Habilitation thesis

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Contents

Overview of the results

The goal of this thesis dissertation is to give an account into two fields I have been working on in the last few years. In the papers collected here, I deal with extension, complementation, and decomposition type issues of positive and symmetric operators in the context of so called anti-dual dual pairs. In this chapter, we briefly present the content, motivation, and background of these articles. The precise definitions and results are found in the Summary of results chapter.

Positive operators on anti-dual pairs ([**42**, **57**])

One of the most natural questions that arises when dealing with partially defined objects in mathematics is whether there exists an extension that has some prescribed properties. A great many authors have studied abstract extension problems for operators on Hilbert spaces, that go at least back to M. G. Krein [**34**] and J. von Neumann [**38**]. (For various different developments of their groundbreaking work see e.g. [**6**–**8**, **11**, **26**, **27**], and the references therein.) The following extension problem was posed by Yu. L. Shmul'yan [**45**]: Assume that a positive operator $A : D \to \mathcal{H}$ is given, where D is a linear subspace of the complex Hilbert space $\mathcal{H}$. Positivity here means that $(Ax \,|\, x) \geq 0$ for all $x \in D$. The question is: under what conditions can we guarantee the existence of an everywhere defined bounded positive extension $\widetilde{A}$ of A? Of course, if there is any then A itself must be bounded. Hence, extending it to the closure by continuity, we may suppose that D is closed. Consider the matrix representation of A with respect to the orthogonal decomposition $\mathcal{H} = D \oplus D^{\perp}$

$$[A] = \begin{bmatrix} A_{11} & * \\ A_{21} & * \end{bmatrix},$$

where $A_{11} : D \to D$ and $A_{21} : D \to D^{\perp}$ arise in the usual way, whereas the second column (of symbols $*$) waits to be filled to obtain a positive operator. It is easy to see that every positive extension of A has representation of the form

$$[\widetilde{A}] = \begin{bmatrix} A_{11} & A_{21}^{*} \\ A_{21} & X \end{bmatrix}, \tag{1}$$

where $A_{11} : D \to D$ and $X : D^{\perp} \to D^{\perp}$ are positive. So, extending A to a positive operator $\widetilde{A}$ is equivalent to find $X \geq 0$ such that the operator matrix (1) is positive. (For a more general completion problem for block operators see [**9**, **58**].) This form also helps us to demonstrate that such an extension need not exist even in the simplest case. Indeed, assume that $\mathcal{H}$ is of the form $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ with a complex Hilbert space K and assume that A_{11} is the zero operator, while $A_{21} = A_{21}^{*}$ is any positive but nonzero bounded operator on K. Then an elementary calculation shows that there is no positive completion at all.

Our main aim was to develop a general extension theory which overcomes the problem of not having orthogonal decomposition when we drop the Hilbert space structure. This level of generality is indeed necessary in our considerations, because we intend to investigate extendibility of "positive" mappings acting on spaces without inner product. In order to introduce the appropriate analogues of standard operator classes, that cover the original Hilbert space setting, and is general enough to be applicable for objects like operator kernels and representable functionals, we are going to consider anti-dual pairs. That is, two appropriately chosen complex linear spaces intertwined by a separating sesquilinear map, called anti-duality. This is a slight modification of the well known dual pair setting, the only difference we make is the conjugate linearity in the second variable. Having an anti-duality at hand also allows us to introduce various topologies, and hence continuity and boundedness of maps acting between the spaces in question. In fact, it turned out that positive and symmetric operators are automatically weakly continuous which suggests that the most adequate topologies for our investigations are the weak topologies.

Our first result is a quite general extension theorem, which can be considered as the main result of the section. This result served as the base of our further investigations throughout. Roughly speaking, this theorem characterizes (in both topological and algebraic ways) operators possessing positive extension to the whole space. In addition, the construction has some useful theoretical consequences, including an explicit formula for the obtained positive extension, as well as for its "quadratic form". Also, we showed that this extension is minimal in some sense, thus we shall called it *Krein-von Neumann extension*, in accordance with the classical literature. We emphasize that in this paper we restricted ourselves to continuous extensions, and thus the operators we dealt with are typically not densely defined. Hence, the maximal (Friedrichs) extension, apart from trivial cases, does not exist. Nevertheless, we showed that the set of positive extensions bounded by a fixed positive operator always possesses a maximal element. We also considered the special case when the anti-duality is the evaluation on the pair of a fixed Banach space and its conjugate topological dual. From that we concluded that Shmul'yan's original result is an immediate consequence of our main theorem. Finally, we applied our results to obtain representable positive extensions of linear functionals given on a left ideal of an involutive algebra.

Self-adjoint extensions ([**56**])

The question whether a self-adjoint extension exists arises naturally in various situations when a partially defined (bounded or unbounded) symmetric operator is given. For classical results, we refer the reader to [**4**, **12**, **26**] and the references therein; for more recent results, see, for example, [**9**, **36**].

The aim of the paper [**56**] was to continue these investigations and to discuss the problem of self-adjoint extendibility of operators acting on anti-dual pairs. The main result, Theorem 14, generalizes Krein's theorem on the existence of a norm preserving self-adjoint extension of a bounded symmetric operator [**34**, Theorem 5.33]. Due to the lack of norm, we considered extensions bounded by a fixed positive operator A. It turned out that extensions preserving the A-bound form an operator interval. As a nice application of Theorem 14, we also generalized a recent result of Yamada [**65**], which was an extension of the Strong Parrott Theorem [**20**, **39**]. In order to demonstrate the effectiveness of our general extension theory, we apply it to obtain hermitian extensions of functionals of an involutive algebra.

Lebesgue decomposition ([**22**, **51**, **53**–**55**])

This section is part of the unification project aiming to find a common framework and generalization for various results obtained in different branches of functional analysis including extension, dilation and decomposition theory. One important class of such results are decomposition theorems analogous to the well known Lebesgue decomposition of measures. What do we mean about analogous? In several cases, transformations of a given system can be grouped into two extreme classes according to the behavior with respect to their qualitative properties. These particular classes are the so-called *regular* transformations (i.e., transformations with “nice" properties) and the so-called *singular* ones (transformations that are hard to deal with). Of course, regularity and singularity may have multiple meanings depending on the context. A decomposition of an object into regular and singular parts is called a Lebesgue-type decomposition.

In order to understand a structure better, it can be effective to characterize its regular and singular elements. This explains why a regular-singular type decomposition theorem may have theoretic importance, especially when the corresponding regular part can be interpreted in a canonical way. The prototype of such results is the celebrated Radon-Nikodym theorem stating that every σ-finite measure splits uniquely into absolutely continuous and singular parts with respect to any other measure, and the absolutely continuous part has an integral representation. Returning to the previous idea, the Radon-Nikodym theorem can be phrased as follows: if we want to decide whether a set function can be represented as a point function, we only need to know if it is absolutely continuous or not. That is to say, in this concrete situation, the appropriate regularity concept is absolute continuity.

In the last 50 years quite a number of authors have made significant contributions to the vast literature of non-commutative Lebesgue-Radon-Nikodym theory – here we mention only Ando [**3**], Gudder [**23**], Inoue [**30**], Kosaki [**32**] and Simon [**46**], and from the recent past Di Bella and Trapani [**16**], Corso [**14**], ter Elst and Sauter [**59**], Hassi et al. [**24**, **25**, **28**], Kosaki [**33**], Sebestyén and Titkos [**44**], Vogt [**64**].

The purpose of [**53**] was to develop and investigate an abstract decomposition theory that can be considered as a common generalization of many of the aforementioned results on Lebesgue-type decompositions. The key observation is that the corresponding absolute continuity and singularity concepts rely only on some topological and algebraic properties of an operator acting between an appropriately chosen vector space and its conjugate dual. So that, the problem of decomposing Hilbert space operators, representable functionals, Hermitian forms and measures can be transformed into the problem of decomposing such an abstract operator.

In this section we are going to investigate Lebesgue decompositions of positive operators on a so called anti-dual pair. Hence, for the readers sake, we gathered in Section 2 the most important facts about anti-dual pairs and operators between them. We also provide here a variant of the famous Douglas factorization theorem. Section 3 contains the main result of the paper (Theorem 22), a direct generalization of Ando's Lebesgue decomposition theorem [**3**, Theorem 1] to the anti-dual pair context. It states that every positive operator on a weak-* sequentially complete anti-dual pair splits into a sum of absolutely continuous and singular parts with respect to another positive operator. We also prove that, when decomposing two positive operators with respect to each other, the corresponding absolute continuous parts are always mutually absolutely continuous.

Given a mathematical structure and an important operation/quantity/relation corresponding to it, a natural question to ask is: how can we describe all maps that respect this operation/quantity/relation? Such and similar problems belong to the gradually enlarging field of *preserver problems*. In the paper [**22**] our goal was to generalize Molnár's result [**37**, Theorem 1.1] about the structure of bijective maps on $\mathscr{B}(\mathcal{H})_+$ (the set of positive operators on the Hilbert space $\mathcal{H}$) that preserve the Lebesgue decomposition in both directions. Molnár proved that the cone is quite rigid in the sense that these maps can be always written in the form

$$A \mapsto SAS^*$$

with a bounded, invertible, linear- or conjugate linear operator $S\colon \mathcal{H} \to \mathcal{H}$. A natural question arises: how can we describe the form of those bijections that preserve absolute continuity (or singularity) of operators in both directions? Clearly, this is a weaker condition than that of Molnár, hence maps considered by Molnár obviously preserve this relation. However, it is not too hard to construct other maps which preserve absolute continuity. For example, one could use the fact that every positive operator is absolutely continuous with respect to every invertible element of $\mathscr{B}(\mathcal{H})_+$, and that invertible elements are the only ones with this property. Therefore, if we leave all positive and not invertible operators fixed, and consider an arbitrary bijection on the subset of invertible and positive operators, then this map preserves absolute continuity in both directions. Despite the existence of such seemingly unstructured maps, it is still possible to describe all maps with this weaker preserver property.

Schur complementation ([58])

Since the first appearance of the name of the "Schur complement" in [**29**], the theory of partitioned matrices (or block operators) is an active field of research in linear algebra and functional analysis. The direction we are interested in is the problem of completing special operator systems. To formulate the central question in the most classical setting, consider the incomplete system $\mathfrak{S} = \left[\begin{smallmatrix} A & B \\ B & * \end{smallmatrix}\right]$ of positive semidefinite $n \times n$ matrices A, B. The task is to find a matrix D for which $\left[\begin{smallmatrix} A & B \\ B & D \end{smallmatrix}\right]$ is a positive semidefinite $2n \times 2n$ matrix. If we denote by A_B the smallest possible solution, then the Schur complement of D in the block-matrix $\left[\begin{smallmatrix} A & B \\ B & D \end{smallmatrix}\right]$ is $D - A_B$. Therefore, to find the Schur complement and to find the minimal operator that makes a system positive is the same problem. In the paper [**58**], we focused our attention on the completion problem.

Because of its wide-range applicability in pure and applied mathematics, a number of authors made a lot of efforts to extend the concept of Schur complement for various settings. We mention first the fundamental work of Pekarev and Šmul'jan [**40**] on the connection between the shorted operator and positive completions of block operators in the context of Hilbert spaces. The corresponding result in Krein spaces has been developed by Contino, Maestripieri, and Marcantognini in [**13**] (see also [**35**]). The relation between extension, completion, and lifting problems of operators on both Hilbert and Krein spaces has been discussed in [**9**]. A quite general approach was developed by Friedrich, Günther and Klotz in [**21**]. They introduced a generalized Schur complement for non-negative 2×2 block matrices whose entries are linear operators on linear spaces. In their considerations the setting is purely algebraic and therefore topology plays a minor role.

In [**58**] we treated the completion problem in the more general setting of anti-dual pairs that covers Hilbert, Krein, and linear spaces. The key idea of our approach was the observation that the block matrix completion problem can be formulated as an operator extension problem. This gave rise to invoke our corresponding Krein–von Neumann extension theory developed in [**57**]. Our aim was two folded: besides solving the block completion problem in a quite general setting, we wanted to demonstrate how the developed method can be applied for structures like rigged Hilbert spaces and involutive algebras.

The cornerstone of [**58**] is a result that provides necessary and sufficient conditions for the positive complementarity of an incomplete operator matrix. Our main result gives an explicit formula for the minimal solution of the completion problem.

Following the method of Pekarev and Šmul'jan, we introduced the notions of parallel sum and difference as an immediate application. Furthermore, by means of these operations we exhibited an alternative description of the Lebesgue decomposition. To demonstrate how this operator approach works in concrete structures, we derive the corresponding results for representable functionals on involutive algebras.

Summary of results

Positive operators on anti-dual pairs ([**42**, **57**])

We start by recalling the notion of anti-duality, which is just a slight technical modification of dual pairing. Although there is no crucial difference between these two notions, we choose anti-duality in order to stay formally as close as possible to the Hilbert space case. Let E, F be complex vector spaces. A function $\langle \cdot, \cdot \rangle : F \times E \to \mathbb{C}$ is called anti-duality if it is sesquilinear (that is to say, linear in the first argument and conjugate linear in the second one) and $\langle \cdot, \cdot \rangle$ separates the points of F and E (i.e., if $\langle f, x \rangle = 0$ for all $x \in E$ then $f = 0_F$ and if $\langle f, x \rangle = 0$ for all $f \in F$ then $x = 0_E$). The triple $((E, F), \langle \cdot, \cdot \rangle)$ is called an anti-dual pair, and it is denoted shortly by $\langle F, E \rangle$. Observe that if $\langle F, E \rangle$ is an anti-dual pair then $\langle E, F \rangle'$ is also an anti-dual pair where $\langle \cdot, \cdot \rangle' : E \times F \to \mathbb{C}$ is given by

$$\langle x, f \rangle' := \overline{\langle f, x \rangle}, \qquad x \in E, \ f \in F. \tag{2}$$

The most natural anti-dual pair is a linear space and a linear subspace of its conjugate algebraic dual, intertwined by the evaluation as anti-duality. In fact, every anti-dual pair can be written in the above form. Indeed, if $\langle F, E \rangle$ is an anti-dual pair, then due to the identification

$$x \mapsto \varphi_x; \qquad \varphi_x(f) := \langle f, x \rangle \qquad \text{for all} \quad f \in F,$$

E may be regarded as a linear subspace of F^*, the algebraic dual of F. Similarly, due to the mapping

$$f \mapsto \psi_f; \qquad \psi_f(x) := \langle f, x \rangle \qquad \text{for all} \quad x \in E,$$

F can be identified as a linear subspace of $\bar{E}^*$, the algebraic anti-dual space of E. Our prototype of anti-dual pairs is the system $((\mathcal{H}, \mathcal{H}), (\cdot \,|\, \cdot))$ where $\mathcal{H}$ is a Hilbert space with inner product $(\cdot \,|\, \cdot)$. This particular anti-dual pair has the useful feature that $\mathcal{H}$ can be identified with its topological dual along the maps $x \mapsto \varphi_x$ and $f \mapsto \psi_f$, according to the Riesz representation theorem. A similar feature is obtained in the general setting if we endow E and F with appropriate topologies. For this purpose the most natural at hand are the *weak topologies* $\sigma(E, F)$ and $\sigma(F, E)$ on E and F, respectively: $\sigma(E, F)$ is the smallest topology making φ_x continuous for all $x \in E$, and similarly, $\sigma(F, E)$ is the smallest topology on F such that all the functionals of the form ψ_f $(f \in F)$ are continuous. Both $(E, \sigma(E, F))$ and $(F, \sigma(F, E))$ are locally convex Hausdorff spaces such that $\bar{E}' = F$ and $F' = E$, where $\bar{E}'$ and F' refer to the topological anti-dual and dual spaces of E and F, respectively. We call an anti dual pair $\langle F, E \rangle$ w^*-sequentially complete, if F is $\sigma(F, E)$ sequentially complete.

Now we turn to investigate special linear operators acting between two sides of anti-dual pairs. If an anti-dual pair $\langle F, E \rangle$ is given, we will use the short notation $A : E \supseteq \operatorname{dom} A \to F$ for *linear* operators acting on a subspace $\operatorname{dom} A$ of E with values in F. We prefer this setting instead of considering duality with conjugate linear operators as in [**1**].

An operator $A : E \supseteq \operatorname{dom} A \to F$ is called *positive* if it satisfies

$$\langle Ax, x\rangle \geq 0 \qquad \text{for all } x \in \operatorname{dom} A,$$

and *symmetric* if

$$\langle Ax, y\rangle = \overline{\langle Ay, x\rangle} \qquad \text{for all } x, y \in \operatorname{dom} A.$$

Note that every positive operator is symmetric. It is obvious that these are direct generalizations of the well-known notions of Hilbert space theory. The main advantage is that this setting allows us to handle structures without Hilbert space structure analogously. As the next example will illustrate, operators on anti-dual pairs appear very naturally for example in noncommutative integration theory.

Example 1. Let $\mathscr{A}$ be a *-algebra with algebraic conjugate-dual $\bar{\mathscr{A}}^*$, and let $\mathscr{I} \subseteq \mathscr{A}$ be a left-ideal. Then $\langle \bar{\mathscr{A}}^*, \mathscr{A}\rangle$ is an anti-dual pair with $\langle f, a\rangle := f(a)$. If a positive linear functional $f : \mathscr{I} \to \mathbb{C}$ is given, we can associate a positive operator $A_f : \mathscr{I} \to \bar{\mathscr{A}}^*$ as $\langle A_f a, b\rangle := f(b^*a)$. It will turn out later that positive extendibility of f to the whole algebra can be characterized by means of A_f. Furthermore, the canonical extension of f itself will be gained from the canonical extension of A_f.

Recall one of the main advantages of weak topology, namely that a linear operator T acting on a topological vector space $(V, \mathscr{T}_V)$ with values in $(F, \sigma(F, E))$ is continuous if and only if the linear functionals

$$\vartheta_x : V \to \mathbb{C}; \qquad \vartheta_x(v) := \langle Tv, x\rangle \tag{3}$$

are continuous for each $x \in E$. For the sake of simplicity we introduce the following terminology: if two anti-dual pairs $\langle F_1, E_1\rangle_1$ and $\langle F_2, E_2\rangle_2$ are given, we will call a map $T : E_1 \to E_2$ *weakly continuous* if T is $\sigma(E_1, F_1)$-$\sigma(E_2, F_2)$ continuous. The set of weakly continuous linear operators $T : E_1 \to E_2$ is denoted by $\mathscr{L}(E_1; E_2)$. By replacing $\langle F_2, E_2\rangle_2$ with $\langle E_2, F_2\rangle_2'$ (see (2)) one can characterize weak continuity of an operator $T : E_1 \to F_2$. Indeed, according to (3), T is weakly continuous if and only if for all $x_2 \in E_2$ there exists $f_1 \in F_1$ such that

$$\langle Tx_1, x_2\rangle_2 = \overline{\langle f_1, x_1\rangle_1} \ (= \langle x_1, f_1\rangle_1') \qquad \text{for all} \quad x_1 \in E_1. \tag{4}$$

The (necessarily weakly continuous) operator $T^* : E_2 \to F_1$ satisfying

$$\langle Tx_1, x_2\rangle_2 = \overline{\langle T^*x_2, x_1\rangle_1}, \qquad x_1 \in E_1, x_2 \in E_2,$$

is called the adjoint of T. In particular, if $E_1 = E_2 =: E$ and $F_1 = F_2 =: F$, the adjoint A^* of an operator $A \in \mathscr{L}(E; F)$ belongs again to $\mathscr{L}(E; F)$ and satisfies $\langle Ax, y\rangle = \overline{\langle A^*y, x\rangle}$ for all $x, y \in E$. Hence it makes sense to speak about self-adjointness $A = A^*$ of an operator $A \in \mathscr{L}(E; F)$. An everywhere defined symmetric operator (that is, an operator $S : E \to F$ such that $\langle Sx, y\rangle = \overline{\langle Sy, x\rangle}$, $x, y \in E$) is automatically weakly continuous, and hence self-adjoint. If $A : E \supseteq \operatorname{dom} A \to F$ is an operator such that $\langle Ax, x\rangle$ is real for all $x \in \operatorname{dom} A$ then the sesquilinear form $\mathfrak{t}_A(x, y) := \langle Ax, y\rangle$ $(x, y \in \operatorname{dom} A)$ is hermitian, thus A is symmetric. Indeed, $\langle Ax, y\rangle = \mathfrak{t}_A(x, y) = \overline{\mathfrak{t}_A(y, x)} = \overline{\langle Ay, x\rangle}$ holds for all $x, y \in \operatorname{dom} A$.

In rest of the paper we are mainly interested in positive operators on anti-dual pairs. In the next example we present the prototype of such positive operators.

Example 2. Let $\langle F, E\rangle$ be an anti-dual pair and $\mathcal{H}$ a Hilbert space. If $T : E \to \mathcal{H}$ is a $\sigma(E, F)$-$\sigma(\mathcal{H}, \mathcal{H})$ continuous linear operator then its adjoint $T^* : \mathcal{H} \to F$ is $\sigma(\mathcal{H}, \mathcal{H})$-$\sigma(F, E)$ continuous so that $T^*T \in \mathscr{L}(E; F)$ is positive:

$$\langle T^*Tx, x\rangle = (Tx \,|\, Tx) \geq 0, \qquad x \in E.$$

We will see later that, under some natural conditions on F, each positive operator $A \in \mathscr{L}(E;F)$ possesses a factorization of the form $A = T^*T$ with a suitable T and $\mathcal{H}$ of Example 2.

The central problem of this section is to provide necessary and sufficient conditions under which a linear operator $A : E \supseteq \operatorname{dom} A \to F$ possesses a positive extension to the whole E. The following set associated to A will play a key role in our treatment:

$$W(A) := \{Ax : x \in \operatorname{dom} A, \langle Ax, x\rangle \leq 1\} \subseteq F. \tag{5}$$

The construction below is motivated by the work of Sebestyén [**41**]. A similar factorization approach has been used by Ando and Nishio in [**4**] who considered extensions of closed positive symmetric operators. Here we emphasize that we do not impose any topological condition, that is, neither closedness of A or $\operatorname{dom} A$, nor density of $\operatorname{dom} A$ is assumed. Another construction of the Krein–von Neumann extension in terms of the boundary conditions was proposed in [**15**, Proposition 4.2].

Theorem 3 ([**57**, Theorem 3.1])**.** *Let $\langle F, E\rangle$ be a w^*-sequentially complete anti-dual pair and let $A : E \supseteq \operatorname{dom} A \to F$ be a linear operator with domain $\operatorname{dom} A$, which is assumed to be only a linear subspace. Then the following statements are equivalent.*

(i) *There is an everywhere defined positive operator $\widetilde{A} : E \to F$ extending A,*
(ii) *$W(A)$ is $\beta(F,E)$-bounded in F,*
(iii) *$W(A)$ is $\sigma(F,E)$-bounded in F,*
(iv) *To any y in E there is $M_y \geq 0$ such that*

$$|\langle Ax, y\rangle|^2 \leq M_y \langle Ax, x\rangle \qquad \text{for all } x \in \operatorname{dom} A. \tag{6}$$

If one (and hence all) of the above conditions is satisfied then there exists a distinguished extension A_N, called the Krein-von Neumann extension of A, which is minimal in the following sense: $A_N \leq \widetilde{A}$ holds for any (everywhere defined) positive extension $\widetilde{A} : E \to F$ of A.

Since not only the result itself, but also the construction of A_N plays an important role in the in this thesis, we sketch the proof of implication (iv)$\Rightarrow$(i).

Endow the range space $\operatorname{ran} A$ with the following inner product:

$$(Ax \,|\, Ay)_A := \langle Ax, y\rangle, \qquad x, y \in E.$$

It can be shown that $(\cdot\,|\,\cdot)_A$ is well defined and positive definite, hence $(\operatorname{ran} A, (\cdot\,|\,\cdot)_A)$ is a pre-Hilbert space. Let $\mathcal{H}_A$ denote its Hilbert completion so that $\operatorname{ran} A \subseteq \mathcal{H}_A$ forms a norm dense linear subspace. The canonical embedding operator

$$J_A(Ax) = Ax, \qquad x \in E, \tag{7}$$

of $\operatorname{ran} A \subseteq \mathcal{H}_A$ into F is weakly continuous by (iv). Hence J_A extends to an everywhere defined weakly continuous operator because of weak-* sequentially completeness of F. We continue to write $J_A \in \mathscr{L}(\mathcal{H}_A, F)$ for this extension. The adjoint operator $J_A^* \in \mathscr{L}(E, \mathcal{H}_A)$ admits the canonical extension property

$$J_A^* x = Ax \in \mathcal{H}_A, \qquad x \in E, \tag{8}$$

from which one obtains that $A_N := J_A J_A^*$ is a positive extension of A.

We mention that the following construction works even if A is everywhere defined on E (i.e., $A \in \mathscr{L}(E;F)$). In that case one obtains a useful factorization of A:

$$A = J_A J_A^*. \tag{9}$$

A simple example below demonstrates that w^*-sequentially completeness of F was really essential in the main theorem.

Example 4. Let $\mathcal{H}$ be a Hilbert space and let A be an unbounded positive self-adjoint operator in $\mathcal{H}$, with (dense) domain $\operatorname{dom} A$. Let $E := \operatorname{dom} A^{1/2}$ and let $F := \mathcal{H}$, then clearly, $\langle F, E\rangle$ is an anti-dual pair with respect to the duality induced by the inner product. But $\langle F, E\rangle$ is not w^*-sequentially complete because E is a proper dense subspace of $\mathcal{H}$. It is readily seen that $A : E \supset \operatorname{dom} A \to F$ fulfills condition (iv) of Theorem 3:

$$|(Ax\,|\,y)|^2 = |(A^{1/2}x\,|\,A^{1/2}y)|^2 \leq \|A^{1/2}y\|^2(Ax\,|\,x), \qquad x \in \operatorname{dom} A, y \in E.$$

Although condition (iv) is satisfied, the statement of Theorem 3 does not remain true. Indeed, assume that A extends to a positive operator $\widetilde{A} : E \to F$. Recall that a self-adjoint operator may not have any proper symmetric extension, hence $A = \widetilde{A}$ and, in particular, $\operatorname{dom} A = \operatorname{dom} A^{1/2}$. But this is impossible because $\operatorname{dom} A \subsetneqq \operatorname{dom} A^{1/2}$ whenever A is unbounded (see [**48**, Corollary 2.4]).

Notice that the set of positive extensions of a given positive operator A has no maximal element (unless $\operatorname{dom} A$ is dense): for example, in the trivial case when $\operatorname{dom} A = \{0\}$, every positive operator is an extension of A. The largest (so-called Friedrichs) extension of a non-densely defined positive operator A becomes a linear relation (that is, a multivalued operator). This fact immediately follows from a description of the Krein–von Neumann extension obtained in [**15**, Proposition 4.2].

The next theorem says that we will get a maximum among continuous positive extensions, bounded by a positive operator B.

Theorem 5 ([**57**, Theorem 3.3])**.** *Let A be a subpositive operator on the w^*-sequentially complete anti-dual pair $\langle F, E\rangle$. Let $B \in \mathscr{L}(E; F)$ be a positive operator such that $A_N \leq B$, then there exists a positive operator $A^B_{max} \in \mathscr{L}(E; F)$, $A^B_{max} \leq B$ such that for every positive extension $\widetilde{A} \in \mathscr{L}(E; F)$ of A, $0 \leq \widetilde{A} \leq B$, one has $\widetilde{A} \leq A^B_{max}$. In other words,*

$$A^B_{max} = \max\{\widetilde{A} \in \mathscr{L}(E; F) : 0 \leq \widetilde{A} \leq B, A \subset \widetilde{A}\}.$$

Furthermore, a positive operator $0 \leq \widetilde{A} \leq B$ is an extension of A if and only if $A_N \leq \widetilde{A} \leq A^B_{max}$:

$$[A_N, A^B_{max}] = \{\widetilde{A} \in \mathscr{L}(E; F) : 0 \leq \widetilde{A} \leq B, A \subset \widetilde{A}\}. \tag{10}$$

The next theorem tells us that the Krein–von Neumann extension preserves certain commutation properties as well.

Theorem 6 ([**57**, Theorem 3.4])**.** *Let $A : \operatorname{dom} A \to F$ be a subpositive operator on the w^*-sequentially complete anti-dual pair $\langle F, E\rangle$. Suppose that there are two weakly continuous operators $B, C \in \mathscr{L}(E)$ leaving $\operatorname{dom} A$ invariant, and that the spectrum of BC restricted to $\operatorname{dom} A$ is bounded. Assume in addition that B and C satisfy*

$$C^*A \subset AB, \qquad \text{and} \qquad B^*A \subset AC, \tag{11}$$

then the Krein-von Neumann extension of A satisfies

$$C^*A_N = A_N B, \qquad \text{and} \qquad B^*A_N = A_N C.$$

Next we are going to investigate the special case when the anti-duality is the evaluation on the pair of a fixed Banach space E and its conjugate topological dual $\bar{E}'$. We will obtain a strengthening of the main result of [**42**], and we will show that Shmul'yan's original result is indeed a corollary of our main theorem.

We remark that the Banach-Steinhaus theorem forces $\bar{E}'$ to be weakly sequentially complete, and hence everything that has been proved in the preceding sections also remains valid for the anti-dual pair $\langle \bar{E}', E\rangle$.

Theorem 7 ([**57**, Theorem 4.2] and [**42**, Theorem 3.1])**.** *Let E be a Banach space, and let $A : E \supseteq \operatorname{dom} A \to \bar{E}'$ be a positive linear operator. Then the following statements are equivalent.*

(i) *A has a bounded positive extension $\widetilde{A} \in \mathscr{L}(E; \bar{E}')$,*

(ii) *There is a constant $M \geq 0$ such that*

$$\|Ax\|^2 \leq M \cdot \langle Ax, x\rangle, \qquad x \in \operatorname{dom} A, \tag{12}$$

(iii) *For any $y \in E$ there exists $M_y \geq 0$ such that*

$$|\langle Ax, y\rangle|^2 \leq M_y \cdot \langle Ax, x\rangle, \qquad x \in \operatorname{dom} A.$$

In any case, there exists the Krein-von Neumann extension A_N of A that is the smallest among the set of positive extensions of A. The norm of A_N satisfies

$$\|A_N\| = \inf\{M \geq 0 : \|Ax\|^2 \leq M \cdot \langle Ax, x\rangle, \quad x \in \operatorname{dom} A\}. \tag{13}$$

If $B, C \in \mathscr{L}(E)$ are continuous operators leaving $\operatorname{dom} A$ *invariant such that $C^*A \subset AB$ and $B^*A \subset AC$ then the Krein-von Neumann extension of A satisfies*

$$C^*A_N = A_N B, \qquad \text{and} \qquad B^*A_N = A_N C. \tag{14}$$

Corollary 8 ([**57**, Corollary 4.4])**.** *Let $A : \operatorname{dom} A \to \mathcal{H}$ be a positive operator satisfying the equivalent conditions of Corollary 7. Then, for every constant $M \geq \|A_N\|$ there is a positive extension A^M_{max} of A with $\|A^M_{max}\| \leq M$ such that for any positive extension $\widetilde{A}$ of A, $\|\widetilde{A}\| \leq M$ one has $\widetilde{A} \leq A^M_{max}$. In other words,*

$$A^M_{max} = \max\{\widetilde{A} \in \mathscr{B}(\mathcal{H}) : \widetilde{A} \geq 0, A \subset \widetilde{A}, \|\widetilde{A}\| \leq M\}.$$

Furthermore, one has equality

$$[A_N, A^M_{max}] = \{\widetilde{A} \in \mathscr{B}(\mathcal{H}) : \widetilde{A} \geq 0, A \subset \widetilde{A}, \|\widetilde{A}\| \leq M\}.$$

In the rest of the section we apply our theory to positive functionals on a *-algebra. Let $\mathscr{A}$ be a (not necessarily unital) complex *-algebra. Recall that a linear functional f on $\mathscr{A}$ is called *representable* if there exist a Hilbert space $\mathcal{H}$, a *-representation (that is, a *-homomorphism) $\pi : \mathscr{A} \to \mathscr{B}(\mathcal{H})$ and a vector $\xi \in \mathcal{H}$ such that

$$f(a) = (\pi(a)\xi \,|\, \xi), \qquad a \in \mathscr{A}.$$

It is clear that every representable functional is positive, nevertheless the converse is not true in general.

Consider now a left ideal $\mathscr{I}$ of $\mathscr{A}$ and a linear functional $f : \mathscr{I} \to \mathbb{C}$. In this section we provide necessary and sufficient conditions under which f admits a representable extension to $\mathscr{A}$ (cf. also [**42**] for the Banach-* algebra setting). Recall that if $f : \mathscr{I} \to \mathbb{C}$ is a linear functional, then we can associate an operator $A : \mathscr{I} \to \bar{\mathscr{A}}^*$ to f by setting

$$\langle Aa, x\rangle := f(x^*a), \qquad x \in \mathscr{A}, a \in \mathscr{I}. \tag{15}$$

Clearly, A is positive if and only if f is positive, i.e., $f(a^*a) \geq 0$ holds for all $a \in \mathscr{I}$.

In what follows, we are exclusively interested in representable extensions of functionals. Recall that f is said to be *Hilbert bounded* if there is constant $M \geq 0$ such that

$$|f(a)|^2 \leq Mf(a^*a) \qquad \text{for all } a \in \mathscr{I}, \tag{16}$$

and *admissible* if for any x in $\mathscr{A}$ there exists $\lambda_x \geq 0$ such that

$$f(a^*x^*xa) \leq \lambda_x f(a^*a) \qquad \text{for all } a \in \mathscr{I}. \tag{17}$$

Theorem 9 ([**57**, Theorem 5.1]). *Let $\mathscr{A}$ be a *-algebra, $\mathscr{I} \subseteq \mathscr{A}$ be a left ideal, and let $f : \mathscr{I} \to \mathbb{C}$ be a linear functional. The following assertions are equivalent:*

(i) *there is a representable functional $\widetilde{f} \in \mathscr{A}^*$ extending f,*
(ii) *f is admissible and Hilbert bounded.*

If there is any, then there is a minimal one (denoted by f_N) among the set of representable extensions.

In case when the algebra has a unit element the theorem can be more easily formulated. In fact, the following simple formula may suggest that extension theory of functionals and extension theory of operators fit nicely together, indeed.

Corollary 10 ([**57**, Theorem 5.3]). *Assume that $\mathscr{A}$ is a unital *-algebra with unit $1 \in \mathscr{A}$. Assume further that $f : \mathscr{I} \to \mathbb{C}$ is admissible. Then f is Hilbert bounded, and its Krein-von Neumann extension satisfies*

$$f_N(x) = \overline{\langle A_N 1, x\rangle}, \qquad x \in \mathscr{A}. \tag{18}$$

In a Banach *-algebra every positive functional $f : \mathscr{I} \to \mathbb{C}$ is automatically admissible according to the following result:

Lemma 11 ([**42**, Lemma 5.1]). *Let $\mathscr{A}$ be a Banach *-algebra, $\mathscr{I} \subseteq \mathscr{A}$ a left ideal, and let $f : \mathscr{I} \to \mathbb{C}$ be a positive linear functional, then*

$$f(a^*x^*xa) \leq r(x^*x)f(a^*a) \qquad \textit{for all } a \in \mathscr{I} \textit{ and } x \in \mathscr{A}.$$

Corollary 12 ([**42**, Theorem 5.3]). *Let $\mathscr{A}$ be a Banach *-algebra and $\mathscr{I} \subseteq \mathscr{A}$ a left ideal. A positive functional $f : \mathscr{I} \to \mathbb{C}$ admits a representable extension to $\mathscr{A}$ if and only if f is Hilbert bounded.*

An analogue of Theorem 5 for positive functionals can be established as follows:

Theorem 13 ([**57**, Theorem 5.4]). *Let $f : \mathscr{I} \to \mathbb{C}$ be an admissible and Hilbert bounded functional and fix any representable functional $g \in \mathscr{A}^*$ such that $f_N \leq g$. Then there is a representable extension $f^g_{max} \in \mathscr{A}^*$ of f such that $f^g_{max} \leq g$, and for every representable extension $\widetilde{f}$ of f one has $\widetilde{f} \leq f^g_{max}$. In other words,*

$$f^g_{max} = \max\{\widetilde{f} \in \mathscr{A}^\sharp : \widetilde{f} \leq g, f \subset \widetilde{f}\}.$$

Furthermore, a representable functional $\widetilde{f} \leq g$ is an extension of f if and only if $f_N \leq \widetilde{f} \leq f^g_{max}$:

$$[f_N, f^g_{max}] = \{\widetilde{f} \in \mathscr{A}^\sharp : \widetilde{f} \leq g, f \subset \widetilde{f}\}.$$

Self-adjoint extensions ([**56**])

M. G. Krein [**34**] proved that every bounded symmetric Hilbert space operator possesses a norm preserving self-adjoint extension. The problem of constructing self-adjoint extensions of a symmetric operator arises in our anti-dual pair setting naturally. Since we cannot speak about norm preservation due to the lack of norm, we need to find a suitable notion to generalize Krein's theorem. Observe that the norm of a self-adjoint operator $S \in \mathscr{B}(\mathcal{H})$ can be expressed by means of the partial order induced by positivity. Namely, $\|S\|$ is the smallest constant $\alpha \geq 0$ such that $-\alpha I \leq S \leq \alpha I$. Based on this observation, a symmetric operator $S_0\colon E \supseteq \operatorname{dom} S_0 \to F$ is called *A-bounded* for a fixed positive operator $A \in \mathscr{L}(E;F)$ if

$$|\langle S_0 x, y\rangle|^2 \leq \alpha^2 \cdot \langle Ax, x\rangle\langle Ay, y\rangle, \qquad x \in \operatorname{dom} S_0, y \in E, \tag{19}$$

holds. The smallest constant α is called the *A-bound* of S_0 and is denoted by $\alpha_A(S_0)$. We will call the extension $S \supset S_0$ *A-bound preserving* if $\alpha_A(S_0) = \alpha_A(S)$.

In the next theorem, which is the main result of this section, we will present a sufficient condition to guarantee for a symmetric linear operator that it possesses a self-adjoint extension. Moreover, we describe the set of all A-bound preserving extensions of a given symmetric operator.

Theorem 14 ([**56**, Theorem 2.1])**.** *Let $\langle F, E\rangle$ be a weak-* sequentially complete anti-dual pair and let $S_0\colon \operatorname{dom} S_0 \to F$ be a symmetric operator,* i.e.,

$$\langle S_0 x, y\rangle = \overline{\langle S_0 y, x\rangle}, \qquad x, y \in \operatorname{dom} S_0.$$

Suppose that S_0 is A-bounded with some positive operator $A \in \mathscr{L}(E;F)$. Then there exist two distinguished self-adjoint extensions $S_m, S_M \in \mathscr{L}(E;F)$ of S_0 such that

$$\alpha_A(S_m) = \alpha_A(S_M) = \alpha_A(S_0).$$

In fact, the interval $[S_m, S_M]$ consists exactly of all self-adjoint extensions $S \supset S_0$ such that $\alpha_A(S) = \alpha_A(S_0)$:

$$[S_m, S_M] = \{S \in \mathscr{L}(E;F) : S_0 \subset S = S^*,\ \alpha_A(S) = \alpha_A(S_0)\}. \tag{20}$$

In the following corollary, we recover the classical result of Krein on self-adjoint norm-preserving extensions.

Corollary 15 ([**56**, Corollary 2.2])**.** *Let $\mathcal{H}$ be a Hilbert space and let $S_0\colon \operatorname{dom} S_0 \to \mathcal{H}$ be a bounded symmetric operator. Then S admits two self-adjoint norm-preserving extensions $S_m, S_M \in \mathscr{B}(\mathcal{H})$ such that the interval $[S_m, S_M]$ consists exactly of all self-adjoint norm-preserving extensions of S_0 :*

$$[S_m, S_M] = \{S \in \mathscr{B}(\mathcal{H}) : S_0 \subset S = S^*, \|S_0\| = \|S\|\}.$$

If a self-adjoint operator $B \in \mathscr{B}(\mathcal{H})$ leaving $\operatorname{dom} S_0$ *invariant satisfies $BS_0 \subset S_0 B$, then*

$$S_m B = BS_m, \qquad S_M B = BS_M. \tag{21}$$

The aim of this section is to generalize Parrott's famous theorem [**39**] on contractive extensions of 2 by 2 block operator-valued matrices, which is one of the crucial results in extension and dilation theory. As an application, we will deduce Yamada's recent result [**65**, Theorem 4] on the extension of the Strong Parrott Theorem [**20**, **60**].

Theorem 16 ([**56**, Theorem 3.1])**.** *Let $\langle F_1, E_1\rangle_1$ and $\langle F_2, E_2\rangle_2$ be two w^*-sequentially complete anti-dual pairs and let $T_1\colon E_1 \supseteq \operatorname{dom} T_1 \to F_2$ and $T_2\colon E_2 \supseteq \operatorname{dom} T_2 \to F_1$ be linear operators such that*

$$\langle T_1 x_1, x_2\rangle_2 = \overline{\langle T_2 x_2, x_1\rangle_1}, \qquad x_1 \in \operatorname{dom} T_1, x_2 \in \operatorname{dom} T_2.$$

Assume, furthermore, that there exist two positive operators $A_i \in \mathscr{L}(E_i; F_i)$ *and constants* $\alpha_i \geq 0$, $(i = 1, 2)$ *such that the following estimates hold true:*

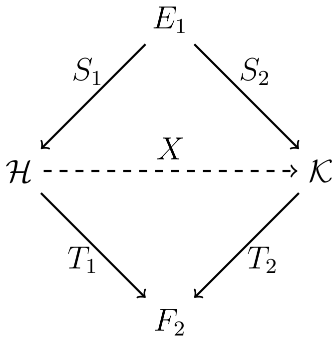
$$|\langle T_1 x_1, y_2 \rangle_2|^2 \leq \alpha_1 \langle A_1 x_1, x_1 \rangle_1 \langle A_2 y_2, y_2 \rangle_2, \qquad x_1 \in \operatorname{dom} T_1, y_2 \in E_2,$$
$$|\langle T_2 x_2, y_1 \rangle_1|^2 \leq \alpha_2 \langle A_1 y_1, y_1 \rangle_1 \langle A_2 x_2, x_2 \rangle_2, \qquad x_2 \in \operatorname{dom} T_2, y_1 \in E_1.$$

Then there exists a $T \in \mathscr{L}(E_1; F_2)$ *such that* $T_1 \subseteq T$ *and* $T_2 \subseteq T^*$ *and that*

$$|\langle T y_1, y_2 \rangle_1|^2 \leq \max\{\alpha_1, \alpha_2\} \cdot \langle A_1 y_1, y_1 \rangle_1 \langle A_2 y_2, y_2 \rangle_2, \qquad y_1 \in E_1, y_2 \in E_2.$$

Corollary 17 ([**56**, Corollary 3.2])**.** *Let* $\langle F_1, E_1 \rangle_1$, $\langle F_2, E_2 \rangle_2$ *be anti-dual pairs and let* $\mathcal{H}$, $\mathcal{K}$ *be Hilbert spaces. For* $S_1 \in \mathscr{L}(E_1, \mathcal{H})$, $S_2 \in \mathscr{L}(E_1, \mathcal{K})$, $T_1 \in \mathscr{L}(\mathcal{H}, F_2)$, *and* $T_2 \in \mathscr{L}(\mathcal{K}, F_2)$, *the following conditions are equivalent:*

(i) $T_1 S_1 = T_2 S_2$, $S_2^* S_2 \leq S_1^* S_1$, *and* $T_1 T_1^* \leq T_2 T_2^*$;
(ii) *there exists* $X \in \mathscr{B}(\mathcal{H}, \mathcal{K})$, $\|X\| \leq 1$, *such that* $X S_1 = S_2$ *and* $T_2 X = T_1$, i.e., X *makes the following diagram commutative:*



Positive functionals play an important role in the representation theory of algebras. Extension of such functionals has been investigated in many different settings. For example, if f is a positive linear functional defined on a closed ideal in a C^*-algebra, then f always admits an extension with the same norm. Positive functionals defined on left-ideals of the full operator algebra possessing normal extension were characterized in [**43**], while positive extendibility of positive functionals defined on left ideals of general *-algebras was studied in [**57**]. Below we demonstrate how our anti-dual pair setting can be used to construct hermitian extensions of linear functionals in the unital *-algebra setting.

Theorem 18 ([**56**, Theorem 4.1])**.** *Let* $\mathscr{A}$ *be a unital* **-algebra,* $\mathscr{I} \subseteq \mathscr{A}$ *a left ideal and* $f \in \mathscr{A}^*$ *a representable positive functional. If* $g_0 \colon \mathscr{I} \to \mathbb{C}$ *is an* f*-bounded symmetric functional with* f*-bound* $\alpha_f(g_0)$*, then there exist two distinguished* f*-bounded hermitian functionals* $g_m, g_M \in \mathscr{A}^*$ *with* f*-bound* $\alpha_f(g_m) = \alpha_f(g_M) = \alpha_f(g_0)$ *extending* g_0*. Furthermore,* $g_m \leq g_M$ *and the interval* $[g_m, g_M]$ *consists of all hermitian* f*-bound preserving extensions of* g_0*:*

$$[g_m, g_M] = \{g \in \mathscr{A}^* : g_0 \subset g = g^*, \alpha_f(g) = \alpha_f(g_0)\}.$$

We remark that Theorem 18 provides only a sufficient condition for the existence of hermitian extensions. On C^*-algebras, the statement of Theorem 18 may be improved in two ways; first, the condition on f of being representable can be replaced by the formally weaker one of being positive. On the other hand, the existence of a dominating positive functional is both necessary and sufficient.

Corollary 19 ([**56**, Corollary 4.2]). *Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{I} \subseteq \mathscr{A}$ a left ideal. A linear functional $g_0\colon \mathscr{I} \to \mathbb{C}$ possesses a continuous hermitian extension g if and only if g_0 is symmetric and f-bounded for some positive functional $f \in \mathscr{A}^*$.*

Lebesgue decomposition ([22, 51, 53–55])

Douglas' factorization theorem. Operators of type $T \in \mathscr{L}(E, \mathcal{H})$ play a peculiar role in the theory of positive operators on anti-dual pairs. In fact, every positive operator A on a weak-* sequentially complete anti-dual pair admits a factorization $A = T^*T$ through a Hilbert space $\mathcal{H}$. Below we describe the range of the adjoint operator $T^* \in \mathscr{L}(\mathcal{H}, F)$. The key result is a variant to Douglas' famous range inclusion theorem [**17**] (for further generalizations to Banach space setting see Barnes [**10**] and Embry [**18**]).

Theorem 20 ([**53**, Theorem 2.1]). *Let $\langle F, E\rangle$ be an anti-dual pair and let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Given two weakly continuous operators $T_j \in \mathscr{L}(\mathcal{H}_j, F)$ ($j = 1, 2$) the following assertions are equivalent:*

(i) $\operatorname{ran} T_1 \subseteq \operatorname{ran} T_2$,

(ii) *there is a constant $\alpha \geq 0$ such that*
$$\|T_1^* x\|^2 \leq \alpha \|T_2^* x\|^2, \qquad x \in E,$$

(iii) *for every $h_1 \in \mathcal{H}_1$ there is a constant $\alpha_{h_1} \geq 0$ such that*
$$|\langle T_1 h_1, x\rangle|^2 \leq \alpha_{h_1} \|T_2^* x\|^2, \qquad x \in E,$$

(iv) *there is a bounded operator $D : \mathcal{H}_1 \to \mathcal{H}_2$ such that*
$$T_1 = T_2 D.$$

Moreover, if any (hence all) of (i)-(iv) is valid, then there is a unique D such that

(a) $\operatorname{ran} D \subseteq (\ker T_2)^\perp$,

(b) $\ker T_1 = \ker D$,

(c) $\|D\|^2 = \inf\{\alpha \geq 0 : \|T_1^* x\|^2 \leq \alpha \|T_2^* x\|^2, (x \in E)\}$.

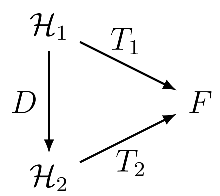


FIGURE 1. Factorization of T_1 along T_2

There are many important objects in operator theory, including the Moore-Penrose pseudoinverse [**5**], the parallel sum [**19**] or the Schur complement [**58**] which can be defined as the Douglas solution of a suitably posed operator equation.

Regarding the structure of Douglas solutions, a natural nonlinear preserver problem might be posed. Let $\varphi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a bijective map with the property that, for every triple A, B, X of bounded operators in $\mathscr{B}(\mathcal{H})$, X is the Douglas solution of the equation $A = BX$ if and only if $Y = \varphi(X)$ is the Douglas solution of the equation $\varphi(A) = \varphi(B)Y$. (Shortly, we can say in that case that φ preserves the Douglas solution in both directions.) The problem is describing the form all such transformation φ. According to the following result, structure of Douglas solution preserving maps is quite rigid:

Theorem 21 ([**54**])**.** *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. A bijective map $\varphi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ preserves the Douglas solution in both directions if and only if there exists a unitary or anti-unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that*

$$\varphi(A) = UAU^*, \qquad A \in \mathscr{B}(\mathcal{H}). \tag{22}$$

Lebesgue decomposition of positive operators. Modeled by the Lebesgue–Radon–Nikodym theory of positive operators on a Hilbert space (see e.g. [**3**] or [**49**]) we can introduce the concepts of absolute continuity and singularity of positive operators on an anti-dual pair. Let A and B be positive operators on an anti-dual pair $\langle F, E\rangle$. We say that B is *absolutely continuous* with respect to A (in notation, $A \ll B$) if for any sequence $(x_n)_{n\in\mathbb{N}}$ of E,

$$\langle Ax_n, x_n\rangle \to 0 \qquad \text{and} \qquad \langle B(x_n - x_m), x_n - x_m\rangle \to 0 \qquad (n, m \to +\infty)$$

imply $\langle Bx_n, x_n\rangle \to 0$. On the other hand, we say that A and B are *mutually singular* (in notation, $A \perp B$) if $C \leq A$ and $C \leq B$ imply $C = 0$ for any positive operator $C \in \mathscr{L}(E; F)$.

The main purpose of this section is to establish an extension of Ando's Lebesgue decomposition theorem [**3**, Theorem 1]. This states that every positive operator B on a weak-* sequentially complete anti-dual pair admits a decomposition $B = B_a + B_s$ where $B_a \ll A$ and $B_s \perp A$.

The construction of the Lebesgue decomposition is based on a the decomposition theory of linear relations between Hilbert spaces. Let us consider the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ and the linear operators J_A, J_B, associated with A and B, respectively, in accordance with the proof of Theorem 3. Introduce the closed linear relation

$$\widehat{B} \coloneqq \overline{\{(Ax, Bx) \in \mathcal{H}_A \times \mathcal{H}_B : x \in E\}} \tag{23}$$

from $\mathcal{H}_A$ to $\mathcal{H}_B$, and denote its multivalued part by $\mathcal{M}$:

$$\mathcal{M} \coloneqq \{\xi \in \mathcal{H}_B : (0, \xi) \in \widehat{B}\}.$$

It can be shown that $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}_B$ and one easily verifies that

$$\mathcal{M} = \{\xi \in \mathcal{H}_B : \exists (x_n)_{n\in\mathbb{N}} \text{ of } E, \langle Ax_n, x_n\rangle \to 0, Bx_n \to \xi \text{ in } \mathcal{H}_B\}. \tag{24}$$

It is easy to check that $B \ll A$ if and only if $\widehat{B}$ is a closed operator, or equivalently, if $\mathcal{M} = \{0\}$. Furthermore, since $\operatorname{ran} A \subseteq \operatorname{dom} \widehat{B}$, the adjoint relation $\widehat{B}^*$ is always a single-valued operator from $\mathcal{H}_B$ to $\mathcal{H}_A$ such that

$$(\operatorname{dom} \widehat{B}^*)^\perp = \mathcal{M}. \tag{25}$$

The main result of the section is the following Lebesgue type decomposition theorem for positive operators:

Theorem 22 ([**53**, Theorem 3.3])**.** *Let A, B be positive operators on a weak-* sequentially complete anti-dual pair $\langle F, E\rangle$. Let P stand for the the orthogonal projection of $\mathcal{H}_B$ onto $\mathcal{M}$, then*

$$B_a \coloneqq J_B(I - P)J_B^* \qquad \text{and} \qquad B_s \coloneqq J_B P J_B^* \tag{26}$$

are positive operators such that $B = B_a + B_s$, B_a is A-absolutely continuous and B_s is A-singular. Furthermore, B_a is the greatest element of the set of those positive operators $C \in \mathscr{L}(E; F)$ such that $C \leq B$ and $C \ll A$.

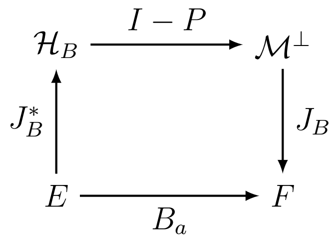


FIGURE 2. Factorization of the absolute continuous part

Suppose now that A, B are positive operators and let $B = B_a + B_s$ be the Lebesgue decomposition of B with respect to A. Here we have $B_a \ll A$. Interchanging the roles of A and B, by the same process we may take the Lebesgue decomposition of A with respect to B, namely, $A = A_a + A_s$. An interesting **feature** of the absolutely continuous parts are that they are absolutely continuous with respect to each other, i.e., $B_a \ll A_a$ and $A_a \ll B_a$. This surprising property was discovered by T. Titkos in context of nonnegative forms [**61**] and measures [**63**]. Theorem 23 below generalizes this fact:

Theorem 23 ([**53**, Theorem 3.6])**.** *Let $\langle F, E \rangle$ be a weak-* sequentially complete anti-dual pair and let $A, B \in \mathscr{L}(E; F)$ be positive operators. Then we have*

$$A_a \ll B_a \qquad \textit{and} \qquad B_a \ll A_a.$$

We have only proved that the canonical absolute continuous parts A_a and B_a have the property of being mutually absolute continuous. However, in contrast to the Lebesgue decomposition of measures, the Lebesgue decomposition of positive operators is not unique in general, so there might exist other Lebesgue-type decompositions differing from what we have constructed in Theorem 22. The statement of Theorem 23 is certainly not true for the absolutely continuous parts of such Lebesgue decompositions.

The parallel sum and characterizations of absolute continuity. Ando's key notion in establishing his Lebesgue-type decomposition theorem was the so called parallel sum of two positive operators. Inspired by his treatment, Hassi, Sebestyén, and de Snoo [**24**] proved an analogous result for nonnegative Hermitian forms by means of the parallel sum as well. Parallel addition may also be defined in various areas of functional analysis, e.g. for measures, representable positive functionals on a *-algebra, and for positive operators from a Banach space to its topological anti-dual, see [**47, 50, 62**]. In what follows we provide a common generalization of those concepts.

The parallel sum $A : B$ of two bounded positive operators on a Hilbert space can be introduced in various ways, see eg. [**2, 19, 40**]. Its quadratic form can be obtained via the formula

$$((A : B)x \,|\, x) = \inf\{(A(x - y) \,|\, x - y) + (By \,|\, y) : y \in \mathcal{H}\}, \tag{27}$$

that uniquely determines the operator $A : B$. Therefore, it seems natural to introduce the parallel sum of two positive operators in the anti-dual pair context as an operator whose quadratic form is (27) (the inner product replaced by anti-duality, of course).

The existence of such an operator is established in the following result:

Theorem 24 ([**53**, Theorem 4.1])**.** *Let $\langle F, E \rangle$ be a weak-* sequentially complete anti-dual pair and let $A, B \in \mathscr{L}(E; F)$ be positive operators. There exists a unique positive operator*

$A:B\in\mathscr{L}(E;F)$, called the parallel sum of A and B, such that

$$\langle (A:B)x,x\rangle=\inf\{(A(x-y)\,|\,x-y)+(By\,|\,y):y\in E\},\qquad x\in E. \tag{28}$$

The parallel sum has a number of theoretical applications. Below we establish only a few of them. Our first result states that the absolutely continuous part might be produced by means of the parallel sum:

Theorem 25 ([**53**, Theorem 4.6])**.** *Let $A,B\in\mathscr{L}(E;F)$ be positive operators on the weak-* sequentially complete anti-dual pair $\langle F,E\rangle$, then*

$$\lim_{n\to\infty}\langle ((nA):B)x,y\rangle=\langle B_a x,y\rangle,\qquad x,y\in E. \tag{29}$$

With the helps of the above result one might establish the following characterization of absolute continuity.

Theorem 26 ([**53**, Theorem 5.1])**.** *Let A,B be positive operators on the weak-* sequentially complete anti-dual pair $\langle F,E\rangle$. The following conditions are equivalent:*

(i) *B is absolutely continuous with respect to A.*
(ii) *B is almost dominated by A, that is, there exists a monotone increasing sequence $(B_n)_{n\in\mathbb{N}}$ of positive operators in $\mathscr{L}(E;F)$ and $(\alpha_n)_{n\in\mathbb{N}}$ of positive numbers such that $B_n\le\alpha_n A$ and $B_n\to B$ pointwise on E.*

We mention here that property (ii) agrees with the original definition of being absolutely continuous according to Ando.

A 'Radon-Nikodym type' characterization of absolute continuity is stated as follows:

Theorem 27 ([**53**, Theorem 5.3])**.** *For every pair $A,B\in\mathscr{L}(E;F)$ of positive operators the following statements are equivalent:*

(i) *B is absolutely continuous with respect to A,*
(ii) *for every $y\in E$ there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in E such that*

$$\langle Bx,y\rangle=\lim_{n\to\infty}\langle Ax,y_n\rangle,\qquad x\in E,$$

and the convergence is uniform on the set $\{x\in E:\langle (A+B)x,x\rangle\le 1\}$.

Characterizations of singularity. The original definition of singularity is rather algebraic, namely it depends on the ordering induced by positivity. However, singularity might be characterized through other properties which reflect some geometric and metric features. Such properties are settled in the next result: For analogous results see [**3**, **25**, **31**, **50**].

Theorem 28 ([**53**, Theorem 6.1])**.** *Let $\langle F,E\rangle$ be a weak-* sequentially complete anti-dual pair and let $A,B\in\mathscr{L}(E;F)$ be positive operators on it. The following assertions are equivalent:*

(i) *A and B are mutually singular,*
(ii) $A:B=0$,
(iii) *the set $\{(Ax,Bx):x\in E\}$ is dense in $\mathcal{H}_A\times\mathcal{H}_B$,*
(iv) *$\xi=0$ is the only vector in $\mathcal{H}_B$ such that $|(Bx\,|\,\xi)_B|^2\le M_\xi\langle Ax,x\rangle$ for every x in E,*
(v) $\mathcal{M}=\mathcal{H}_B$,
(vi) *for every x in E there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that*

$$\langle Ax_n,x_n\rangle\to 0\qquad\text{and}\qquad\langle B(x-x_n),x-x_n\rangle\to 0.$$

Lebesgue decomposition of positive operators on Hilbert spaces. Let $\mathcal{H}$ be a complex Hilbert space with inner product $(\cdot\,|\,\cdot)$, then $\langle\mathcal{H},\mathcal{H}\rangle$ forms a weak-* sequentially complete anti-dual pair with $\langle\cdot,\cdot\rangle := (\cdot\,|\,\cdot)$. Therefore, everything what has been said so far remains valid for $\langle\mathcal{H},\mathcal{H}\rangle$ and the positive operators on it. In particular, we retrieve Ando's main results [**3**, Theorem 2 and 6] immediately from Theorems 22, 26 and 35:

Theorem 29 ([**53**, Theorem 8.1])**.** *Let A, B be bounded positive operators on a complex Hilbert space $\mathcal{H}$ and let $B_a := \lim_{n\to\infty}(nA) : B$ where the limit is taken in the strong operator topology and let $B_s := B - B_a$. Then*

$$B = B_a + B_s \tag{30}$$

is a Lebesgue-type decomposition, i.e., B_a is A-absolutely continuous and B_s is A-singular. B_a is maximal among those positive operators $C \geq 0$ such that $C \leq B$ and $C \ll A$. The Lebesgue decomposition (30) *is unique if and only if $B_a \leq \alpha A$ for some constant $\alpha \geq 0$.*

Lebesgue decomposition of forms. Let $\mathfrak{D}$ be a complex vector space and let $\mathfrak{t}$, $\mathfrak{w}$ be nonnegative Hermitian forms on it. Let us denote by $\bar{\mathfrak{D}}^*$ the algebraic dual space of $\mathfrak{D}$, then $\langle\bar{\mathfrak{D}}^*,\mathfrak{D}\rangle$ forms a weak-* sequentially complete anti-dual pair and

$$\langle Tx, y\rangle := \mathfrak{t}(x,y), \qquad \langle Wx, y\rangle := \mathfrak{w}(x,y), \qquad x, y \in \mathfrak{D}$$

define two positive operators $T, W : \mathfrak{D} \to \bar{\mathfrak{D}}^*$. We recall that the form $\mathfrak{t}$ is called $\mathfrak{w}$-almost dominated if there is a monotonically nondecreasing sequence of forms $\mathfrak{t}_n$ such that $\mathfrak{t}_n \leq \alpha_n\,\mathfrak{w}$ for some $\alpha_n \geq 0$ and $\mathfrak{t}_n \to \mathfrak{t}$ pointwise. Similarily, $\mathfrak{t}$ is called $\mathfrak{w}$-closable if for every sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathfrak{D}$ such that $\mathfrak{w}(x_n, x_n) \to 0$ and $\mathfrak{t}(x_n - x_m, x_n - x_m) \to 0$ it follows that $\mathfrak{t}(x_n, x_n) \to 0$.

It is immediate to conclude that the form $\mathfrak{t}$ is $\mathfrak{w}$-closable if and only if the operator T is W-absolutely continuous. Similarly, $\mathfrak{t}$ is $\mathfrak{w}$-almost dominated precisely when T is W-almost dominated. Consequently, from Theorem 26 it follows that the notions of closability and almost dominatedness are equivalent (cf. also [**24**, Theorem 3.8]). The map $\mathfrak{t} \mapsto T$ between nonnegative hermiatian forms and positive operators on $\mathfrak{D}$ is a bijection, so from Theorems 22 and 35 we conclude the following result (see [**24**, Theorem 2.11 and 4.6]):

Theorem 30 ([**53**, Theorem 8.2])**.** *Let $\mathfrak{t}, \mathfrak{w}$ be nonnegative Hermitian forms on a complex vector space $\mathfrak{D}$ and let $\mathfrak{t}_a(x,x) := \lim_{n\to\infty}((n\,\mathfrak{t}) : \mathfrak{s})(x,x)$, $x \in \mathfrak{D}$ and $\mathfrak{t}_s := \mathfrak{t} - \mathfrak{t}_a$. Then*

$$\mathfrak{t} = \mathfrak{t}_a + \mathfrak{t}_s \tag{31}$$

is a Lebesgue-type decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$, i.e., $\mathfrak{t}_a$ is $\mathfrak{w}$-absolutely continuous and $\mathfrak{t}_s$ is $\mathfrak{w}$-singular. Furthermore, $\mathfrak{t}_a$ is maximal among those forms $\mathfrak{s}$ such that $\mathfrak{s} \leq \mathfrak{t}$ and $\mathfrak{s} \ll \mathfrak{w}$. The Lebesgue decomposition (31) *is unique if and only if $\mathfrak{t}_a \leq \alpha\,\mathfrak{w}$ for some constant $\alpha \geq 0$.*

Lebesgue decomposition of representable functionals. Let $\mathscr{A}$ be a *-algebra (with or without unit), i.e., an algebra endowed with an involution. A functional $f : \mathscr{A} \to \mathbb{C}$ is called representable if there is a triple $(\mathcal{H}_f, \pi_f, \zeta_f)$ such that $\mathcal{H}_f$ is a Hilbert space, $\zeta_f \in \mathcal{H}_f$ and $\pi_f : \mathscr{A} \to \mathscr{B}(\mathcal{H}_f)$ is a *-algebra homomorphism such that

$$f(a) = (\pi_f(a)\zeta_f \,|\, \zeta_f)_f, \qquad a \in \mathscr{A}.$$

A straightforward verification shows that every representable functional f is positive hence the map $A : \mathscr{A} \to \bar{\mathscr{A}}^*$ defined by

$$\langle Aa, b\rangle := f(b^*a), \qquad a, b \in \mathscr{A} \tag{32}$$

is a positive operator. (Note however that not every positive operator A arises from a representable functional f in the above way.) Denote by $\mathcal{H}_A$ the corresponding auxiliary Hilbert space. It is easy to show that $\pi_f : \mathscr{A} \to \mathscr{B}(\mathcal{H}_A)$, $a \mapsto \pi_f(a)$ is a *-homomorphism, where the bounded operator $\pi_f(a)$ arises from the densely defined one given by

$$\pi_f(a)(Ab) := A(ab), \qquad b \in \mathscr{A}.$$

It follows from the representability of f that $|f(a)|^2 \leq Cf(a^*a)$, $a \in \mathscr{A}$, for some constant $C \geq 0$ and hence

$$Aa \mapsto f(a), \qquad a \in \mathscr{A}$$

defines a continuous linear functional from $\operatorname{ran} A \subseteq \mathcal{H}_A$ to $\mathbb{C}$. The corresponding representing functional ζ_f satisfies

$$(Aa \,|\, \zeta_f)_A = f(a), \qquad a \in \mathscr{A},$$

and admits the useful property $\pi_f(a)\zeta_f = Aa$. It follows therefore that

$$f(a) = (\pi_f(a)\zeta_f \,|\, \zeta_f)_A, \qquad a \in \mathscr{A}.$$

Let g be another representable functional on $\mathscr{A}$. We say that g is f-absolutely continuous if for every sequence $(a_n)_{n\in\mathbb{N}}$ of $\mathscr{A}$ such that $f(a_n^*a_n) \to 0$ and $g((a_n - a_m)^*(a_n - a_m)) \to 0$ it follows that $g(a_n^*a_n) \to 0$. Furthermore, g and f are singular with respect to each other if $h = 0$ is the only representable functional such that $h \leq f$ and $h \leq g$.

Denote by $B : \mathscr{A} \to \bar{\mathscr{A}}^*$ the positive operator associated with g and let $(\mathcal{H}_B, \pi_g, \zeta_g)$ the corresponding GNS-triplet obtained along the above procedure. Let us introduce $\mathcal{M} \subseteq \mathcal{H}_B$ and P as in Section 3. Then $\mathcal{M}$ and $\mathcal{M}^\perp$ are both π_g-invariant, so

$$g_s(a) := (\pi_g(a)P\zeta_g \,|\, P\zeta_g)_B, \qquad g_a(a) := (\pi_g(a)(I-P)\zeta_g \,|\, (I-P)\zeta_g)_B$$

are representable functionals on $\mathscr{A}$ such that

$$\langle B_a a, b\rangle = g_a(b^*a), \qquad \langle B_s a, b\rangle = g_s(b^*a). \tag{33}$$

It is clear therefore that $g_a \ll f$ and $g_s \perp f$. If $\mathscr{A}$ has a unit element 1 then the absolutely continuous and singular parts can be written in a much simpler form:

$$g_a(a) = \overline{\langle B_a 1, a\rangle}, \qquad g_s(a) = \overline{\langle B_s 1, a\rangle}, \qquad a \in \mathscr{A}.$$

After these observations we can state the corresponding Lebesgue decomposition theorem of representable functionals [**23**, Corollary 3]:

Theorem 31 ([**51**, Theorem 3.3] and [**53**, Theorem 8.4])**.** *Let f, g be representable functionals on the *-algebra $\mathscr{A}$, then g_a and g_s are representable functionals such that $g = g_a + g_s$, where g_a is f-absolutely continuous and g_s is f-singular. Furthermore, g_a is is maximal among those representable functionals h such that $h \leq g$ and $h \ll f$.*

Lebesgue decomposition of additive set functions. Let X be a non-empty set and $\mathscr{R}$ be an algebra of sets on X. Let α be a non-negative finitely additive measure and denote by $\mathscr{S}$ the unital *-algebra of $\mathscr{R}$-measurable functions, then α induces a positive operator $A : \mathscr{S} \to \bar{\mathscr{S}}^*$ by

$$\langle A\varphi, \psi\rangle := \int \varphi\bar{\psi}\, d\alpha, \qquad \varphi, \psi \in \mathscr{S}.$$

We notice that we can easily recover α from A, namely

$$\alpha(R) = \langle A\chi_R, \chi_R\rangle, \qquad R \in \mathscr{R}. \tag{34}$$

However, not every positive operator $A : \mathscr{S} \to \bar{\mathscr{S}}^*$ induces a finitely additive measure, as it turns out from the next statement.

Proposition 32. *If $A : \mathscr{S} \to \bar{\mathscr{S}}^*$ is a positive operator then* (34) *defines an additive set function if and only if*

$$\langle A|\varphi|, |\varphi|\rangle = \langle A\varphi, \varphi\rangle, \qquad \varphi \in \mathscr{S}. \tag{35}$$

Assume that we are given another nonnegative additive set function β on $\mathscr{R}$, then β is called absolutely continuous with respect to α if for each $\varepsilon > 0$ there exists some $\delta > 0$ such that $R \in \mathscr{R}$ and $\alpha(R) < \delta$ imply $\beta(R) < \varepsilon$. Furthermore, α and β are mutually singular if $\gamma = 0$ is the only nonnegative additive set function such that $\gamma \le \alpha$ and $\gamma \le \beta$.

The Lebesgue decomposition of β with respect to α can be obtained by means of their induced positive operators.

Our claim is to prove that the Lebesgue decomposition of β with respect to α can also be derived from that of the induced positive operators. To this aim we note first that singularity of A and B obviously implies the singularity of α and β. It is less obvious that A-absolute continuity of B implies the α-absolute continuity of β (cf. also [**52**, Lemma 3.1]). To see this consider a sequence $(R_n)_{n\in\mathbb{N}}$ of $\mathscr{R}$ such that $\alpha(R_n) \to 0$. Clearly,

$$(J_A^*\chi_{R_n} \,|\, J_A^*\chi_{R_n})_A = \langle A\chi_{R_n}, \chi_{R_n}\rangle \to 0.$$

Since $(J_B^*\chi_{R_n} \,|\, J_B^*\chi_{R_n})_B \le \beta(X)$, the sequence $(J_B^*\chi_{R_n})_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_B$, and for every $\xi \in \operatorname{dom} \widehat{B}^*$,

$$(J_B^*\chi_{R_n} \,|\, \xi)_B = (J_A^*\chi_{R_n} \,|\, \widehat{B}^*\xi)_A \to 0.$$

Consequently, $J_B^*\chi_{R_n} \to 0$ weakly in $\mathcal{H}_B$, and hence $B\chi_{R_n} \to 0$ in $\bar{\mathscr{S}}^*$ with respect to the weak-* topology $\sigma(\bar{\mathscr{S}}^*, \mathscr{S})$. This implies that

$$\beta(R_n) = \langle B\chi_{R_n}, 1\rangle \to 0,$$

hence $\beta \ll \alpha$.

Theorem 33 ([**53**, Theorem 8.6])**.** *Let $\alpha, \beta : \mathscr{R} \to \mathbb{R}_+$ be nonnegative additive set functions. There exist two nonnegative additive set functions β_a, β_s such that $\beta = \beta_a + \beta_s$, where β_a is α-absolutely continuous and β_s is α-singular.*

Uniqueness. It was pointed out by Ando [**3**] that the Lebesgue decomposition among positive operators on an infinite dimensional Hilbert space is not unique. Since anti-dual pairs are even more general, we expect the same in our case. The reason why non-uniqueness occurs in the non-commutative integration theory is that absolute continuity is not hereditary: $B \ll A$ and $C \le B$ do not imply $C \ll A$. In fact, it may even happen that $C \ne 0$ and $C \perp A$. More explicitly, we have the following result:

Proposition 34 ([**53**, Proposition 7.1])**.** *Let A, B be positive operators on the weak-* sequentially complete anti-dual pair $\langle F, E\rangle$. Suppose that B is A-absolutely continuous but not A-dominated, i.e., there is no $\alpha \ge 0$ such that $B \le \alpha A$. Then there is a non-zero positive operator $B' \le B$ such that $B' \perp A$.*

The next result gives a complete characterization of uniqueness of the Lebesgue decomposition. We mention that this is a direct generalization of Ando's uniqueness result [**3**, Theorem 6]. We also refer the reader to [**28**, Theorem 7.8], [**24**, Theorem 4.6], [**31**, Theorem 2.8 and 2.9]; cf. also [**31**, Theorem 3.6 and Corollary 3.7]

Theorem 35 ([**53**, Theorem 7.2]). *Let $\langle F, E\rangle$ be a weak-* sequentially complete anti-dual pair and let $A, B \in \mathscr{L}(E;F)$ be positive operators. The following statements are equivalent:*

(i) *the Lebesgue-decomposition of B into A-absolutely continuous and A-singular parts is unique,*
(ii) $\operatorname{dom}\widehat{B}^* \subseteq \mathcal{H}_B$ *is closed,*
(iii) *the map $Ax \mapsto (I-P)Bx$ is norm continuous between $\mathcal{H}_A$ and $\mathcal{H}_B$,*
(iv) $B_a \leq \alpha A$ *for some $\alpha \geq 0$,*

Using Theorem 35 above one can easily exhibit some counterexamples showing non-uniqueness of Lebesgue decomposition of positive operators on Hilbert spaces as well nonnegative forms. However, the question of uniqueness is much more sophisticated in the context of representable functionals. of the Lebesgue-type decomposition. According to Kosaki [**32**], the Lebesgue decomposition of representable functionals is not necessarily unique, even in the case of von Neumann algebras. Kosaki's counterexample was rather complicated, a much more simpler one might be given:

Example 36 ([**55**, Example 6.6]). Assume $\mathcal{H}$ is infinite dimensional Hilbert spae and consider an orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ in it. Let $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ be two monotone decreasing sequences in ℓ^1 with positive coefficients such that $\alpha_n/\beta_n \to \infty$. Set

$$Fx := \sum_{n=1}^{\infty} \alpha_n (x\,|\,e_n)e_n, \qquad Gx := \sum_{n=1}^{\infty} \beta_n (x\,|\,e_n)e_n, \qquad x \in \mathcal{H}, \tag{36}$$

and define f, g by

$$f(T) := \operatorname{Tr}(FT), \qquad g(T) := \operatorname{Tr}(GT), \qquad T \in \mathscr{B}(\mathcal{H}).$$

Letting

$$F_n x := \sum_{k=1}^{n} \alpha_k (x\,|\,e_k)e_k$$

we have $F_n \leq F_{n+1} \to F$ in operator norm and also $F_n \leq \dfrac{\alpha_n}{\beta_n} G$. Hence F is G-absolutely continuous, i.e., $[G]F = F$. On the other hand, $F \leq cG$ is impossible because $\alpha_n/\beta_n \to \infty$. The G-Lebesgue decomposition of F is therefore not unique, in accordance with Theorem 35. Hence the g-Lebesgue decomposition of f fails to be unique as well.

As the above example shows, the Lebesgue-type decomposition of representable functionals is not necessarily unique even over von Neumann algebras. Nevertheless, it is possible to give a nontrivial sufficient condition for the uniqueness in terms of the regular part. As we shall see, this property can be necessary in some particular cases.

Theorem 37 ([**55**, Theorem 6.1]). *Let f and g be representable functionals on $\mathscr{A}$. If the f-absolutely continuous part g_a of g satisfies $g_a \leq cg$ holds for some $c \geq 0$, then the Lebesgue-type decomposition of g with respect to f is unique.*

Corollary 38. *Let $\mathscr{A}$ be a finite dimensional *-algebra and let f, g be representable positive functionals on $\mathscr{A}$. Then the Lebesgue decomposition of g with respect to f is always unique.*

Maps preserving absolute continuity and singularity. In the paper [**22**] we investigated so called singularity and absolute continuity preserving bijections. We say that a bijective map $\varphi : \mathscr{B}(\mathcal{H})_+ \to \mathscr{B}(\mathcal{H})_+$ preserves absolute continuity in both directions if

$$A \ll B \quad \iff \quad \varphi(A) \ll \varphi(B) \qquad \text{for all } A, B \in \mathscr{B}(\mathcal{H})_+.$$

Similarly, we say that a bijection $\varphi : \mathscr{B}(\mathcal{H})_+ \to \mathscr{B}(\mathcal{H})_+$ preserves singularity in both directions if

$$A \perp B \quad \iff \quad \varphi(A) \perp \varphi(B) \qquad \text{for all } A, B \in \mathscr{B}(\mathcal{H})_+.$$

To formulate our results, we need some further notation. With calligraphic letters we always denote linear (not necessarily closed) subspaces of $\mathcal{H}$ and we use the symbol $\mathrm{Lat}(\mathcal{H})$ for the set of all subspaces. A special subset of $\mathrm{Lat}(\mathcal{H})$ formed by operator ranges is denoted by

$$\mathrm{Lat}_{\mathrm{op}}(\mathcal{H}) := \{\mathcal{M} \subseteq \mathcal{H} : \exists\, S \in \mathscr{B}(\mathcal{H}),\ \operatorname{ran} S = \mathcal{M}\} = \{\operatorname{ran} A^{1/2} : A \in \mathscr{B}(\mathcal{H})_+\},$$

where the second identity is due to the range equality

$$\operatorname{ran} S = \operatorname{ran}(SS^*)^{1/2} \quad \text{for all } S \in \mathscr{B}(\mathcal{H}). \tag{37}$$

It is known that $\mathrm{Lat}_{\mathrm{op}}(\mathcal{H})$ forms a lattice and that $\mathrm{Lat}_{\mathrm{op}}(\mathcal{H}) \subsetneqq \mathrm{Lat}(\mathcal{H})$, for more information see [**19**].

For every positive integer n we set $\mathrm{Lat}_n(\mathcal{H})$ and $\mathrm{Lat}_{-n}(\mathcal{H})$ to be the set of all n-dimensional and n-codimensional operator ranges, respectively:

(a) $\mathrm{Lat}_n(\mathcal{H}) := \{\mathcal{M} \in \mathrm{Lat}_{\mathrm{op}}(\mathcal{H}) : \dim \mathcal{M} = n\} = \{\mathcal{M} \in \mathrm{Lat}(\mathcal{H}) : \dim \mathcal{M} = n\}$,
(b) $\mathrm{Lat}_{-n}(\mathcal{H}) := \{\mathcal{M} \in \mathrm{Lat}_{\mathrm{op}}(\mathcal{H}) : \operatorname{codim} \mathcal{M} = n\}$.

Observe also that $\mathrm{Lat}_{-n}(\mathcal{H})$ consists of all n codimensional *closed* subspaces of $\mathcal{H}$. We use the symbol $\mathbf{B}_+^n(\mathcal{H})$ to denote the set of all bounded positive operators with n dimensional range. We also introduce the following subset of $\mathscr{B}(\mathcal{H})_+$ which is associated with an operator range $\mathcal{M} \in \mathrm{Lat}_{\mathrm{op}}(\mathcal{H})$:

$$\mathcal{R}_{1/2}(\mathcal{M}) := \{C \in \mathscr{B}(\mathcal{H})_+ : \operatorname{ran} C^{1/2} = \mathcal{M}\}.$$

Note that $\mathcal{R}_{1/2}(\mathcal{M})$ is never empty according to (37).

In [**22**] we gave a complete description of bijections that preserve absolute continuity in both directions, and of those that preserve singularity in both directions. It turned out that these maps have the same structure.

Theorem 39 ([**22**, Theorem A]). *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and assume that $\varphi : \mathscr{B}(\mathcal{H})_+ \to \mathscr{B}(\mathcal{H})_+$ is a bijective map. Then the following four statements are equivalent:*

(i) *φ preserves absolutely continuity in both directions,*
(ii) *φ preserves singularity in both directions,*
(iii) *there exists a bounded, invertible, linear- or conjugate linear operator $T : \mathcal{H} \to \mathcal{H}$ such that*

$$\operatorname{ran} \varphi(A)^{1/2} = \operatorname{ran} TA^{1/2} \qquad \text{for all } A \in \mathscr{B}(\mathcal{H})_+, \tag{38}$$

(iv) *there exists a bounded, invertible, linear- or conjugate linear operator $T : \mathcal{H} \to \mathcal{H}$ and a family $\{Z_A : A \in \mathscr{B}(\mathcal{H})_+\}$ of invertible positive operators such that*

$$\varphi(A) = (TAT^*)^{1/2} Z_A (TAT^*)^{1/2} \qquad \text{for all } A \in \mathscr{B}(\mathcal{H})_+. \tag{39}$$

If $\dim \mathcal{H} < \infty$, then $\mathrm{Lat}(\mathcal{H}) = \mathrm{Lat}_{\mathrm{op}}(\mathcal{H})$, every operator has closed range, and $\operatorname{ran} A = \operatorname{ran} A^{1/2}$ holds for all $A \in \mathscr{B}(\mathcal{H})_+$. Therefore the notions of absolute continuity and singularity simplify considerably. In particular, the characterization of absolute continuity reduces to

$$A \ll B \quad \iff \quad \operatorname{ran} A \subseteq \operatorname{ran} B,$$

for every pair A, B of positive operators. Similarly, the range characterization of singularity reduces to

$$A \perp B \quad \iff \quad \operatorname{ran} A \cap \operatorname{ran} B = \{0\}.$$

Furthermore, we have $\mathcal{R}_{1/2}(\mathcal{M}) = \{C \in \mathscr{B}(\mathcal{H})_+ : \operatorname{ran} C = \mathcal{M}\}$ for all $\mathcal{M} \in \mathrm{Lat}(\mathcal{H})$. Therefore the finite dimensional version of Theorem A can be proved much more easily using the fundamental theorem of projective geometry provided that $\dim H > 2$. However, we point out that the result we get is slightly different, as T is not necessarily linear- or conjugate linear anymore.

Theorem 40 ([**22**, Theorem B])**.** *Let $\mathcal{H}$ be a complex Hilbert space such that $3 \leq \dim \mathcal{H} < +\infty$ and let $\varphi : \mathscr{B}(\mathcal{H})_+ \to \mathscr{B}(\mathcal{H})_+$ be a bijective map. Then the following three statements are equivalent:*

(i) *φ preserves absolutely continuity in both directions,*
(ii) *φ preserves singularity in both directions,*
(iii) *there is a semilinear bijection $T : \mathcal{H} \to \mathcal{H}$ such that*

$$\operatorname{ran} \varphi(A) = \operatorname{ran} TA \quad \text{for all } A \in \mathscr{B}(\mathcal{H})_+.$$

Finally, in case when $\dim \mathcal{H} = 2$, the fundamental theorem of projective geometry cannot be applied. However, one can prove easily that points (i) and (ii) are both equivalent with the following condition:

(iii') $\varphi(0) = 0$, φ maps the set of all invertible positive operators bijectively onto itself, and there is a bijection $\Psi\colon \mathrm{Lat}_1(\mathcal{H}) \to \mathrm{Lat}_1(\mathcal{H})$ such that

$$\operatorname{ran} \varphi(A) = \Psi(\operatorname{ran} A) \quad \text{for all } A \in \mathbf{B}^1_+(\mathcal{H}).$$

Schur complementation ([58])

The next theorem provides necessary and suphicient conditions for positivity of an incomplete operator matrix of the form $\left[\begin{smallmatrix} A & B^* \\ B & * \end{smallmatrix}\right]$.

Theorem 41 ([**58**, Theorem 2.1])**.** *Let $\langle F_1, E_1\rangle$ and $\langle F_2, E_2\rangle$ be weak-* sequentially complete anti-dual pairs and let $A \in \mathscr{L}(E_1; F_1)$ and $B \in \mathscr{L}(E_1; F_2)$ be weakly continuous linear operators such that $A \geq 0$. Then the following assertions are equivalent:*

(i) *There is a positive operator $C \in \mathscr{L}(E_2; F_2)$ such that the operator matrix $\left[\begin{smallmatrix} A & B^* \\ B & C \end{smallmatrix}\right]$ is positive.*
(ii) *For every $y_2 \in E_2$ there exists a constant $M_{y_2} \geq 0$ such that*

$$|\langle Bx_1, y_2\rangle|^2 \leq M_{y_2} \cdot \langle Ax_1, x_1\rangle \qquad \text{for all } x_1 \in E_1.$$

(iii) *For the canonical embedding operator $J_A : \mathcal{H}_A \to F_1$ constructed in (7) the following range inclusion holds*

$$\operatorname{ran} B^* \subseteq \operatorname{ran} J_A.$$

If any of the above conditions is fulfilled, then the linear operator

$$S_0 : \mathcal{H}_A \supseteq \operatorname{ran} A \to F_2; \qquad Ax_1 \mapsto Bx_1, \qquad x_1 \in E_1 \tag{40}$$

is well defined and weakly continuous. Furthermore, its unique continuous extension $S \in \mathscr{L}(\mathcal{H}_A; F_2)$ *possesses the property that*

$$A_B := SS^* \tag{41}$$

is the smallest positive operator that makes $\left[\begin{smallmatrix} A & B^* \\ B & * \end{smallmatrix}\right]$ *positive. The quadratic form of* A_B *is given by*

$$\langle SS^* y_2, y_2\rangle = \sup\big\{|\langle Bx_1, y_2\rangle|^2 : x_1 \in E_1, \langle Ax_1, x_1\rangle \leq 1\big\} \tag{42}$$

$$= \sup\big\{\langle Bx_1, y_2\rangle + \langle B^* y_2, x_1\rangle - \langle Ax_1, x_1\rangle : x_1 \in E_1\big\}. \tag{43}$$

Definition 42. We refer to the operator A_B in (41) as the *complement* of A with respect to B. If $C \in \mathscr{L}(E_2; F_2)$ is any positive operator that makes the system $\left[\begin{smallmatrix} A & B^* \\ B & C \end{smallmatrix}\right]$ positive, then $C - A_B$ is called the *Schur complement* of C in the block matrix $\left[\begin{smallmatrix} A & B^* \\ B & C \end{smallmatrix}\right]$.

As a straightforward consequence of Theorem 41 we retrieve the classical result [**40**, §1 Condition 2' and Theorem 1.1] of Pekarev and Šmul'jan.

Corollary 43 ([**58**, Corollary 2.3])**.** *Let* $\mathcal{H}$ *be a Hilbert space and let* $A, B \in \mathscr{B}(\mathcal{H})$ *be bounded operators. Assume further that* A *is positive. Then the following assertions are equivalent:*

(i) *There exists a* $C \in \mathscr{B}(\mathcal{H})$ *such that* $\left[\begin{smallmatrix} A & B^* \\ B & C \end{smallmatrix}\right]$ *is positive,*
(ii) $B^*B \leq mA$ *for some constant* $m \geq 0$*,*
(iii) $\operatorname{ran} B^* \subseteq \operatorname{ran} A^{1/2}$.

As an immediate consequence of Theorem .... we gain a new equivalent definition the parallel sum. Namely, $A : B$ can be obtained as the Schur complement of A in the block matrix $\left[\begin{smallmatrix} A+B & A \\ A & A \end{smallmatrix}\right]$:

Proposition 44. *Let* $A, B \in \mathscr{L}(E; F)$ *be positive operators. Then*

$$A : B = A - (A + B)_A.$$

Another useful transformation of two positive operators is the so called *parallel difference*. It might be defined in an analogous way to parallel sum, namely as the Schur complement of A in the block matrix $\left[\begin{smallmatrix} A-B & A \\ A & A \end{smallmatrix}\right]$, provided it exists.

Definition 45. Assume that $A, B \in \mathscr{L}(E; F)$ are operators such that $(A - B)_A$ does exist. Then the operator defined by

$$B \div A := (A - B)_A - A \tag{44}$$

is called the *parallel difference* of B and A. The quadratic form of $B \div A$ can be calculated as

$$\langle (B \div A)y, y\rangle = \sup\{\langle B(x + y), x + y\rangle - \langle Ax, x\rangle : x \in E\}. \tag{45}$$

As a significant application of parallel sum and parallel difference we reprove the Lebesgue decomposition theorem:

Theorem 46 ([**58**, Theorem 2.7])**.** *Assume that* A *and* B *are positive operators belonging to* $\mathscr{L}(E; F)$*. Then the operator* $(A : B) \div A$ *is equal to the absolute continuous part* B_a *of* B*. In particular,*

$$B = (A : B) \div A + [B - (A : B) \div A]$$

is idential with the a Lebesgue decomposition of B *with respect to* A*.*

As a nontrivial application of Theorem 41 one may introduce the Schur complement of representable positive functionals:

Theorem 47 ([**58**, Theorem 3.4])**.** *Let f, g be linear functionals on a *-algebra $\mathscr{A}$. Suppose that f is representable and that there is a constant $C \geq 0$ such that*

$$|g(a)|^2 \leq Cf(a^*a), \qquad a \in \mathscr{A}. \tag{46}$$

Then there exists a representable positive functional h such that $f + g + g^* + h$ is representable and

$$f(a^*a) + g(b^*a) + \overline{g(b^*a)} + h(b^*b) \geq 0 \tag{47}$$

for all a, b in $\mathscr{A}$. Furthermore, there is a smallest h possessing this property.

Definition 48. We call the smallest representable functional satisfying (47) the *complement of f with respect to g*, and we denote it by f_g.

The complement f_g can be calculated as

$$f_g(a) = (\pi_f(a)\eta_g \,|\, \eta_g)_f, \qquad a \in \mathscr{A},$$

where $\eta_g \in \mathcal{H}_f$ is the representing vector of the bounded linear functional

$$\mathcal{H}_f \to \mathbb{C}; \qquad \pi_f(a)\xi_f \mapsto g(a).$$

Similarly to the case of positive operators the parallel sum, parallel difference, as well as the Lebesgue decomposition of representable functionals may be obtained by means of the Schur complement.

Theorem 49 ([**58**, Definition 3.7-8 and Theorem 3.10])**.** *Let f, g be representable positive functionals on a *-algebra $\mathscr{A}$.*

(a) *Then $(f + g)_f$ and $(f : g) := f - (f + g)_f$ are representable functionals and one has*
$$(f : g)(a^*a) = \inf\{f((a + b)^*(a + b)) + g(b^*b) : b \in \mathscr{A}\}.$$
(b) *If $f \geq g$ and the complement $(f - g)_f \in \mathscr{A}^\sharp$ exists then the parallel difference $g \div f := (f - g)_f - f$ is a representable functional such that*
$$(g \div f)(a^*a) = \sup\{g((a + b)^*(a + b)) - f(b^*b) : b \in \mathscr{A}\}.$$
(c) *The representable functional $g_a = (g : f) \div f$ is representable and f-absolutely continuous and the decomposition*
$$g = g_a + (g - g_a)$$
is a Lebesgue-type decomposition of g with respect to f.

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