$, we write $P_i < P'_i$ if $P_{ij} \leq P'_{ij}$ for all $j \in N$ and there is $k \in N$ such that $P_{ik} < P'_{ik}$.

Consider a financial network $F = (N, z, L, d)$. A *payment matrix* $P \in \mathcal{M}$ describes the mutual payments to be made by the agents, that is, P_{ij} is the monetary amount to be paid by agent $i \in N$ to agent $j \in N$. Given a payment matrix $P \in \mathcal{M}$, the *asset value* $a_i(P)$ of agent $i \in N$ is given by

$$a_i(P) = z_i + \sum_{j \in N} P_{ji}.$$

Subtracting the payments as made by an agent from the asset value yields an agent's equity. The *equity* $e_i(P)$ of an agent $i \in N$ is given by

$$e_i(P) = a_i(P) - \sum_{j \in N} P_{ij} = z_i + \sum_{j \in N} (P_{ji} - P_{ij}).$$

It follows immediately from the above expression that the sum over agents of their equities is the same as the sum over agents of their initial endowments. We have that

$$\sum_{i \in N} e_i(P) = \sum_{i \in N} z_i. \tag{2.1}$$

The analysis of financial networks is complicated because the mutual liability structure may result in contagion effects of default.

Our first aim is to define a clearing payment matrix. To do so, we define *feasible payments* of agent $i \in N$ as payments which belong to the image $d^i(\mathbb{R}_+)$ of the division rule d^i of agent i. A payment matrix is feasible if every row $i \in N$ of the matrix belongs to the feasible set of payments of agent i, that is, payments are made in accordance with the division rules. The set of *feasible payment matrices* $\mathcal{P}$ is therefore defined as

$$\mathcal{P} = \{P \in \mathcal{M} \mid \forall i \in N,\ P_i \in d^i(\mathbb{R}_+)\}.$$

The following definition of a clearing payment matrix is due to Csóka and Herings (2021). It extends the definition of Eisenberg and Noe (2001) for proportional division rules in a continuous setting. For a discrete setting with a smallest unit of account, see Csóka and Herings (2018).

Definition 2.1. The matrix $P \in \mathcal{M}$ is a *clearing payment matrix* of the financial network $F = (N, z, L, d)$ if it satisfies the following three properties:

1. *Feasibility:* $P \in \mathcal{P}$.
2. *Limited liability:* For every $i \in N$, $e_i(P) \geq 0$.
3. *Priority of creditors:* For every $i \in N$, if $P_i < L_i$, then $e_i(P) = 0$.

Limited liability requires all agents to end up with non-negative equity. Priority of creditors states that agents are only allowed to default if their equity is equal to zero.

Csóka and Herings (2021) prove the following result, which relates clearing payment matrices to the solution of a particular system of equations.

Theorem 2.2. *Let $F = (N, z, L, d)$ be a financial network. The payment matrix $P \in \mathcal{M}$ is a clearing payment matrix of F if and only if it solves the system of equations:*

$$P_{ij} = d^i_j(a_i(P)), \quad i, j \in N.$$

When calculating the clearing payment matrix as the solution to a system of equations, we take for every agent the value of the estate equal to the agent's asset value and next use the respective division rule to spend this asset value. Notice that agent $i \in N$ is treated as a claimant on its own estate $a_i(P)$ with a claim equal to $L_{ii} = 0$, so a clearing payment matrix P satisfies $P_{ii} = 0$.

3 Multiplicity of Clearing Payment Matrices

We start by presenting two examples to show that clearing payment matrices need not be unique.

Example 3.1. We consider a financial network $F = (N, z, L, d)$ with three agents $N = \{1, 2, 3\}$, zero endowments $z = (0, 0, 0)$, and a liability matrix equal to

$$L = \begin{bmatrix} 0 & 4 & 8 \\ 8 & 0 & 4 \\ 4 & 8 & 0 \end{bmatrix}.$$

We examine the possible clearing payment matrices for the four common specifications of division rules: proportional, priority, constrained equal awards, and constrained equal losses.

We start with some general observations. Let P be a clearing payment matrix of F. By Definition 2.1, a clearing payment matrix satisfies limited liability, so, for every $i \in N$, it holds that $e_i(P) \geq 0$. Since by Equation (2.1) the sum over all agents of their equities is equal to the sum over all agents of their initial endowments, so $\sum_{i \in N} e_i(P) = \sum_{i \in N} z_i = 0$, it follows that, for every $i \in N$, $e_i(P) = 0$. The condition of priority of creditors in Definition 2.1 is therefore automatically satisfied and to find a clearing payment matrix, we should therefore identify those payment matrices where all agents end up with zero equity while satisfying feasibility. A final observation is that in any clearing payment matrix the estates of the agents are all between 0 and 12.

Assume all agents use proportional division rules and let P be a clearing payment matrix of F. The estates of the agents satisfy the following system of equations:

$$E_i = \sum_{j \in N \setminus \{i\}} P_{ji} = \tfrac{2}{3} E_{i+1} + \tfrac{1}{3} E_{i-1}, \quad i \in N,$$

where we use the convention that agent 0 is identified with agent 3 and agent 4 with agent 1. It now follows from Gaussian elimination that $E_1 = E_2 = E_3$. Since estates of the agents are between 0 and 12, the set of clearing payment matrices when all agents use proportional division rules is given by

$$\mathcal{P}^{\text{prop}} = \{P \in \mathcal{M} \mid \exists E \in [0, 12],\ \forall i \in N,\ P_i = \tfrac{1}{12} E L_i\}.$$

There is a one-dimensional, convex set of clearing payment matrices, ranging from no payments at all to full payments by all agents.

Next assume all agents use priority division rules, where the permutation is chosen such that larger liabilities have priority. The estates of the agents now satisfy the equations

$$E_i = \min\{E_{i+1}, 8\} + \max\{E_{i-1} - 8, 0\}, \quad i \in N. \tag{3.1}$$

Suppose not all estates are equal. Let $j \in N$ be such that $E_j < E_{j+1}$. It follows from the system of equations in (3.1) that $E_j \geq 8$, since the equation corresponding to E_j cannot hold with equality if $E_j < 8$ and $E_j < E_{j+1}$. We also have that

$$E_{j+1} = \min\{E_{j+2}, 8\} + \max\{E_j - 8, 0\} \leq 8 + E_j - 8 = E_j,$$

a contradiction to $E_j < E_{j+1}$. Consequently, it follows that all estates are equal, so $0 \leq E_1 = E_2 = E_3 \leq 12$.

The set of clearing payment matrices when all agents use priority division rules with the highest claim having priority is therefore given by

$$\begin{aligned}\mathcal{P}^{\text{prior}} &= \{P \in \mathcal{M} \mid \exists E \in [0,8],\ \forall i \in N,\ P_{i,i-1} = E \text{ and } P_{i,i+1} = 0\}, \\ &\quad \cup \{P \in \mathcal{M} \mid \exists E \in [8,12],\ \forall i \in N,\ P_{i,i-1} = 8 \text{ and } P_{i,i+1} = E - 8\}.\end{aligned}$$

There is again a one-dimensional multiplicity of clearing payment matrices, ranging from no payments at all to full payments by all agents.

We now study the case of constrained equal award division rules. If the maximal estate across agents is less than or equal to 8, then the estates of the agents satisfy the following system of equations:

$$E_i = \sum_{j \in N \setminus \{i\}} P_{ji} = \tfrac{1}{2} E_{i+1} + \tfrac{1}{2} E_{i-1}, \quad i \in N.$$

It then follows that all estates must be equal, so $0 \leq E_1 = E_2 = E_3 \leq 8$. Any of these values of the estate generates a clearing payment matrix.

Next consider the case where the maximal estate across agents is strictly greater than 8. Let $j \in N$ be such that $E_j > 8$. Since $E_j = P_{j+1,j} + P_{j-1,j} \leq P_{j+1,j} + 4$, it holds that $P_{j+1,j} > 4$, so $E_{j+1} > 8$. We therefore find that all estates are strictly greater than 8. The system of equations becomes

$$E_i = E_{i+1} - 4 + 4 = E_{i+1}, \quad i \in N,$$

so solutions are given by $8 \leq E_1 = E_2 = E_3 \leq 12$. The set of clearing payment matrices when all agents use constrained equal awards division rules is therefore given by

$$\begin{aligned}\mathcal{P}^{\text{cea}} &= \{P \in \mathcal{M} \mid \exists E \in [0,8],\ \forall i \in N,\ P_{i,i-1} = \tfrac{1}{2} E \text{ and } P_{i,i+1} = \tfrac{1}{2} E\}, \\ &\quad \cup \{P \in \mathcal{M} \mid \exists E \in [8,12],\ \forall i \in N,\ P_{i,i-1} = E - 4 \text{ and } P_{i,i+1} = 4\}.\end{aligned}$$

We again find a one-dimensional multiplicity of clearing payment matrices, ranging from no payments to full payments.

We finally examine the constrained equal losses division rules. If the maximal estate across agents is less than or equal to 4, then the estates of the agents satisfy the following system of equations:

$$E_i = \sum_{j \in N \setminus \{i\}} P_{ji} = E_{i+1}, \quad i \in N,$$

so solutions are given by $0 \leq E_1 = E_2 = E_3 \leq 4$. Consider next the case where at least one estate, say E_j, exceeds 4. Then agent j makes a payment greater than 4 to agent $j-1$, so

E_{j-1} exceeds 4. It now follows that all estates exceed 4. We obtain the following system of equations:

$$E_i = 4 + \tfrac{1}{2}(E_{i+1} - 4) + \tfrac{1}{2}(E_{i-1} - 4) = \tfrac{1}{2}E_{i+1} + \tfrac{1}{2}E_{i-1},$$

so solutions are given by $4 \leq E_1 = E_2 = E_3 \leq 12$. The set of clearing payment matrices when all agents use constrained equal losses division rules is therefore given by

$$\begin{aligned} \mathcal{P}^{\text{cel}} &= \{P \in \mathcal{M} \mid \exists E \in [0,4],\ \forall i \in N,\ P_{i,i-1} = E \text{ and } P_{i,i+1} = 0\}, \\ &\quad \cup \{P \in \mathcal{M} \mid \exists E \in [4,12],\ \forall i \in N,\ P_{i,i-1} = \tfrac{1}{2}E + 2 \text{ and } P_{i,i+1} = \tfrac{1}{2}E - 2\}. \end{aligned}$$

Again a one-dimensional multiplicity of payment matrices results, which ranges from a least to a greatest clearing payment matrix. △

The next example shows that multiplicity of clearing payment matrices may depend on the division rules that are being used. This example also demonstrates the possibility of multiple clearing payment matrices when all agents have strictly positive endowments.

Example 3.2. We consider a financial network $F = (N, z, L, d)$ with three agents $N = \{1, 2, 3\}$, endowments $z = (3, 6, 7)$, and a liability matrix equal to

$$L = \begin{bmatrix} 0 & 6 & 4 \\ 12 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

The highest possible asset value of agent 2 results when agent 1 pays the full liability $L_{12} = 6$ to agent 2, which leads to asset value $a_2(P) = z_2 + L_{12} = 6 + 6 = 12$ of agent 2. Since agent 2 has liabilities of 12 towards agent 1 and liabilities of 5 towards agent 3, agent 2 will always default and end up with zero equity due to priority of creditors, irrespective of the division rules in place.

We next examine the set of clearing payment matrices for the four most common specifications of division rules and start with the case of proportional division rules. From the system of equations presented in Theorem 2.2, we have that

$$\begin{aligned} P_{12} &= \min\{\tfrac{3}{5}(3 + P_{21}), 6\}, \\ P_{21} &= \tfrac{12}{17}(6 + P_{12}). \end{aligned}$$

This system of equations has $P_{12} = 6$ and $P_{21} = 144/17$ as its unique solution. We find that the unique clearing payment matrix in the presence of proportional division rules and the resulting vector of equities are given by

$$P^{\text{prop}} \approx \begin{bmatrix} 0 & 6 & 4 \\ 8.47 & 0 & 3.53 \\ 0 & 0 & 0 \end{bmatrix} \qquad e(P^{\text{prop}}) \approx \begin{bmatrix} 1.47 \\ 0.00 \\ 14.53 \end{bmatrix}.$$

In case of priority division rules with higher claims having priority, Theorem 2.2 leads to the following two equations:

$$\begin{aligned} P_{12} &= \min\{3 + P_{21}, 6\}, \\ P_{21} &= 6 + P_{12}. \end{aligned}$$

The unique solution is given by $P_{12} = 6$ and $P_{21} = 12$. We find that with priority division rules the unique clearing payment matrix and resulting vector of equities are given by

$$P^{\text{prior}} = \begin{bmatrix} 0 & 6 & 4 \\ 12 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad e(P^{\text{prior}}) = \begin{bmatrix} 5 \\ 0 \\ 11 \end{bmatrix}.$$

We continue with the examination of constrained equal awards division rules.

Suppose there is a clearing payment matrix such that the asset value of agent 1 is below 8. This implies $P_{21} = a_1(P) - z_1 < 8 - 3 = 5$. When using the constrained equal awards division rule, agent 2 only makes a payment to agent 1 less than 5 if the asset value of agent 2 is below 10. Theorem 2.2 yields the following equations:

$$\begin{aligned} P_{12} &= \tfrac{1}{2}(3 + P_{21}), \\ P_{21} &= \tfrac{1}{2}(6 + P_{12}). \end{aligned}$$

The only solution to this system of equations has $P_{12} = 4$ and $P_{21} = 5$, which is incompatible with an asset value of agent 1 below 8. Consequently, any clearing payment matrix results in an asset value of agent 1 of at least 8.

We next examine the existence of a clearing payment matrix where the asset value of agent 1 is at least equal to 8. To obtain such an asset value, the payment of agent 2 to agent 1 must at least be equal to 5. Under the constrained equal awards division rule, the asset value of agent 2 must then at least be equal to 10. The result of Theorem 2.2 gives rise to the following two equations:

$$\begin{aligned} P_{12} &= \min\{3 + P_{21} - 4, 6\}, \\ P_{21} &= 6 + P_{12} - 5 = P_{12} + 1. \end{aligned}$$

We find a continuum of solutions, with the value of P_{12} ranging between 4 and 6 and $P_{21} = P_{12} + 1$. For every $E \in [8, 10]$, we obtain a clearing payment matrix and resulting equities

$$P^{\text{cea}} = \begin{bmatrix} 0 & E-4 & 4 \\ E-3 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \qquad e(P^{\text{cea}}) = \begin{bmatrix} 0 \\ 0 \\ 16 \end{bmatrix}.$$

We conclude with the case of constrained equal losses division rules. Agent 1 has endowments equal to 3, makes at least a payment of 2 to agent 2, so will achieve equal losses on the payments to agents 2 and 3. The asset value of agent 2 is at least equal to 8, so also agent 2 will achieve equal losses on the payments to agents 1 and 3. Theorem 2.2 gives rise to the following two equations:

$$\begin{aligned} P_{12} &= \min\{2 + \tfrac{1}{2}(3 + P_{21} - 2), 6\} = \min\{2\tfrac{1}{2} + \tfrac{1}{2}P_{21}, 6\}, \\ P_{21} &= 7 + \tfrac{1}{2}(6 + P_{12} - 7) = 6\tfrac{1}{2} + \tfrac{1}{2}P_{12}. \end{aligned}$$

The unique solution is given by $P_{12} = 6$ and $P_{21} = 19/2$. We obtain the following market clearing payment matrix and corresponding equity:

$$P^{\text{cel}} = \begin{bmatrix} 0 & 6 & 4 \\ 9.5 & 0 & 2.5 \\ 0 & 0 & 0 \end{bmatrix} \qquad e(P^{\text{prior}}) = \begin{bmatrix} 2.5 \\ 0.0 \\ 13.5 \end{bmatrix}.$$

$\triangle$

In Example 3.2, different division rules imply significantly different structural properties as far as clearing payment matrices are concerned. Constrained equal award division rules lead to a one-dimensional multiplicity of clearing payment matrices, whereas the clearing payment matrix is uniquely determined under the other division rules. Agent 1 defaults in almost all clearing payment matrices for constrained equal awards division rules, but not when any of the other division rules are used. Agent 2 fully defaults with respect to agent 3 under the priority division rule, fully pays the liability to agent 3 under constrained equal awards rules, whereas the claim of agent 3 on agent 2 is partially paid for under the other division rules.

A complete lattice is a partially ordered non-empty set in which every non-empty subset has a supremum and an infimum. In both Example 3.1 and in Example 3.2, the set of clearing payments matrices is a complete lattice. This turns out to be a general result as has been shown in Csóka and Herings (2021).

Theorem 3.3. *Let $F = (N, z, L, d)$ be a financial network. The set of clearing payment matrices of F is a complete lattice. In particular, there exists a least clearing payment matrix P^- and a greatest clearing payment matrix P^+.*

Eisenberg and Noe (2001) have shown Theorem 3.3 for the case of proportional division rules. Csóka and Herings (2018) prove a similar result in a discrete set-up.

4 Centralized Clearing as a Programming Problem

Csóka and Herings (2018) show in a discrete set-up that decentralized clearing results in the least clearing payment matrix. Ketelaars and Borm (2021) consider the continuous set-up and show that decentralized clearing processes converge to the least clearing payment matrix under mild conditions. Consider a decentralized clearing process where all agents simultaneously make the largest payments that are compatible with their cash at hand. In Example 3.1 all agents start with zero endowments, there are no positive feasible payments, and the decentralized clearing process stops at the least clearing payment matrix with zero payments. For the case of constrained equal awards division rules in Example 3.2, the decentralized clearing process is illustrated in Table 1.

z	L			P^1			P^2			$\ldots$	P^{10}			$\ldots$	P^-		
3	0	6	4	0	1.5	1.5	0	3	3	$\ldots$	0	3.996	3.996	$\ldots$	0	4	4
6	12	0	5	3	0	3	3.75	0	3.75	$\ldots$	4.995	0	4.995	$\ldots$	5	0	5
7	0	0	0	0	0	0	0	0	0	$\ldots$	0	0	0	$\ldots$	0	0	0

Table 1: The sequence of payment matrices using constrained equal awards division rules in Example 3.2.

In P^1, both agent 1 and agent 2 make equal payments to their creditors. Then, the new asset value of agent 1 becomes 3 and the new asset value of agent 2 becomes 1.5. In the next iteration, agents again make additional payments $P^2 - P^1$ in accordance with the constrained equal awards division rules. The payment matrices along the sequence show total payments made so far. P^{10} is rounded to three decimals. The process takes infinitely many steps and converges to the least clearing payment matrix P^-.

As Examples 3.1 and 3.2 demonstrate, the amount of default can be significantly higher in the least clearing payment matrix than in the greatest clearing payment matrix. This triggers the natural question whether it is possible to find the greatest clearing payment matrix. Since decentralized clearing processes end up in the least clearing payment matrix, doing so requires a centralized approach. We show in this section that the greatest clearing payment matrix can be found by solving a particular maximization problem. For the division rules considered in this paper, this maximization problem can be written as a linear programming problem or an integer linear programming problem.

Throughout this section, $\mathbb{1}$ denotes a vector of ones of appropriate dimension. Theorems 4.1, 4.3, and 4.5 correspond to unpublished parts of Csóka and Herings (2017).

Theorem 4.1. *Let $F = (N, z, L, d)$ be a financial network. The greatest clearing payment matrix of F is the unique solution to the following maximization problem:*

$$\begin{aligned} &\max_{P \in \mathcal{P}} \textstyle\sum_{i \in N} \sum_{j \in N} P_{ij}, \\ &\textit{subject to} \\ &z + P^\top \mathbb{1} - P\mathbb{1} \geq 0. \end{aligned} \tag{4.1}$$

Proof. Let P' be a solution to (4.1) and let some $i \in N$ be given. We show that $P'_i = d^i(a_i(P'))$.

If $P'_i = L_i$, then we have that

$$a_i(P') = z_i + \sum_{j \in N} P'_{ji} \geq \sum_{j \in N} P'_{ij} = \sum_{j \in N} L_{ij},$$

where the inequality follows from (4.1). From the definition of a division rule, it follows that $d^i(a_i(P')) = L_i$, so it holds that $P'_i = d^i(a_i(P'))$.

Consider the case $P'_i < L_i$. We show that $e_i(P') = 0$. Suppose $e_i(P') > 0$. Since $P' \in \mathcal{P}$ there exists $E' \in \mathbb{R}_+$ such that $P'_i = d^i(E')$. Since d^i is continuous and $e_i(P') > 0$ there exists $\varepsilon > 0$ such that

$$z_i + \textstyle\sum_{j \in N} P'_{ji} - \sum_{j \in N} d^i_j(E' + \varepsilon) \geq 0.$$

The payment matrix $P'' \in \mathcal{P}$ defined by

$$\begin{aligned} P''_i &= d^i(E' + \varepsilon), & \\ P''_j &= P'_j, & j \neq i, \end{aligned}$$

satisfies the constraints in (4.1) and leads to a strictly higher value of the objective function than P', a contradiction. Consequently, it holds that $e_i(P') = 0$.

Since $P' \in \mathcal{P}$ there exists $E' \in \mathbb{R}_+$ such that $P'_i = d^i(E')$ and from $P'_i < L_i$ and the definition of a division rule, we have $\sum_{j \in N} d^i_j(E') = E'$. Since $e_i(P') = 0$, we therefore have that

$$E' = \sum_{j \in N} d^i_j(E') = \sum_{j \in N} P'_{ij} = z_i + \sum_{j \in N} P'_{ji} = a_i(P').$$

It follows that $P'_i = d^i(E') = d^i(a_i(P'))$.

We use Theorem 2.2 to conclude that P' is a clearing payment matrix.

Let P be any clearing payment matrix. By feasibility, it holds that $P \in \mathcal{P}$. By limited liability, it holds that, for every $i \in N$,

$$e_i(P) = z_i + \sum_{j \in N} P_{ji} - \sum_{j \in N} P_{ij} \geq 0.$$

Any clearing payment matrix therefore satisfies the constraints in (4.1). We have that P' is the clearing payment matrix with the largest sum of the payments made, so we can use Theorem 3.3 to conclude that P' must be the greatest clearing payment matrix. □

The maximization over $P \in \mathcal{P}$ in the program (4.1) guarantees that payments are feasible. The constraint ensures that no agent ends up with negative equity. The property that an agent is not allowed to default when having positive equity follows from the fact that the solution maximizes the objective function. Otherwise, it would be possible to increase the value of the objective function by having the defaulting agent make additional payments. The maximization of the objective function also guarantees that the greatest clearing payment matrix is selected.

When the financial network has proportional division rules, the greatest clearing payment matrix can be found as the solution to a linear programming problem. The following result has been shown in Eisenberg and Noe (2001). It follows as a special case of Theorem 4.1 when the feasibility constraint $P \in \mathcal{P}$ is replaced by explicit constraints that ensure payments are made according to proportional division rules.

Theorem 4.2. *Let $F = (N, z, L, d)$ be a financial network with proportional division rules. The greatest clearing payment matrix of F is the unique solution to the following linear programming problem:*

$$\begin{array}{ll} \max_{P \in \mathbb{R}_+^{N \times N}, \lambda \in \mathbb{R}_+^N} \sum_{i \in N} \sum_{j \in N} P_{ij}, & \\ \textit{subject to} & \\ P_{ij} = \lambda_i L_{ij}, & i, j \in N, \\ \lambda_i \leq 1, & i \in N, \\ z + P^\top \mathbb{1} - P \mathbb{1} \geq 0. & \end{array} \tag{4.2}$$

The first and second constraint in the linear program (4.2) guarantee that payments are proportional to the liabilities and at most equal to those liabilities. These constraints replace the requirement $P \in \mathcal{P}$ of maximization problem (4.1). Demange (2018) uses a similar program to create a threat index by calculating the marginal effects of endowment increases.

Also for constrained equal awards division rules, we can replace the requirement $P \in \mathcal{P}$ of the program in (4.1) by a set of simple constraints. We define, for every $i \in N$, $\overline{L}_i = \max_{j \in N} L_{ij}$. Using Theorem 4.1, the following result follows in a straightforward way.

Theorem 4.3. *Let $F = (N, z, L, d)$ be a financial network with constrained equal awards division rules. The greatest clearing payment matrix of F is the unique solution P^+ to the following maximization problem:*

$$\begin{array}{ll} \max_{P \in \mathbb{R}_+^{N \times N}, \lambda \in \mathbb{R}_+^N} \sum_{i \in N} \sum_{j \in N} P_{ij}, & \\ \textit{subject to} & \\ P_{ij} = \min\{\lambda_i, L_{ij}\}, & i, j \in N, \\ \lambda_i \leq \overline{L}_i, & i \in N, \\ z + P^\top \mathbb{1} - P \mathbb{1} \geq 0. & \end{array} \tag{4.3}$$

The program in (4.3) maximizes the total payments as made by the agents subject to three conditions. The first condition expresses that agent i pays all of its claimants the amount λ_i, except when λ_i would exceed the value of the claim. This yields the feasibility condition of clearing payment matrices under the constrained equal awards rule. The second condition serves to pin down a unique value of λ_i in all circumstances. It is possible to omit this constraint in the optimization problem, although one loses the property that λ_i is uniquely determined as well as the interpretation of λ_i as the highest payment made by agent i. The third condition requires that no agent ends up with negative equity.

It is well-known that the constraint in (4.3) involving a minimum operator can be avoided by introducing binary decision variables q_{ij} for every $i, j \in N$. If $q_{ij} = 0$, then the payment P_{ij} is equal to L_{ij} and if $q_{ij} = 1$, then P_{ij} is equal to $\lambda_i \leq L_{ij}$. This leads to the following result.

Theorem 4.4. *Let $F = (N, z, L, d)$ be a financial network with constrained equal awards division rules. The greatest clearing payment matrix of F is the unique solution P^+ to the following integer linear programming problem:*

$$
\begin{array}{ll}
\max_{P \in \mathbb{R}^{N\times N}_+, \lambda \in \mathbb{R}^N_+, q \in \{0,1\}^{N\times N}} \sum_{i\in N} \sum_{j\in N} P_{ij}, & \\
\textit{subject to} & \\
P_{ij} \leq \lambda_i, & i, j \in N, \\
P_{ij} \leq L_{ij}, & i, j \in N, \\
P_{ij} \geq \lambda_i - \overline{L}_i(1 - q_{ij}), & i, j \in N, \\
P_{ij} \geq L_{ij} - \overline{L}_i q_{ij}, & i, j \in N, \\
\lambda_i \leq \overline{L}_i, & i \in N, \\
z + P^\top \mathbb{1} - P\mathbb{1} \geq 0. &
\end{array}
\tag{4.4}
$$

Proof. We show first that any $(P, \lambda, q) \in \mathbb{R}^{N\times N}_+ \times \mathbb{R}^N_+ \times \{0, 1\}^{N\times N}$ satisfying the constraints in (4.4) is such that, for every $i, j \in N$, $P_{ij} = \min\{\lambda_i, L_{ij}\}$. We show next that for any $(P, \lambda) \in \mathbb{R}^{N\times N}_+ \times \mathbb{R}^N_+$ satisfying the constraints in (4.3) there is $q \in \{0, 1\}^{N\times N}$ such that (P, λ, q) satisfies the constraints in (4.4).

Let $(P, \lambda, q) \in \mathbb{R}^{N\times N}_+ \times \mathbb{R}^N_+ \times \{0, 1\}^{N\times N}$ satisfy the constraints in (4.4). Let $i, j \in N$ be such that $q_{ij} = 0$. The constraints $P_{ij} \leq \lambda_i$, $P_{ij} \leq L_{ij}$, and $P_{ij} \geq L_{ij} - \overline{L}_i q_{ij} = L_{ij}$ imply $P_{ij} = \min\{\lambda_i, L_{ij}\}$. Let $i, j \in N$ be such that $q_{ij} = 1$. The constraints $P_{ij} \leq \lambda_i$, $P_{ij} \leq L_{ij}$, and $P_{ij} \geq \lambda_i - \overline{L}_i(1 - q_{ij}) = \lambda_i$ imply $P_{ij} = \min\{\lambda_i, L_{ij}\}$.

Let $(P, \lambda) \in \mathbb{R}^{N\times N}_+ \times \mathbb{R}^N_+$ satisfy the constraints in (4.3). For every $i, j \in N$, if $P_{ij} < L_{ij}$, then define $q_{ij} = 1$, and if $P_{ij} = L_{ij}$, then define $q_{ij} = 0$. We show that (P, λ, q) satisfies the constraints in (4.4). Since $P_{ij} = \min\{\lambda_i, L_{ij}\}$, it follows that $P_{ij} \leq \lambda_i$ and $P_{ij} \leq L_{ij}$. If $P_{ij} < L_{ij}$, then $q_{ij} = 1$ and $P_{ij} = \lambda_i = \lambda_i - \overline{L}_i(1 - q_{ij})$. Clearly, it holds that

$P_{ij} \geq 0 \geq L_{ij} - \overline{L}_i$. If $P_{ij} = L_{ij}$, then $q_{ij} = 0$ and $P_{ij} \geq 0 \geq \lambda_i - \overline{L}_i = \lambda_i - \overline{L}_i(1 - q_{ij})$. It also holds that $P_{ij} = L_{ij} = L_{ij} - \overline{L}_i q_{ij}$. □

To obtain elegant formulations, we have treated all payments P_{ij} for $i, j \in N$ in the same way in the optimization problems (4.3) and (4.4). Of course, there is no need to introduce explicit variables P_{ij} and q_{ij} when $L_{ij} = 0$ since we can simply substitute $P_{ij} = 0$.

Also for the constrained equal losses rule, we can replace the requirement $P \in \mathcal{P}$ of the program in (4.1) by a set of simple constraints. Using Theorem 4.1, we obtain the following result in a straightforward way.

Theorem 4.5. *Let $F = (N, z, L, d)$ be a financial network with constrained equal losses division rules. The greatest clearing payment matrix of F is the unique solution to the following maximization problem:*

$$\begin{array}{lll}
\max_{P \in \mathbb{R}_+^{N \times N}, \mu \in \mathbb{R}_+^N} \sum_{i \in N} \sum_{j \in N} P_{ij}, & & \\
\textit{subject to} & & \\
P_{ij} = \max\{L_{ij} - \mu_i, 0\}, & i, j \in N, & \\
\mu_i \leq \overline{L}_i, & i \in N, & \\
z + P^\top \mathbb{1} - P\mathbb{1} \geq 0. & &
\end{array} \tag{4.5}$$

The program in (4.5) maximizes the total payments as made by the agents subject to three conditions. The first condition expresses that agent i pays all creditors the amount of their claim minus μ_i, except when μ_i exceeds the value of the claim. This corresponds to the feasibility condition of clearing payment matrices under the constrained equal losses rule. Similar to the case of constrained equal awards division rules, the second condition serves to pin down the value of μ_i. The only case where μ_i would not be uniquely determined without this constraint is when agent i does not make any payments in the greatest clearing payment matrix, a case that can only occur if $z_i = 0$ and i does not receive any payments from any of the other agents or if i does not have any creditors, both rather contrived situations. The third condition requires that no agent ends up with negative equity.

Similar to the case for constrained equal awards division rules, it is possible to avoid the constraint in (4.5) involving the maximum operator by introducing binary decision variables q_{ij} for every $i, j \in N$. If $q_{ij} = 0$, then the payment P_{ij} is equal to 0 and if $q_{ij} = 1$, then P_{ij} is equal to $L_{ij} - \mu_i$. This leads to the following result.

Theorem 4.6. *Let $F = (N, z, L, d)$ be a financial network with constrained equal losses division rules. The greatest clearing payment matrix of F is the unique solution P^+ to the* *following integer linear programming problem:*

$$
\begin{array}{ll}
\max_{P\in\mathbb{R}_+^{N\times N},\mu\in\mathbb{R}_+^N,q\in\{0,1\}^{N\times N}} \sum_{i\in N}\sum_{j\in N} P_{ij}, & \\
\textit{subject to} & \\
P_{ij} \geq L_{ij} - \mu_i, & i,j \in N, \\
P_{ij} \leq L_{ij} - \mu_i + \overline{L}_i(1-q_{ij}), & i,j \in N, \\
P_{ij} \leq \overline{L}_i q_{ij}, & i,j \in N, \\
\mu_i \leq \overline{L}_i, & i \in N, \\
z + P^\top \mathbb{1} - P\mathbb{1} \geq 0. &
\end{array}
\tag{4.6}
$$

Proof. We show first that any $(P,\mu,q) \in \mathbb{R}_+^{N\times N} \times \mathbb{R}_+^N \times \{0,1\}^{N\times N}$ satisfying the constraints in (4.6) is such that, for every $i,j \in N$, $P_{ij} = \max\{L_{ij}-\mu_i, 0\}$. We show next that for any $(P,\mu) \in \mathbb{R}_+^{N\times N} \times \mathbb{R}_+^N$ satisfying the constraints in (4.5) there is $q \in \{0,1\}^{N\times N}$ such that (P,μ,q) satisfies the constraints in (4.6).

Let $(P,\mu,q) \in \mathbb{R}_+^{N\times N} \times \mathbb{R}_+^N \times \{0,1\}^{N\times N}$ satisfy the constraints in (4.6). Let $i,j \in N$ be such that $q_{ij} = 0$. The constraints $P_{ij} \geq 0$, $P_{ij} \geq L_{ij} - \mu_i$, and $P_{ij} \leq \overline{L}_i q_{ij} = 0$ imply $P_{ij} = \max\{L_{ij}-\mu_i, 0\}$. Let $i,j \in N$ be such that $q_{ij} = 1$. The constraints $P_{ij} \geq 0$, $P_{ij} \geq L_{ij} - \mu_i$, and $P_{ij} \leq L_{ij} - \mu_i + \overline{L}_i(1-q_{ij}) = L_{ij} - \mu_i$ imply $P_{ij} = \max\{L_{ij}-\mu_i, 0\}$.

Let $(P,\mu) \in \mathbb{R}_+^{N\times N} \times \mathbb{R}_+^N$ satisfy the constraints in (4.5). For every $i,j \in N$, if $P_{ij} > 0$, then define $q_{ij} = 1$, and if $P_{ij} = 0$, then define $q_{ij} = 0$. We show that (P,μ,q) satisfies the constraints in (4.6). Since $P_{ij} = \max\{L_{ij}-\mu_i, 0\}$, it follows that $P_{ij} \geq L_{ij} - \mu_i$. If $P_{ij} > 0$, then $q_{ij} = 1$ and $P_{ij} = L_{ij} - \mu_i = L_{ij} - \mu_i + \overline{L}_i(1-q_{ij})$. Clearly, it holds that $P_{ij} = L_{ij} - \mu_i \leq L_{ij} \leq \overline{L}_i = \overline{L}_i q_{ij}$. If $P_{ij} = 0$, then $q_{ij} = 0$ and $P_{ij} = 0 \leq \overline{L}_i - \mu_i \leq L_{ij} - \mu_i + \overline{L}_i = L_{ij} - \mu_i + \overline{L}_i(1-q_{ij})$. It also holds that $P_{ij} = 0 = \overline{L}_i q_{ij}$. □

We finally turn to priority division rules. The following result follows immediately from Theorem 4.1.

Theorem 4.7. *Let $F = (N,z,L,d)$ be a financial network with priority division rules. The greatest clearing payment matrix of F is the unique solution P^+ to the following maximization problem:*

$$
\begin{array}{ll}
\max_{P\in\mathbb{R}_+^{N\times N},E\in\mathbb{R}_+^N} \sum_{i\in N}\sum_{j\in N} P_{ij}, & \\
\textit{subject to} & \\
P_{ij} = \max\{0, \min\{E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}, L_{ij}\}\}, & i,j \in N, \\
E_i \leq \sum_{k\in N} L_{ik}, & i \in N, \\
z + P^\top \mathbb{1} - P\mathbb{1} \geq 0. &
\end{array}
\tag{4.7}
$$

The program in (4.7) maximizes the total payments as made by the agents subject to three conditions. The first condition expresses that agent i pays all creditors at most their claim or what is left after creditors having priority are paid off. The maximum makes sure that in case the latter amount is negative, no payments are made. This corresponds to the feasibility condition of clearing payment matrices under the priority rule. The second condition serves to pin down the value of E_i for solvent agents. The third condition requires that no agent ends up with negative equity.

The next result demonstrates that the problem of finding the greatest clearing payment matrix can be written as an integer linear programming problem as well in the case of priority division rules.

Theorem 4.8. *Let $F = (N, z, L, d)$ be a financial network with priority division rules. The greatest clearing payment matrix of F is the unique solution P^+ to the following integer linear programming problem:*

$$\max_{P \in \mathbb{R}_+^{N\times N}, E \in \mathbb{R}_+^N, q \in \{0,1\}^{N\times N}, r \in \{0,1\}^{N\times N}} \sum_{i\in N}\sum_{j\in N} P_{ij}, \tag{4.8}$$

subject to

$$P_{ij} \leq L_{ij}, \quad i,j \in N, \tag{4.9}$$

$$P_{ij} \leq E_i - \sum_{k\in N \mid \pi(k)<\pi(j)} L_{ik} + \sum_{k\in N} L_{ik}(1-q_{ij}), \quad i,j \in N, \tag{4.10}$$

$$P_{ij} \leq \overline{L}_i q_{ij}, \quad i,j \in N, \tag{4.11}$$

$$P_{ij} \geq E_i - \sum_{k\in N \mid \pi(k)<\pi(j)} L_{ik} - \sum_{k\in N} L_{ik}(1-r_{ij}), \quad i,j \in N, \tag{4.12}$$

$$P_{ij} \geq L_{ij} - \overline{L}_i r_{ij}, \quad i,j \in N, \tag{4.13}$$

$$E_i \leq \sum_{k\in N} L_{ik}, \quad i \in N, \tag{4.14}$$

$$z + P^\top \mathbb{1} - P\mathbb{1} \geq 0. \tag{4.15}$$

Proof. We show first that any $(P, E, q, r) \in \mathbb{R}_+^{N\times N} \times \mathbb{R}_+^N \times \{0,1\}^{N\times N} \times \{0,1\}^{N\times N}$ satisfying the constraints in (4.8) is such that

$$P_{ij} = \max\{0, \min\{E_i - \sum_{k\in N \mid \pi(k)<\pi(j)} L_{ik}, L_{ij}\}\}, \quad i,j \in N.$$

We show next that for any $(P, E) \in \mathbb{R}_+^{N\times N} \times \mathbb{R}_+^N$ satisfying the constraints in (4.7) there is $(q, r) \in \{0,1\}^{N\times N} \times \{0,1\}^{N\times N}$ such that (P, E, q, r) satisfies the constraints in (4.8).

Let $(P, E, q, r) \in \mathbb{R}_+^{N\times N} \times \mathbb{R}_+^N \times \{0,1\}^{N\times N} \times \{0,1\}^{N\times N}$ satisfy the constraints in (4.8). Fix some $(i, j) \in N \times N$. We distinguish three cases.

Case 1. $P_{ij} = 0$.

If $L_{ij} = 0$, then it clearly holds that $P_{ij} = \max\{0, \min\{E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}, L_{ij}\}\}$. Assume $L_{ij} > 0$. It follows from (4.13) that $r_{ij} = 1$. From (4.12) we obtain that $P_{ij} \geq E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}$, so $P_{ij} = \max\{0, \min\{E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}, L_{ij}\}\}$ as desired.

Case 2. $0 < P_{ij} < L_{ij}$.

It follows from (4.11) that $q_{ij} = 1$ and from (4.13) that $r_{ij} = 1$. We use (4.10) and (4.12) to conclude that $P_{ij} = E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}$. We conclude that $P_{ij} = \max\{0, \min\{E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}, L_{ij}\}\}$.

Case 3. $0 < P_{ij} = L_{ij}$.

It follows from (4.11) that $q_{ij} = 1$, so from (4.10) that $P_{ij} \leq E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}$, and we conclude that $P_{ij} = \max\{0, \min\{E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}, L_{ij}\}\}$.

Let $(P, E) \in \mathbb{R}_+^{N\times N} \times \mathbb{R}_+^N$ satisfy the constraints in (4.7). For every $i, j \in N$, we define $q_{ij} \in \{0, 1\}$ and $r_{ij} \in \{0, 1\}$ as follows. If $P_{ij} = L_{ij} = 0$, then define $q_{ij} = r_{ij} = 0$. If $P_{ij} = 0 < L_{ij}$, then define $q_{ij} = 0$ and $r_{ij} = 1$. If $0 < P_{ij} < L_{ij}$, then define $q_{ij} = r_{ij} = 1$. Finally, if $0 < P_{ij} = L_{ij}$, then define $q_{ij} = 1$ and $r_{ij} = 0$. We verify next that the inequalities in (4.9)–(4.13) are satisfied. To do so, we fix $(i, j) \in N \times N$ and distinguish four cases.

Case 1. $P_{ij} = L_{ij} = 0$.

The inequalities in (4.9)–(4.13) reduce to

$$\begin{aligned}
&0 \leq 0,\\
&0 \leq E_i + \textstyle\sum_{k\in N|\pi(k)\geq\pi(j)} L_{ik},\\
&0 \leq 0,\\
&0 \geq E_i - \textstyle\sum_{k\in N|\pi(k)<\pi(j)} L_{ik} - \sum_{k\in N} L_{ik},\\
&0 \geq 0.
\end{aligned}$$

These equalities are clearly satisfied, where the fourth inequality uses the fact that $E_i \leq \sum_{k\in N} L_{ik}$.

Case 2. $P_{ij} = 0 < L_{ij}$.

The inequalities in (4.9)–(4.13) reduce to

$$\begin{aligned}
&0 \leq L_{ij},\\
&0 \leq E_i + \textstyle\sum_{k\in N|\pi(k)\geq\pi(j)} L_{ik},\\
&0 \leq 0,\\
&0 \geq E_i - \textstyle\sum_{k\in N|\pi(k)<\pi(j)} L_{ik},\\
&0 \geq L_{ij} - \overline{L}_i.
\end{aligned}$$

Since $0 = P_{ij} = \max\{0, \min\{E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}, L_{ij}\}\}$ and $L_{ij} > 0$, it follows that $E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik} \leq 0$, so the fourth inequality above holds. The other inequalities hold trivially.

Case 3. $0 < P_{ij} < L_{ij}$.

The inequalities in (4.9)–(4.13) reduce to

$$
\begin{aligned}
&P_{ij} \leq L_{ij},\\
&P_{ij} \leq E_i - \textstyle\sum_{k\in N|\pi(k)<\pi(j)} L_{ik},\\
&P_{ij} \leq \overline{L}_i,\\
&P_{ij} \geq E_i - \textstyle\sum_{k\in N|\pi(k)<\pi(j)} L_{ik},\\
&P_{ij} \geq L_{ij} - \overline{L}_i.
\end{aligned}
$$

Since $P_{ij} = \max\{0, \min\{E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}, L_{ij}\}\}$ and $0 < P_{ij} < L_{ij}$, it follows that $P_{ij} = E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}$. It is now easily verified that all inequalities above hold.

Case 4. $0 < P_{ij} = L_{ij}$.

The inequalities in (4.9)–(4.13) reduce to

$$
\begin{aligned}
&L_{ij} \leq L_{ij},\\
&L_{ij} \leq E_i - \textstyle\sum_{k\in N|\pi(k)<\pi(j)} L_{ik},\\
&L_{ij} \leq \overline{L}_i,\\
&L_{ij} \geq E_i - \textstyle\sum_{k\in N|\pi(k)<\pi(j)} L_{ik} - \sum_{k\in N} L_{ik},\\
&L_{ij} \geq L_{ij}.
\end{aligned}
$$

From $0 < L_{ij} = P_{ij} = \max\{0, \min\{E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik}, L_{ij}\}\}$ it follows that $E_i - \sum_{k\in N|\pi(k)<\pi(j)} L_{ik} \geq L_{ij}$. We have shown the second inequality above. The fourth inequality above follows from $E_i \leq \sum_{k\in N} L_{ik}$. The other inequalities are trivially true. $\square$

We assumed in Theorems 4.3–4.8 that all agents use the same division rule. A closer inspection of the proofs reveals that this feature is not used. We can therefore obtain similar results if agents use heterogeneous division rules within the classes of proportional, constrained equal awards, constrained equal losses, and priority division rules.

5 Conclusion

We consider financial networks with perfectly liquid non-negative endowments, liabilities, and agent-specific bankruptcy rules. The set of clearing payment matrices is a complete lattice, so has a least and a greatest element. We illustrate by means of examples that there can be infinitely many clearing payment matrices and that multiplicity of clearing payment matrices depends on the division rules that are in place. Previous research has shown that decentralized clearing leads to the selection of the least clearing payment matrix. We show how a centralized approach can be used to select the greatest clearing payment matrix. We present a programming approach to calculate the greatest clearing payment matrix. We also show that for proportional division rules, this programming problem can be written as a linear programming problem. For common division rules like constrained equal awards, constrained equal losses, and priority division rules, we show how the programming problem can be written as an integer linear programming problem.

There are many possibilities for further research. The Eisenberg and Noe (2001) model has been extended in various ways. For setups with default costs, see Rogers and Veraart (2013), Roukny, Battiston, and Stiglitz (2018), and Jackson and Pernoud (2020). Schuldenzucker, Seuken, and Battiston (2020) introduce credit default swaps and show how these can lead to multiplicity of clearing payment matrices. Cifuentes, Ferrucci, and Shin (2005) analyze a related direct externality in financial networks, when agents' endowments also contain one illiquid asset. Defaulting agents firesale the illiquid asset, which reduces the other agents' value of endowments as well. In this setting, Amini, Filipović, and Minca (2016) give conditions for uniqueness of the clearing payment matrix and the corresponding asset prices. Feinstein (2017) generalizes those conditions for multiple illiquid assets. If these conditions are not satisfied, there is again scope for multiplicity of clearing payment matrices. An examination of the trade-off between decentralized and centralized clearing is therefore highly relevant for the various extensions of the baseline model.

6 References

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