SHARP MORREY-SOBOLEV INEQUALITIES AND EIGENVALUE PROBLEMS ON RIEMANNIAN-FINSLER MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

ALEXANDRU KRISTÁLY, ÁGNES MESTER, AND ILDIKÓ I. MEZEI

Abstract. Combining the sharp isoperimetric inequality established by Z. Balogh and A. Kristály [*Math. Ann.*, in press, doi.org/10.1007/s00208-022-02380-1] with an anisotropic symmetrization argument, we establish sharp Morrey-Sobolev inequalities on n-dimensional Finsler manifolds having nonnegative n-Ricci curvature. A byproduct of this method is a Hardy-Sobolev-type inequality in the same geometric setting. As applications, by using variational arguments, we guarantee the existence/multiplicity of solutions for certain eigenvalue problems and elliptic PDEs involving the Finsler-Laplace operator. Our results are also new in the Riemannian setting.

1. Introduction

Most of elliptic PDEs are studied over Sobolev spaces which are usually embedded into certain Lebesgue spaces; this fact is described quantitatively by Sobolev inequalities. Within this theory, a prominent class of Sobolev inequalities is provided by those defined on curved structures. Motivated mainly by the Yamabe problem, Aubin [2] initiated in the early seventies the so-called *AB-program*, i.e., to determine the best constants within such Sobolev inequalities on Riemannian manifolds. It turned out that this study deeply depends on the curvature of the ambient space, and it is still a very active area of geometric analysis. A comprehensive work in this topic is provided by Hebey [15] and subsequent references.

Roughly speaking, two main classes of Sobolev inequalities can be distinguished, depending on the curvature restriction of (noncompact) Riemannian manifolds, having: (a) nonpositive sectional curvature, or (b) Ricci curvature bounded from below.

In case (a), Sobolev inequalities similar to Euclidean ones hold on Cartan-Hadamard manifolds[1], having the same sharp constants as in their Euclidean counterparts, see e.g. Druet, Hebey and Vaugon [12], Hebey [15] and Muratori and Soave [25]. One of the main tools to prove such Sobolev inequalities is a Schwarz-type symmetrization 'from manifolds to Euclidean spaces', combined with sharp isoperimetric inequalities, known as the Cartan-Hadamard conjecture, which is valid in low-dimensions; see e.g. Ghomi and Spruck [14], and Kloeckner and Kuperberg [17].

In case (b), the existence of a lower bound for the volume of small balls which is uniform w.r.t. their center characterizes the validity of Sobolev inequalities, see Hebey [15, Chapter 3]. In the particular case when the Ricci curvature is nonnegative, rigidity phenomena occur, i.e., a Sobolev inequality with the same Sobolev constant as in its Euclidean counterpart is supported on such a manifold if and only if the manifold is isometric to the Euclidean space, see Ledoux [23]. Quantitatively speaking, a close constant in a Sobolev inequality to its optimal Euclidean counterpart implies 'topologically closer' manifold to the Euclidean space, described by the trivialization of homotopy groups, see do Carmo and Xia [13] (and its nonsmooth version for $CD(0,N)$ spaces in Kristály [20]).

Very recently, Balogh and Kristály [4] proved sharp L^p-Sobolev inequalities on n-dimensional Riemannian manifolds having nonnegative Ricci curvature and Euclidean volume growth, whenever $1 < p < n$. They used symmetrization arguments and a sharp isoperimetric inequality

---

2000 *Mathematics Subject Classification.* Primary: 58J05, 53C23; Secondary: 53C60.

Key words and phrases. Morrey-Sobolev inequality; Riemann-Finsler manifolds; Ricci curvature; sharp isoperimetric inequality; anisotropic symmetrization.

A. Kristály is supported by the UEFISCDI/CNCS grant PN-III-P4-ID-PCE2020-1001 (Romania) and Excellence Scholarship Program ÓE-KP-2-2022 (Hungary). I.I. Mezei is supported by the UEFISCDI/CNCS grant PN-III-P4-ID-PCE2020-1001 (Romania). Á. Mester is supported by the National Research, Development and Innovation Fund of Hungary, financed under the K_18 funding scheme, Project No. 127926 (Hungary).

[1]Complete, simply connected Riemannian manifolds with nonpositive sectional curvature.

recently proved by Brendle [7], and alternatively, by themselves [4]. We notice that the sharp isoperimetric inequality in [4] is valid even for generic $CD(0,N)$ spaces, thus in particular, for reversible Finsler manifolds with nonnegative n-Ricci curvature (for short, $\mathsf{Ric}_n \geq 0$, see §2).

Based on the sharp isoperimetric inequality in [4], the main purpose of the present paper is to adapt a suitable Schwarz-type symmetrization argument to Finsler manifolds with $\mathsf{Ric}_n \geq 0$ and to establish (possibly sharp) functional inequalities. In particular, we aim to handle the complementary case $p > n$ w.r.t. [4] by establishing sharp Morrey-Sobolev inequalities on such Finsler structures. We remark that our results are also new in the Riemannian setting.

In order to give a flavor of our results, let (M,F) be a noncompact, complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n \geq 0$, endowed with the canonical volume form $\mathrm{d}v_F$ and the induced Finsler metric $d_F : M \times M \to \mathbb{R}$, and let F^* be the polar transform of F; for these notions, see §2. Let $B_x(r) = \{z \in M : d_F(x,z) < r\}$ be the geodesic ball with center $x \in M$ and radius $r > 0$. The *asymptotic volume ratio of* (M,F) is defined as

$$\mathsf{AVR}_F = \lim_{r\to\infty} \frac{\mathrm{Vol}_F(B_x(r))}{\omega_n r^n},$$

where $\mathrm{Vol}_F(S) = \displaystyle\int_S \mathrm{d}v_F$ for any measurable set $S \subset M$, while $\omega_n = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2})$ denotes the volume of the Euclidean open unit ball in $\mathbb{R}^n$. Note that $\mathsf{AVR}_F \in [0,1]$ is well defined, i.e., it is independent of the choice of $x \in M$. Also, by the generalized Bishop-Gromov volume growth inequality, see Shen [31], we have that the mapping $r \mapsto \frac{\mathrm{Vol}_F(B_x(r))}{r^n}$ is nonincreasing on $(0,\infty)$ for every $x \in M$. We say that (M,F) has *Euclidean volume growth* whenever $\mathsf{AVR}_F > 0$.

The key result is a *Pólya-Szegő inequality* on Finsler manifolds with $\mathsf{Ric}_n \geq 0$, involving the asymptotic volume ratio AVR_F (see Theorem 3.1), whose proof is based on the sharp isoperimetric inequality from [4] and a symmetrization argument in the spirit of Aubin [2]. Although the latter symmetrization is well known, a careful adaptation is needed 'from manifolds to normed spaces'. The first main consequence is the following sharp Morrey-Sobolev inequality with *support-bound*:

Theorem 1.1. *Let (M,F) be a noncompact, complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n \geq 0$ and Euclidean volume growth $0 < \mathsf{AVR}_F \leq 1$. If $p > n \geq 2$, then one has*

$$\|u\|_{L^\infty(M)} \leq \mathsf{T}_F^{\mathsf{MS}}\, \mathrm{Vol}_F(\mathrm{supp}\, u)^{\frac{1}{n}-\frac{1}{p}} \left(\int_M F^*(x,Du(x))^p \mathrm{d}v_F\right)^{\frac{1}{p}}, \quad \forall u \in C_0^\infty(M), \tag{1.1}$$

where the constant

$$\mathsf{T}_F^{\mathsf{MS}} = n^{-\frac{1}{p}} \omega_n^{-\frac{1}{n}} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}} \mathsf{AVR}_F^{-\frac{1}{n}}$$

is sharp.

A counterpart of Theorem 1.1 is the following sharp Morrey-Sobolev inequality with L^1-*bound*:

Theorem 1.2. *Let (M,F) be a noncompact, complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n \geq 0$ and $0 < \mathsf{AVR}_F \leq 1$, and for any $p > n \geq 2$, consider the constant $\eta = \frac{np}{np+p-n}$. Then, for every $u \in C_0^\infty(M)$ one has*

$$\|u\|_{L^\infty(M)} \leq \mathsf{C}_F^{\mathsf{MS}} \left(\int_M |u(x)| \mathrm{d}v_F\right)^{1-\eta} \left(\int_M F^*(x,Du(x))^p \mathrm{d}v_F\right)^{\frac{\eta}{p}}, \tag{1.2}$$

where the constant

$$\mathsf{C}_F^{\mathsf{MS}} = (n\omega_n^{\frac{1}{n}})^{-\frac{np'}{n+p'}} \left(\frac{1}{n}+\frac{1}{p'}\right)\left(\frac{1}{n}-\frac{1}{p}\right)^{\frac{(n-1)p'-n}{n+p'}} \left(\mathsf{B}\left(\frac{1-n}{n}p'+1, p'+1\right)\right)^{\frac{n}{n+p'}} \mathsf{AVR}_F^{-\frac{\eta}{n}}$$

is sharp. Hereafter, $p' = \frac{p}{p-1}$, and $\mathsf{B}(\cdot,\cdot)$ denotes the Euler beta-function.

Theorems 1.1 and 1.2 are also new in the Riemannian setting and can be viewed as new pieces within the aforementioned *AB-program* of Aubin [2]. Note that Kristály [18] proved that whenever (1.1) and (1.2) hold with generic constants instead of $\mathsf{C}_F^{\mathsf{MS}}$ and $\mathsf{T}_F^{\mathsf{MS}}$, then there is a non-collapsing phenomenon of the metric balls on the Riemannian manifold. In the Euclidean case (when the asymptotic volume ratio is 1), Theorems 1.1 and 1.2 reduce to well known results of Talenti [32]. A natural question arises: are there nonzero extremal functions in (1.1) and (1.2)? At this moment, we can provide such an answer within the class of Riemannian manifolds: equality occurs in (1.1) and (1.2) for a nonzero, enough smooth function if and only if the Riemannian manifold is isometric to the Euclidean space $\mathbb{R}^n$, see Theorem 4.1.

A natural byproduct of our arguments is the validity of Sobolev inequalities involving *singular terms* within the same geometric setting as above; in fact, we can prove the following Hardy-Sobolev-type inequality:

Theorem 1.3. *Let (M,F) be a noncompact, complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n \geq 0$ and Euclidean volume growth $0 < \mathsf{AVR}_F \leq 1$. If $n > p > 1$, we have for every $x_0 \in M$ that*

$$\int_M F^*(x, Du(x))^p \mathrm{d}v_F \geq \mathsf{AVR}_F^{\frac{p}{n}} \left(\frac{n-p}{p}\right)^p \int_M \frac{|u(x)|^p}{d_F(x_0,x)^p} \mathrm{d}v_F, \quad \forall u \in C_0^\infty(M). \tag{1.3}$$

As before, Theorem 1.3 is also new in the Riemannian framework; although expected, we do not know the sharpness of (1.3) unless we are in the Euclidean setting. Theorem 1.3 can be viewed as a counterpart of Hardy-Sobolev inequalities on Cartan-Hadamard manifolds, established e.g. by Berchio, Ganguly, Grillo and Pinchover [6], D'Ambrosio and Dipierro [11], Huang, Kristály and Zhao [16], Kristály [19], and Zhao [34]. An improved version of Theorem 1.3 for $p = 2$ is stated in Theorem 4.2, which is a Brezis-Poincaré-Vázquez inequality on Finsler manifolds.

The second purpose of the paper – which is not detailed here – is to show the applicability of the aforementioned functional inequalities. First, by using Theorem 1.1 and a variational argument à la Ricceri [29], in Theorem 5.1 we provide a multiplicity result for an elliptic PDE involving the p-Finsler-Laplace operator. Second, by applying the Hardy-Sobolev inequality (Theorem 1.3) and the Brezis-Poincaré-Vázquez inequality (Theorem 4.2), we provide a sufficient condition to guarantee the existence of a positive solution for a sub-critical elliptic PDE involving the 2-Finsler-Laplace operator, see Theorem 5.3.

Finally, we emphasize the richness of those Finsler manifolds where our results can be applied. Beside Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth (extensively studied in [4]), we provide a whole class of non-Riemannian Finsler manifolds modeled over $\mathbb{R}^n$ with the required properties. More precisely, we endow the space $\mathbb{R}^{n-1}$ ($n \geq 3$) with a complete Riemannian metric g having nonnegative Ricci curvature and assume that the induced warped metric $\tilde{g}$ on $\mathbb{R}^{n-1} \times \mathbb{R}$, defined by $\tilde{g}_{(x,t)}(v,w) = \sqrt{g_x(v,v) + w^2}$, $(x,t) \in \mathbb{R}^n, (v,w) \in T_x\mathbb{R}^{n-1} \times T_t\mathbb{R}$, has Euclidean volume growth $0 < \mathsf{AVR}_{\tilde{g}} \leq 1$. Then the parameter-depending Finsler manifold $(\mathbb{R}^n, F_\varepsilon)$, with $\varepsilon > 0$, defined by means of the Finsler metric $F_\varepsilon : T\mathbb{R}^n \to [0,\infty)$,

$$F_\varepsilon((x,t),(v,w)) = \sqrt{g_x(v,v) + w^2 + \varepsilon\sqrt{g_x(v,v)^2 + w^4}}, \quad (x,t) \in \mathbb{R}^n, (v,w) \in T_x\mathbb{R}^{n-1} \times T_t\mathbb{R},$$

has the properties that $\mathsf{Ric}_n \geq 0$ and $0 < \mathsf{AVR}_{F_\varepsilon} \leq 1$; for details, see §4.4.

The paper is organized as follows. In Section 2 we recall those auxiliary notions that are used throughout the whole paper. Section 3 is devoted to the anisotropic symmetrization, by proving among others, a Pólya-Szegő inequality involving the asymptotic volume ratio AVR_F together with a Hardy-Littlewood-Pólya inequality. In Section 4 we prove Morrey-Sobolev and Hardy-Sobolev inequalities on Finsler manifolds with $\mathsf{Ric}_n \geq 0$ and $\mathsf{AVR}_F \in (0,1]$, and discuss the geometric properties of the non-Riemannian Finsler manifolds $(\mathbb{R}^n, F_\varepsilon)$ introduced in the previous paragraph. Applications of Sobolev inequalities are presented in Section 5 for two elliptic PDEs, both involving the Finsler-Laplace operator.

2. Preliminaries on Finsler geometry

Let M be a connected n-dimensional smooth manifold and $TM = \bigcup_{x\in M} T_xM$ its tangent bundle. The pair (M,F) is called a Finsler manifold if $F : TM \to [0,\infty)$ is a continuous function such that

(i) F is C^∞ on $TM \setminus \{0\}$;
(ii) $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda \geq 0$ and all $(x,y) \in TM$;

(iii) the $n\times n$ Hessian matrix $\Big(g_{ij}(x,y)\Big) := \Big(\big[\frac{1}{2}F^2(x,y)\big]_{y^iy^j}\Big)$ is positive definite for all $(x,y)\in TM\setminus\{0\}$.

If, in addition, $F(x,\lambda y)=|\lambda|F(x,y)$ holds for every $\lambda\in\mathbb{R}$ and $(x,y)\in TM$, then the Finsler manifold is called reversible.

A curve $\gamma:[0,r]\to M$ is called a geodesic if its velocity field $\dot\gamma$ is parallel along the curve, i.e., $D_{\dot\gamma}\dot\gamma=0$, where D denotes the covariant derivative induced by the Chern connection, see Bao, Chern and Shen [5, Chapter 2]. (M,F) is said to be complete if every geodesic $\gamma:[0,r]\to M$ can be extended to a geodesic defined on $\mathbb{R}$.

The distance function $d_F: M\times M\to[0,\infty)$ is defined by

$$d_F(x_1,x_2)=\inf_{\gamma\in\Lambda(x_1,x_2)}\int_0^r F(\gamma(t),\dot\gamma(t))\mathrm{d}t,$$

where $\Lambda(x_1,x_2)$ denotes the set of all piecewise C^∞ curves $\gamma:[0,r]\to M$ such that $\gamma(0)=x_1$ and $\gamma(r)=x_2$. Clearly, $d_F(x_1,x_2)=0$ if and only if $x_1=x_2$, and d_F verifies the triangle inequality, as well. However, d_F is symmetric if and only if (M,F) is a reversible Finsler manifold.

Let $B_x(1)=\Big\{(y^i)\in\mathbb{R}^n:\ F\Big(x,\sum_{i=1}^n y^i\frac{\partial}{\partial x^i}\Big)<1\Big\}\subset\mathbb{R}^n$, and define the ratio

$$\sigma_F(x)=\frac{\omega_n}{\mathrm{Vol}(B_x(1))},$$

where Vol denotes in the sequel the canonical Euclidean volume, and $\omega_n=\pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2})$ is the volume of the n-dimensional Euclidean open unit ball. The Busemann-Hausdorff volume form is defined as

$$\mathrm{d}v_F(x)=\sigma_F(x)dx^1\wedge\cdots\wedge dx^n, \tag{2.1}$$

see Shen [30, Section 2.2].

For a fixed point $x\in M$ let $y,v\in T_xM$ be two linearly independent tangent vectors. Then the flag curvature is defined as

$$\mathrm{K}^y(y,v)=\frac{g_y(\mathrm{R}(y,v)v,y)}{g_y(y,y)g_y(v,v)-g_y(y,v)^2},$$

where g is the fundamental tensor induced by the Hessian matrices (g_{ij}), and R is the Chern curvature tensor, see Bao, Chern and Shen [5, Chapter 3]. The Ricci curvature is defined as

$$\mathsf{Ric}(y)=F^2(x,y)\sum_{i=1}^{n-1}\mathrm{K}^y(y,e_i),$$

where $\{e_1,...,e_{n-1},\frac{y}{F(x,y)}\}$ is an orthonormal basis of T_xM with respect to g_y.

Let $\{e_i\}_{i=1,...,n}$ be a basis for T_xM. The mean distortion $\mu: TM\setminus\{0\}\to(0,\infty)$ is defined by $\mu(x,y)=\frac{\mathrm{Vol}(B_x(1))}{\omega_n}\sqrt{\det(g_{ij}(x,y))}$. The mean covariation $\mathcal{H}: TM\setminus\{0\}\to\mathbb{R}$ is defined by $\mathcal{H}(x,y)=\frac{d}{dt}(\ln\mu(\gamma_{(x,y)}(t),\dot\gamma_{(x,y)}(t)))|_{t=0}$, where $\gamma_{(x,y)}$ is the geodesic such that $\gamma_{(x,y)}(0)=x$ and $\dot\gamma_{(x,y)}(0)=y$. We say that (M,F) has nonnegative n-Ricci curvature, denoted by $\mathsf{Ric}_n\geq 0$, if $\mathsf{Ric}\geq 0$ and the mean covariation $\mathcal{H}$ is identically zero. Note that Finsler manifolds of Berwald type (i.e., the coefficients of the Chern connection $\Gamma^k_{ij}(x,y)$ do not depend on $y\in T_xM$) endowed with the Busemann-Hausdorff measure have vanishing mean covariation, see Shen [31]; this class contains both Riemannian manifolds and Minkowski spaces.

Endowed with the canonical volume form $\mathrm{d}v_F$, the n-dimensional Finsler manifold (M,F) verifies for every $x\in M$ that

$$\lim_{r\to 0^+}\frac{\mathrm{Vol}_F(B_x(r))}{\omega_n r^n}=1. \tag{2.2}$$

In addition, if (M,F) is a complete Finsler manifold with $\mathsf{Ric}_n\geq 0$, the Bishop-Gromov volume comparison principle states that $r\mapsto\frac{\mathrm{Vol}_F(B_x(r))}{r^n}$ is a nonincreasing function on $(0,\infty)$, see Shen [31, Theorem 1.1]. In particular, from the latter property and (2.2), it turns out that $\mathsf{AVR}_F\in[0,1]$.

The polar transform $F^* : T^*M \to [0,\infty)$ is defined as the dual metric of F, i.e.,

$$F^*(x,\alpha) = \sup_{y\in T_xM\setminus\{0\}} \frac{\alpha(y)}{F(x,y)}.$$

Let $u : M \to \mathbb{R}$ be a differentiable function in the distributional sense. The gradient of u is defined as $\nabla_F u(x) = J^*(x, Du(x))$, where $Du(x) \in T_x^*M$ denotes the (distributional) derivative of u at $x \in M$ and J^* is the Legendre transform given by

$$J^*(x,\alpha) = \sum_{i=1}^n \frac{\partial}{\partial \alpha_i}\left(\frac{1}{2}F^{*2}(x,\alpha)\right)\frac{\partial}{\partial x^i}.$$

In local coordinates, we have

$$Du(x) = \sum_{i=1}^n \frac{\partial u}{\partial x^i}(x)\mathrm{d}x^i \quad \text{and} \quad \nabla_F u(x) = \sum_{i,j=1}^n g_{ij}^*(x, Du(x))\frac{\partial u}{\partial x^i}(x)\frac{\partial}{\partial x^j},$$

where (g_{ij}^*) is the Hessian matrix $\Big(g_{ij}^*(x,\alpha)\Big) = \left(\left[\frac{1}{2}F^{*2}(x,\alpha)\right]_{\alpha^i\alpha^j}\right)$, see Ohta and Sturm [27, Lemma 1.1]. Therefore, the gradient operator ∇_F is usually nonlinear.

We recall the eikonal equation, i.e., if $x_0 \in M$ is fixed, then by Ohta and Sturm [27], one has

$$F^*(x, Dd_F(x_0,x)) = F(x, \nabla_F d_F(x_0,x)) = Dd_F(x_0,x)(\nabla_F d_F(x_0,x)) = 1 \text{ for a.e. } x\in M. \quad (2.3)$$

If X is a vector field on M, then, in local coordinates, the divergence of X is defined as $\mathrm{div}(X) = \frac{1}{\sigma_F}\frac{\partial}{\partial x^i}(\sigma_F X^i)$. The p-Finsler-Laplace operator is given by

$$\Delta_{F,p}u(x) = \mathrm{div}\big(F^*(x,Du(x))^{p-2}\cdot \nabla_F u(x)\big),$$

while the divergence theorem reads as

$$\int_M v(x)\Delta_{F,p}u(x)\,\mathrm{d}v_F = -\int_M F^*(x,Du(x))^{p-2}\cdot Dv(x)\big(\nabla_F u(x)\big)\,\mathrm{d}v_F, \quad \forall v\in C_0^\infty(M), \quad (2.4)$$

see Ohta and Sturm [27]. Note that in general, $\Delta_{F,p}$ is nonlinear. When $p=2$, the 2-Finsler-Laplace operator is simply denoted by $\Delta_F \coloneqq \Delta_{F,2}$.

Finally, let Ω be an open subset of M. The Sobolev space on Ω associated with the Finsler structure F is defined by

$$W_F^{1,p}(\Omega) = \left\{u\in W_{\mathrm{loc}}^{1,p}(\Omega) : \int_\Omega F^*(x,Du(x))^p\,\mathrm{d}v_F < +\infty\right\},$$

while $W_{0,F}^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_F^{1,p}(\Omega)} = \left(\int_\Omega |u(x)|^p\,\mathrm{d}v_F + \int_\Omega F^*(x,Du(x))^p\,\mathrm{d}v_F\right)^{\frac{1}{p}}.$$

3. Anisotropic symmetrization on Finsler manifolds with $\mathsf{Ric}_n \geq 0$

In what follows, let (M,F) be a noncompact, complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n \geq 0$, endowed with the canonical volume form $\mathrm{d}v_F$ and the induced Finsler metric $d_F : M\times M \to \mathbb{R}$. In particular, $(M, d_F, \mathrm{d}v_F)$ is a metric measure space satisfying the $CD(0,n)$ condition, see Ohta [26]. Consequently, by Balogh and Kristály [4, Theorem 1.1], for every bounded open set $\Omega\subset M$ with smooth boundary, we have the sharp isoperimetric inequality

$$\mathcal{P}_F(\partial\Omega) \geq n\omega_n^{\frac{1}{n}}\mathsf{AVR}_F^{\frac{1}{n}}\mathrm{Vol}_F(\Omega)^{\frac{n-1}{n}}, \quad (3.1)$$

where $\mathcal{P}_F(\partial\Omega)$ stands for the anisotropic perimeter of $\partial\Omega$, defined as $\mathcal{P}_F(\partial\Omega) = \displaystyle\int_{\partial\Omega}\mathrm{d}\sigma_F$, where $\mathrm{d}\sigma_F$ stands for the $(n-1)$-dimensional Lebesgue measure induced by $\mathrm{d}v_F$.

Beside the Riemannian setting, see Brendle [7] and Balogh and Kristály [4], one can characterize the equality in (3.1) in the case of the simplest non-Riemannian Finsler structures, namely on Minkowski spaces. To be precise, let $(\mathbb{R}^n, H)$ be a Finsler manifold endowed with the

Lebesgue measure $\mathrm{d}v_H$, such that $H:\mathbb{R}^n\to[0,\infty)$ is a smooth, absolutely homogeneous norm. A Wulff-shape associated to the norm H is the set

$$W_H(R):=\{x\in\mathbb{R}^n : H(x)<R\}, \tag{3.2}$$

for any number $R>0$. In the sequel, we assume, without loss of generality, that the set $W_H(1)$ has measure $\mathrm{Vol}(W_H(1))=\omega_n$; in this case, we say that $H:\mathbb{R}^n\to[0,\infty)$ is a normalized Minkowski norm. Accordingly, it turns out that $\sigma_H=1$, i.e., $\mathrm{Vol}_H(W_H(1))=\mathrm{Vol}(W_H(1))=\omega_n$, which yields both $\mathrm{d}v_H(x)=\mathrm{d}x$ in (2.1) and $\mathsf{AVR}_H=1$. Then, due to Cabré, Ros-Oton, and Serra [9, Theorem 1.2], we have

$$\mathcal{P}_H(\partial\Omega)=n\omega_n^{\frac{1}{n}}\,\mathrm{Vol}_H(\Omega)^{\frac{n-1}{n}}, \tag{3.3}$$

if and only if Ω has a Wulff-shape, i.e., $\Omega=W_H(R)$ for some $R>0$ (up to translations).

Let $\mathscr{C}(M)$ be the space of continuous functions $u:M\to[0,\infty)$ with compact support $S\subset M$, where S is smooth enough and u is of class C^2 having only non-degenerate critical points in the interior of S. Based on classical Morse theory and density arguments (see Aubin [2]), it is enough to consider test functions $u\in\mathscr{C}(M)$ in order to handle generic Sobolev inequalities.

The anisotropic rearrangement of a bounded set $\Omega\subset M$ w.r.t. the normalized Minkowski norm H is a Wulff-shape

$$\Omega_H^\star:=W_H(R), \tag{3.4}$$

where $R>0$ is chosen such that $\mathrm{Vol}_F(\Omega)=\mathrm{Vol}_H(\Omega_H^\star)=\mathrm{Vol}(\Omega_H^\star)$.

Similarly to Druet, Hebey and Vaugon [12], and Alvino, Ferone, Trombetti and Lions [1], for every function $u:M\to[0,\infty)$ belonging to $\mathscr{C}(M)$, one can associate its anisotropic rearrangement $u_H^\star:\mathbb{R}^n\to[0,\infty)$, defined as

$$u_H^\star(x):=v(\omega_nH(x)^n),\quad \forall x\in\mathbb{R}^n \tag{3.5}$$

for some nonincreasing function $v:[0,\infty)\to[0,\infty)$, such that for every $t\geq 0$, one has

$$\mathrm{Vol}_F(\{x\in M : u(x)>t\})=\mathrm{Vol}_H(\{x\in\mathbb{R}^n : u_H^\star(x)>t\}). \tag{3.6}$$

Note that by definition, we have

$$\mathrm{Vol}_F(\mathrm{supp}(u))=\mathrm{Vol}_H(\mathrm{supp}(u_H^\star)), \tag{3.7}$$

while by the layer cake representation (see Lieb and Loss [24, Theorem 1.13]), it follows that

$$\|u\|_{L^q(M)}=\|u_H^\star\|_{L^q(\mathbb{R}^n)},\quad \forall q\in(0,\infty]. \tag{3.8}$$

Furthermore, by using the isoperimetric inequality (3.1), we can prove the following Pólya-Szegő-type result, which represents a crucial ingredient in our arguments:

Theorem 3.1. *Let (M,F) be a noncompact, complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n\geq 0$ and $0<\mathsf{AVR}_F\leq 1$, and let $H:\mathbb{R}^n\to[0,\infty)$ be a normalized Minkowski norm. Then, for every $u\in\mathscr{C}(M)$ and $p>1$, we have*

$$\int_M F^*(x,Du(x))^p\mathrm{d}v_F\geq \mathsf{AVR}_F^{\frac{p}{n}}\int_{\mathbb{R}^n}H^*(Du_H^\star(x))^p\mathrm{d}v_H. \tag{3.9}$$

Proof. We adapt the arguments of Hebey [15] to the anisotropic setting. Let $0<t<\max_M u$ be arbitrarily fixed, and consider the sets

$$\Omega_t=\{x\in M : u(x)>t\}\subset M\ \text{ and }\ \Omega_t^\star=\{x\in\mathbb{R}^n : u_H^\star(x)>t\}\subset\mathbb{R}^n,$$

and the level sets

$$\Gamma_t=u^{-1}(t)\ \text{ and }\ \Gamma_t^\star=(u_H^\star)^{-1}(t),$$

which turn out to be the boundaries of Ω_t and $\Omega_t^\star$, respectively.

By definition of $u_H^\star$, it follows that the set $\Omega_t^\star$ has a Wulff-shape such that

$$\mathrm{Vol}_F(\Omega_t)=\mathrm{Vol}_H(\Omega_t^\star)=:\mathcal{V}(t). \tag{3.10}$$

Then, the isoperimetric inequality (3.1) and the equality (3.3) in case of Wulff-shapes on a Minkowski space implies that

$$\begin{aligned}\mathcal{P}_F(\Gamma_t) &\geq n\omega_n^{\frac{1}{n}}\mathsf{AVR}_F^{\frac{1}{n}}\mathrm{Vol}_F(\Omega_t)^{\frac{n-1}{n}} = n\omega_n^{\frac{1}{n}}\mathsf{AVR}_F^{\frac{1}{n}}\mathrm{Vol}_H(\Omega_t^\star)^{\frac{n-1}{n}}\\ &= \mathsf{AVR}_F^{\frac{1}{n}}\cdot\mathcal{P}_H(\Gamma_t^\star). \end{aligned}\tag{3.11}$$

By using relation (3.10) and the co-area formula proved by Shen [30, Theorem 3.3.1, p. 46], it follows that

$$\mathcal{V}(t) = \int_t^\infty\left(\int_{\Gamma_s}\frac{1}{F^*(x,Du(x))}\mathrm{d}\sigma_F\right)\mathrm{d}s = \int_t^\infty\left(\int_{\Gamma_s^\star}\frac{1}{H^*(Du_H^\star(x))}\mathrm{d}\sigma_H\right)\mathrm{d}s,$$

where $\mathrm{d}\sigma_F$ and $\mathrm{d}\sigma_H$ denote the $(n-1)$-dimensional Hausdorff measures induced by $\mathrm{d}v_F$ and $\mathrm{d}v_H$, respectively. It follows that

$$\mathcal{V}'(t) = -\int_{\Gamma_t}\frac{1}{F^*(x,Du(x))}\mathrm{d}\sigma_F = -\int_{\Gamma_t^\star}\frac{1}{H^*(Du_H^\star(x))}\mathrm{d}\sigma_H. \tag{3.12}$$

On the one hand, since $u_H^\star$ is anisotropically symmetric in $H(x)$, the quantity $H^*(Du_H^\star)$ is constant on $\Gamma_t^\star$, thus

$$\mathcal{V}'(t) = -\frac{\mathcal{P}_H(\Gamma_t^\star)}{H^*(Du_H^\star(x))}\quad\text{for every } x\in\Gamma_t^\star. \tag{3.13}$$

On the other hand, applying Hölder's inequality and using relation (3.12), for every $p>1$, we have

$$\begin{aligned}\mathcal{P}_F(\Gamma_t) &= \int_{\Gamma_t}\mathrm{d}\sigma_F = \int_{\Gamma_t}\frac{1}{F^*(x,Du(x))^{\frac{p-1}{p}}}F^*(x,Du(x))^{\frac{p-1}{p}}\mathrm{d}\sigma_F\\ &\leq \left(-\mathcal{V}'(t)\right)^{\frac{p-1}{p}}\left(\int_{\Gamma_t}F^*(x,Du(x))^{p-1}\mathrm{d}\sigma_F\right)^{\frac{1}{p}}.\end{aligned}$$

Accordingly, relations (3.11) and (3.13) yield that

$$\begin{aligned}\int_{\Gamma_t}F^*(x,Du(x))^{p-1}\mathrm{d}\sigma_F &\geq \mathcal{P}_F(\Gamma_t)^p\left(-\mathcal{V}'(t)\right)^{1-p} \geq \mathsf{AVR}_F^{\frac{p}{n}}\cdot\mathcal{P}_H(\Gamma_t^\star)^p\left(\frac{\mathcal{P}_H(\Gamma_t^\star)}{H^*(Du_H^\star(x))}\right)^{1-p}\\ &= \mathsf{AVR}_F^{\frac{p}{n}}\int_{\Gamma_t^\star}H^*(Du_H^\star(x))^{p-1}\mathrm{d}\sigma_H.\end{aligned}$$

Applying the co-area formula once again, we obtain

$$\begin{aligned}\int_M F^*(x,Du(x))^p\mathrm{d}v_F &= \int_0^\infty\left(\int_{\Gamma_t}F^*(x,Du(x))^{p-1}\mathrm{d}\sigma_F\right)\mathrm{d}t\\ &\geq \mathsf{AVR}_F^{\frac{p}{n}}\int_0^\infty\left(\int_{\Gamma_t^\star}H^*(Du_H^\star(x))^{p-1}\mathrm{d}\sigma_H\right)\mathrm{d}t\\ &= \mathsf{AVR}_F^{\frac{p}{n}}\int_{\mathbb{R}^n}H^*(Du_H^\star(x))^p\mathrm{d}v_H,\end{aligned}$$

which concludes the proof. □

A Hardy-Littlewood-Pólya-type argument (see Lieb and Loss [24, Theorem 3.4]) combined with a careful application of the Bishop-Gromov comparison principle yields the following rearrangement inequality:

Proposition 3.1. *Let (M,F) be a complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n\geq 0$, $x_0\in M$ be any fixed point and $H:\mathbb{R}^n\to[0,\infty)$ be a normalized Minkowski norm. Let $p>1$. Then for every decreasing function $f:[0,\infty)\to[0,\infty)$, one has*

$$\int_M u(x)^p f(d_F(x_0,x))\mathrm{d}v_F \leq \int_{\mathbb{R}^n}u_H^\star(x)^p f(H(x))\mathrm{d}v_H,\quad \forall u\in\mathscr{C}(M). \tag{3.14}$$

Proof. Let us denote the distance function from the point $x_0 \in M$ by $d_{x_0}(x) := d_F(x_0, x), \forall x \in M$. By Fubini's theorem, (3.14) is equivalent to

$$\int_0^\infty \int_0^\infty \int_M \chi_{\{u^p>t\}}(x)\chi_{\{f\circ d_{x_0}>s\}}(x)\mathrm{d}v_F \,\mathrm{d}t\,\mathrm{d}s \le \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \chi_{\{{u_H^\star}^p>t\}}(x)\chi_{\{f\circ H>s\}}(x)\mathrm{d}v_H \,\mathrm{d}t\,\mathrm{d}s,$$

where χ_S denotes the characteristic function of a set $S \neq \emptyset$.

Now let t and $s \in [0, \infty)$ be arbitrarily fixed, and define the set $S_{t,\rho} := \{x \in M : \chi_{\{u^p>t\}}(x) > \rho\}$ for every $\rho \ge 0$. By using the fact that f is decreasing, the layer cake representation implies that

$$\begin{aligned} I_{t,s}(u, f) &:= \int_M \chi_{\{u^p>t\}}(x)\chi_{\{f\circ d_{x_0}>s\}}(x)\mathrm{d}v_F \\ &= \int_{B_{x_0}(f^{-1}(s))} \chi_{\{u^p>t\}}(x)\mathrm{d}v_F \\ &= \int_0^\infty \mathrm{Vol_F}\{x \in B_{x_0}(f^{-1}(s)) : \chi_{\{u^p>t\}}(x) > \rho\}\mathrm{d}\rho \\ &= \int_0^\infty \mathrm{Vol_F}\big(B_{x_0}(f^{-1}(s)) \cap S_{t,\rho}\big)\mathrm{d}\rho. \end{aligned}$$

As (M, F) has nonnegative n-Ricci curvature, by the Bishop-Gromov comparison principle (see Shen [31]) we have that $\mathrm{Vol}_F(B_{x_0}(R)) \le \mathrm{Vol}(B_0(R)) = \mathrm{Vol}(W_H(R))$ for every $R > 0$, where $B_0(R) \subset \mathbb{R}^n$ denotes the Euclidean ball with center in the origin and radius $R > 0$, while $W_H(R) \subset \mathbb{R}^n$ has a Wulff-shape defined in (3.2). Therefore, using relation (3.6) and a layer cake representation again, we obtain

$$\begin{aligned} I_{t,s}(u, f) &\le \int_0^\infty \min\Big\{\mathrm{Vol_F}\big(B_{x_0}(f^{-1}(s))\big), \mathrm{Vol_F}\big(S_{t,\rho}\big)\Big\}\mathrm{d}\rho \\ &\le \int_0^\infty \min\Big\{\mathrm{Vol}\big(B_0(f^{-1}(s))\big), \mathrm{Vol}\big(S^\star_{t,\rho}\big)\Big\}\mathrm{d}\rho = \int_0^\infty \mathrm{Vol}\Big(W_H(f^{-1}(s)) \cap S^\star_{t,\rho}\Big)\mathrm{d}\rho \\ &= \int_{\mathbb{R}^n} \chi_{\{f\circ H>s\}}(x)\chi_{\{{u_H^\star}^p>t\}}(x)\mathrm{d}v_H, \end{aligned}$$

where $S^\star_{t,\rho}$ denotes the anisotropic rearrangement of $S_{t,\rho}$ w.r.t the norm H, in the sense of (3.4). This completes the proof. □

In particular, Proposition 3.1 yields the following Hardy-type rearrangement inequality:

$$\int_M \frac{u(x)^p}{d_F(x_0, x)^p}\mathrm{d}v_F \le \int_{\mathbb{R}^n} \frac{u_H^\star(x)^p}{H(x)^p}\mathrm{d}v_H, \quad \forall u \in \mathscr{C}(M). \tag{3.15}$$

Remark 3.1. In fact, for a fixed test function $u \in \mathscr{C}(M)$, we can consider multiple rearrangements, which are all equimeasurable in the following sense. Let H_1 and H_2 be two reversible normalized Minkowski norms on $\mathbb{R}^n$. We can associate to u its anisotropic rearrangements w.r.t. both H_1 and H_2, in the form of $u^\star_{H_1}, u^\star_{H_2} : \mathbb{R}^n \to [0, \infty)$. Then, we have

$$\int_{\mathbb{R}^n} H_1^*(Du^\star_{H_1}(x))^q \mathrm{d}v_{H_1} = \int_{\mathbb{R}^n} H_2^*(Du^\star_{H_2}(x))^q \mathrm{d}v_{H_2}, \tag{3.16}$$

for any $q \in (0, \infty)$. Indeed, by using definition (3.5), there exists a nonincreasing function $g : [0, \infty) \to [0, \infty)$ such that $u^\star_{H_1}(x) = g(H_1(x))$ and $u^\star_{H_2}(x) = g(H_2(x))$. As the Finsler structures H_1 and H_2 are absolutely homogeneous, we have

$$H_i^*(Du^\star_{H_i}(x)) = H_i^*\big(g'(H_i(x))DH_i(x)\big) = |g'(H_i(x))|H_i^*(DH_i(x)), \text{ for } i = 1, 2.$$

Applying the eikonal equation $H_i^*(DH_i(x)) = 1$ for every $x \in \mathbb{R}^n \setminus \{0\}$, then using a change of variables, it follows that for every $q > 0$, we have

$$\int_{\mathbb{R}^n} H_i^*(Du^\star_{H_i}(x))^q \mathrm{d}v_{H_i} = \int_{\mathbb{R}^n} |g'(H_i(x))|^q \mathrm{d}v_{H_i} = n\omega_n \int_0^\infty |g'(\rho)|^q \rho^{n-1}\mathrm{d}\rho,$$

where we used the fact that both Minkowski norms H_i are normalized, $i = 1, 2$.

In the case when $H(x) = |x|$ is the standard Euclidean norm, $u^\star_{|\cdot|}$ turns out to be the usual radially symmetric rearrangement of u, for which the sets $\{x \in \mathbb{R}^n : u^\star_{|\cdot|}(x) > t\}$ are Euclidean open balls with center $0 \in \mathbb{R}^n$. This radial symmetrization has proven to be particularly useful when showing the Finslerian counterparts of several Euclidean Sobolev-type inequalities, see §4.

On the other hand, the anisotropic rearrangement (3.5) w.r.t. an arbitrary reversible Minkowski norm H can be a key technique when considering rigidity results in the spirit of Balogh and Kristály [4] and Kristály [19]. More specifically, we believe that in certain sharp Sobolev inequalities, equality holds for some nonzero extremal function $u \in \mathscr{C}(M)$ if and only if $\mathsf{AVR}_F = 1$; in particular, when (M, F) is a Finsler manifold of Berwald type, this would imply that (M, F) is isometric to a Minkowski space $(\mathbb{R}^n, H)$.

Finally, we present the following auxiliary result, whose proof, similarly to Balogh and Kristály [4, Lemma 3.1], follows in a straightforward way by the layer cake representation:

Lemma 3.1. (see [4]) *Let (M, F) be a complete, reversible Finsler manifold, $R > 0$ and $x_0 \in M$ an arbitrarily fixed point. If $f : [0, R] \to \mathbb{R}$ is a C^1-function on $(0, R)$, then*

$$\int_{B_{x_0}(R)} f(d_F(x_0, x)) \mathrm{d}v_F = f(R) \mathrm{Vol}_F(B_{x_0}(R)) - \int_0^R f'(r) \mathrm{Vol}_F(B_{x_0}(r)) \mathrm{d}r.$$

4. Morrey-Sobolev and Hardy-Sobolev Inequalities

4.1. Morrey-Sobolev interpolation inequality: sharp support-bound. Let $p > n \geq 2$. In the Euclidean case, Talenti [32, Theorem 2.E] proved the following Morrey-Sobolev inequality

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq \mathsf{T}_{p,n} \, \mathrm{Vol}(\mathrm{supp}\, u)^{\frac{1}{n} - \frac{1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{4.1}$$

where $\mathrm{supp}\, u \subset \mathbb{R}^n$ denotes the support of u, and $\mathrm{Vol}(\mathrm{supp}\, u)$ stands for the Euclidean volume of the set $\mathrm{supp} u$; moreover, the constant

$$\mathsf{T}_{p,n} = n^{-\frac{1}{p}} \omega_n^{-\frac{1}{n}} \left(\frac{p-1}{p-n} \right)^{\frac{1}{p'}} \tag{4.2}$$

is sharp and achieved by the function

$$u(x) = \left(1 - |x|^{\frac{p-n}{p-1}}\right)_+.$$

The counterpart of (4.1) on Finsler manifolds with nonnegative n-Ricci-curvature is given by Theorem 1.1.

Proof of Theorem 1.1. Let $u \in C_0^\infty(M)$ be arbitrarily fixed, and consider its spherically symmetric rearrangement $u^\star_{|\cdot|} : \mathbb{R}^n \to [0, \infty)$ w.r.t. the Euclidean norm $|\cdot|$. Then, by relations (3.7)-(3.9) and (4.1), it follows that

$$\begin{aligned} \|u\|_{L^\infty(M)} = \|u^\star_{|\cdot|}\|_{L^\infty(\mathbb{R}^n)} &\leq \mathsf{T}_{p,n} \, \mathrm{Vol}(\mathrm{supp}\, u^\star_{|\cdot|})^{\frac{1}{n} - \frac{1}{p}} \|\nabla u^\star_{|\cdot|}\|_{L^p(\mathbb{R}^n)} \\ &\leq \mathsf{T}_{p,n} \, \mathsf{AVR}_F^{-\frac{1}{n}} \mathrm{Vol}_F(\mathrm{supp}\, u)^{\frac{1}{n} - \frac{1}{p}} \left(\int_M F^*(x, Du(x))^p \mathrm{d}v_F \right)^{\frac{1}{p}}, \end{aligned}$$

which is precisely (1.1).

We assume by contradiction that there exists a constant $\mathcal{C} < \mathsf{T}_F^{\mathsf{MS}} = \mathsf{T}_{p,n} \, \mathsf{AVR}_F^{-\frac{1}{n}}$ such that

$$\|u\|_{L^\infty(M)} \leq \mathcal{C} \, \mathrm{Vol}_F(\mathrm{supp}\, u)^{\frac{1}{n} - \frac{1}{p}} \left(\int_M F^*(x, Du(x))^p \mathrm{d}v_F \right)^{1/p}, \quad \forall u \in C_0^\infty(M). \tag{4.3}$$

We fix $x_0 \in M$ and $R > 0$, and define the function $u_R : M \to [0, \infty)$ by

$$u_R(x) = \left(1 - \left(\frac{d_F(x_0, x)}{R}\right)^{\frac{p-n}{p-1}}\right)_+, \; x \in M.$$

By the eikonal equation (2.3) and Lemma 3.1, it follows that

$$\begin{aligned}\int_M F^*(x,Du_R(x))^p \mathrm{d}v_F &= \frac{1}{R^p}\left(\frac{p-n}{p-1}\right)^p \int_{B_{x_0}(R)} \left(\frac{d_F(x_0,x)}{R}\right)^{(1-n)p'} \mathrm{d}v_F \\ &= \frac{1}{R^p}\left(\frac{p-n}{p-1}\right)^p \Big(\mathrm{Vol}_F(B_{x_0}(R))- \\ &\quad -(1-n)p'\int_0^1 t^{(1-n)p'-1}\mathrm{Vol}_F(B_{x_0}(Rt))\mathrm{d}t\Big).\end{aligned}$$

The latter relation and Lebesgue's dominated convergence theorem imply that

$$\lim_{R\to\infty}\frac{1}{R^{n-p}}\int_M F^*(x,Du_R(x))^p\mathrm{d}v_F = n\omega_n\left(\frac{p-n}{p-1}\right)^{p-1}\mathsf{AVR}_F.$$

Note that $\|u_R\|_{L^\infty(M)}=1$ and $\operatorname{supp} u_R = B_{x_0}(R)$. If we use u_R as a test function in (4.3), an argument via the latter limit gives that

$$1\leq \mathcal{C}\,\omega_n^{\frac{1}{n}} n^{\frac{1}{p}}\left(\frac{p-n}{p-1}\right)^{\frac{1}{p'}}\mathsf{AVR}_F^{\frac{1}{n}},$$

which is equivalent to $\mathcal{C}\geq \mathsf{T}_{p,n}\,\mathsf{AVR}_F^{-\frac{1}{n}} = \mathsf{T}_F^{\mathsf{MS}}$, contradicting our initial assumption. □

4.2. Morrey-Sobolev interpolation inequality: sharp L^1-bound. The following Morrey-Sobolev inequality is proved by Talenti [32, Theorem 2.C], stating that for every $p>n\geq 2$, one has

$$\|u\|_{L^\infty(\mathbb{R}^n)}\leq \mathsf{C}_{p,n}\|u\|_{L^1(\mathbb{R}^n)}^{1-\eta}\|\nabla u\|_{L^p(\mathbb{R}^n)}^{\eta},\quad \forall u\in C_0^\infty(\mathbb{R}^n), \tag{4.4}$$

where

$$\eta=\frac{np}{np+p-n}. \tag{4.5}$$

Moreover, the constant

$$\mathsf{C}_{p,n}=(n\omega_n^{\frac{1}{n}})^{-\frac{np'}{n+p'}}\left(\frac{1}{n}+\frac{1}{p'}\right)\left(\frac{1}{n}-\frac{1}{p}\right)^{\frac{(n-1)p'-n}{n+p'}}\left(\mathsf{B}\left(\frac{1-n}{n}p'+1,p'+1\right)\right)^{\frac{n}{n+p'}} \tag{4.6}$$

is sharp and achieved by the function

$$u(x)=\begin{cases}\displaystyle\int_{|x|}^1 r^{\frac{1-n}{p-1}}(1-r^n)^{\frac{1}{p-1}}\mathrm{d}r, & \text{if } |x|\leq 1;\\ 0, & \text{otherwise.}\end{cases}$$

We prove the counterpart of (4.4) on Finsler manifolds with nonnegative n-Ricci-curvature:

Proof of Theorem 1.2. Let $u\in C_0^\infty(M)$ be arbitrarily fixed, and consider its spherically symmetric rearrangement $u^\star_{|\cdot|}:\mathbb{R}^n\to[0,\infty)$ w.r.t. the Euclidean norm $|\cdot|$. By applying relations (3.8) and (3.9), and using Talenti's inequality (4.4), it follows that

$$\begin{aligned}\|u\|_{L^\infty(M)} = \|u^\star_{|\cdot|}\|_{L^\infty(\mathbb{R}^n)} &\leq \mathsf{C}_{p,n}\|u^\star_{|\cdot|}\|_{L^1(\mathbb{R}^n)}^{1-\eta}\|\nabla u^\star_{|\cdot|}\|_{L^p(\mathbb{R}^n)}^{\eta} \\ &\leq \mathsf{C}_{p,n}\|u\|_{L^1(M)}^{1-\eta}\mathsf{AVR}_F^{-\frac{\eta}{n}}\left(\int_M F^*(x,Du(x))^p\mathrm{d}v_F\right)^{\frac{\eta}{p}},\end{aligned} \tag{4.7}$$

which is exactly (1.2).

As for the optimality of the constant, we assume that $\mathsf{C}_F^{\mathsf{MS}}=\mathsf{C}_{p,n}\mathsf{AVR}_F^{-\frac{\eta}{n}}$ is not sharp in (1.2), i.e., there exists $\mathcal{C}<\mathsf{C}_F^{\mathsf{MS}}$ such that

$$\|u\|_{L^\infty(M)}\leq\mathcal{C}\left(\int_M|u(x)|\mathrm{d}v_F\right)^{1-\eta}\left(\int_M F^*(x,Du(x))^p\mathrm{d}v_F\right)^{\frac{\eta}{p}},\ \forall u\in C_0^\infty(M). \tag{4.8}$$

Let $h,H:(0,1]\to\mathbb{R}$ be the functions

$$h(r)=r^{\frac{1-n}{p-1}}(1-r^n)^{\frac{1}{p-1}}\quad\text{and}\quad H(s)=\int_0^s h(r)\mathrm{d}r.$$

Let $x_0 \in M$ and $R > 0$ be fixed, and consider the function $u_R : M \to [0, \infty)$ defined by

$$u_R(x) = \begin{cases} H(1) - H\left(\frac{d_F(x_0, x)}{R}\right), & \text{if } x \in B_{x_0}(R); \\ 0, & \text{otherwise.} \end{cases}$$

First, we have that

$$\|u_R\|_{L^\infty(M)} = H(1) = \int_0^1 h(r)\mathrm{d}r = \frac{1}{n}\mathsf{B}\left(\frac{1-n}{n}p' + 1, p'\right). \tag{4.9}$$

By using Lemma 3.1 and a change of variables, it turns out that

$$\begin{aligned} \int_M |u_R(x)|\mathrm{d}v_F &= \int_{B_{x_0}(R)} \left(H(1) - H\left(\frac{d_F(x_0, x)}{R}\right)\right)\mathrm{d}v_F \\ &= \frac{1}{R}\int_0^R \mathrm{Vol}_F(B_{x_0}(r))H'\left(\frac{r}{R}\right)\mathrm{d}r \\ &= \int_0^1 \mathrm{Vol}_F(B_{x_0}(Rt))h(t)\mathrm{d}t. \end{aligned} \tag{4.10}$$

On the other hand, since

$$Du_R(x) = -\frac{1}{R}H'\left(\frac{d_F(x_0, x)}{R}\right)Dd_F(x_0, x) \ \text{ for a.e. } x \in B_{x_0}(R),$$

the eikonal equation (2.3) and the absolute homogeneity of the Finsler structure F yield that

$$\begin{aligned} \int_M F^*(x, Du_R(x))^p\mathrm{d}v_F &= \frac{1}{R^p}\int_{B_{x_0}(R)} h^p\left(\frac{d_F(x_0, x)}{R}\right)\mathrm{d}v_F \\ &= -\frac{1}{R^p}\int_0^1 \mathrm{Vol}_F(B_{x_0}(Rt)) \cdot (h^p)'(t)\mathrm{d}t, \end{aligned} \tag{4.11}$$

where we used Lemma 3.1 and a change of variables.

By density reasons, the function u_R can be used as a test function in (4.8), i.e.,

$$\|u_R\|_{L^\infty(M)} \leq \mathcal{C}\left(\int_M |u_R(x)|\mathrm{d}v_F\right)^{1-\eta}\left(\int_M F^*(x, Du_R(x))^p\mathrm{d}v_F\right)^{\frac{\eta}{p}}.$$

Furthermore, Lebesgue's dominated convergence theorem and relations (4.10) and (4.11) imply that

$$\lim_{R\to\infty}\frac{1}{R^n}\int_M |u_R(x)|\mathrm{d}v_F = \omega_n\mathsf{AVR}_F\int_0^1 t^n h(t)\mathrm{d}t = \omega_n\mathsf{AVR}_F\,\frac{1}{n}\mathsf{B}\left(\frac{1-n}{n}p' + 2, p'\right),$$

and

$$\begin{aligned} \lim_{R\to\infty}\frac{1}{R^{n-p}}\int_M F^*(x, Du_R(x))^p\mathrm{d}v_F &= -\omega_n\mathsf{AVR}_F\int_0^1 t^n(h^p)'(t)\mathrm{d}t \\ &= \omega_n\mathsf{AVR}_F\,\mathsf{B}\left(\frac{1-n}{n}p' + 1, p' + 1\right). \end{aligned}$$

Therefore, by using the latter limits and relations (4.9) and (4.6), a straightforward manipulation of the above terms implies

$$\mathsf{C}_{p,n} \leq \mathcal{C}\,\mathsf{AVR}_F^{1-\eta+\frac{\eta}{p}}.$$

Since

$$1 - \eta + \frac{\eta}{p} = \frac{\eta}{n},$$

see (4.5), the latter inequality contradicts our initial assumption $\mathcal{C} < \mathsf{C}_F^{\mathsf{MS}}$, which yields the sharpness of $\mathsf{C}_F^{\mathsf{MS}}$ in (1.2). □

Concerning the equality in the Morrey-Sobolev inequalities (1.1) and (1.2), we can state the following rigidity result in the case when $(M, F) = (M, g)$ is a Riemannian manifold; in particular, it turns out that the existence of nonzero extremal functions implies that the manifold is isometric to the Euclidean space:

Theorem 4.1. *Let (M,g) be a noncompact, complete n-dimensional Riemannian manifold having* $\mathsf{Ric} \geq 0$, $0 < \mathsf{AVR}_g \leq 1$, *and let* $2 \leq n < p$. *Then the following statements are equivalent:*

(i) *Equality holds in (1.1) for some nonzero and nonnegative function* $u \in \mathscr{C}(M)$;
(ii) *Equality holds in (1.2) for some nonzero and nonnegative function* $u \in \mathscr{C}(M)$;
(iii) (M,g) *is isometric to the Euclidean space* $(\mathbb{R}^n, g_0)$.

Proof. The proofs of the equivalences (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iii) are analogous; we shall present the latter. Suppose that equality holds in (1.2) for some nonzero and nonnegative function $u \in \mathscr{C}(M)$. Consequently, equalities hold in the chain of inequalities (4.7), therefore we have equality in the Pólya-Szegő inequality (3.9) as well. As the latter inequality is rigid in the Riemannian case, see Balogh and Kristály [4, Proposition 3.1], we obtain that

$$\mathsf{AVR}_g = 1,$$

i.e., (M,g) is isometric to the Euclidean space $(\mathbb{R}^n, g_0)$, see e.g. Petersen [28]. The converse is trivial. □

Remark 4.1. A natural question arises on the validity of Theorem 4.1 not only for functions belonging to $\mathscr{C}(M)$ but to the appropriate Sobolev spaces associated to the Morrey-Sobolev inequalities (1.1) and (1.2). Such a question requires a deeper analysis, since usually the 'small' subspace of functions where Sobolev inequalities can be easily obtained do not contain the expected extremal functions, while after the approximation/density arguments we cannot track back the equality cases in the proof; see e.g. Brothers and Ziemer [8], Balogh and Kristály [4]. In our case (Theorem 4.1) however, the corresponding extremal functions belong to $\mathscr{C}(M)$.

In the spirit of the latter result, one should ask whether there exist similar rigidity statements in the general, Finslerian setting as well. More specifically, we formulate the following question:

> *If (M,F) is a noncompact, complete n-dimensional reversible Finsler manifold with* $\mathsf{Ric}_n \geq 0$, $0 < \mathsf{AVR}_F \leq 1$, *and* $2 \leq n < p$, *is it true that if there exists a nonzero and nonnegative extremal function* $u \in \mathscr{C}(M)$ *of inequality* (1.1) (*or* (1.2), *respectively*), *then* $\mathsf{AVR}_F = 1$ (*thus, the manifold is a locally Minkowski space*)?

We believe that the anisotropic rearrangement (3.5) w.r.t. an arbitrary reversible Minkowski norm H shall be a key ingredient when considering such problems. However, the lack of a rigid isoperimetric inequality impedes this investigation; indeed, the characterization of the equality in the isoperimetric inequality (3.1) – and hence in the Pólya-Szegő inequality (3.9) – is currently available only on Riemannian manifolds, see Brendle [7], and Balogh and Kristály [4].

4.3. Hardy-Sobolev-type inequalities. In this section we consider Sobolev inequalities involving a Hardy-type singular term of the form $x \mapsto d_F(x_0,x)^{-p}$, where $x_0 \in M$ is any fixed point, $p > 1$.

Proof of Theorem 1.3. Due to a density reason, since F is reversible, we may assume without loss of generality that $u \geq 0$. Let us define the symmetric rearrangement of u w.r.t. the Euclidean norm $|\cdot|$, i.e., $u^\star_{|\cdot|} : \mathbb{R}^n \to [0,\infty)$. Using the rearrangement inequalities (3.9), (3.15), and the classical Euclidean Hardy-Sobolev inequality, see e.g. Balinsky, Evans and Lewis [3, Corollary 1.2.6], we have

$$\begin{aligned}
\int_M F^*(x, Du(x))^p \mathrm{d}v_F &\geq \mathsf{AVR}_F^{\frac{p}{n}} \int_{\mathbb{R}^n} |\nabla u^\star_{|\cdot|}(x)|^p \mathrm{d}x \\
&\geq \mathsf{AVR}_F^{\frac{p}{n}} \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{u^\star_{|\cdot|}(x)^p}{|x|^p} \mathrm{d}x \\
&\geq \mathsf{AVR}_F^{\frac{p}{n}} \left(\frac{n-p}{p}\right)^p \int_M \frac{|u(x)|^p}{d_F(x_0,x)^p} \mathrm{d}v_F,
\end{aligned}$$

which concludes the proof. □

Remark 4.2. The sharpness of the Hardy-Sobolev inequality (1.3) is open in the generic Finsler setting. This fact can be attributed to the lack of extremal functions in the Euclidean case, thus arguments similar to Theorems 1.1 and 1.2 no longer yield the expected conclusion.

In the particular case when $p = 2$, $\Omega \subset M$ is a smooth bounded open set and $x_0 \in \Omega$, we have the following *Brezis-Poincaré-Vázquez* inequality:

Theorem 4.2. *Let (M, F) be a noncompact, complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n \geq 0$, $0 < \mathsf{AVR}_F \leq 1$, and $n \geq 2$. Let $\Omega \subset M$ be a smooth, bounded open set with $x_0 \in \Omega$ arbitrarily fixed. If $\mu \in \left[0, \frac{(n-2)^2}{4}\mathsf{AVR}_F^{\frac{2}{n}}\right]$, then for every $u \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega F^*(x, Du(x))^2 \mathrm{d}v_F - \mu \int_\Omega \frac{u(x)^2}{d_F(x_0, x)^2} \mathrm{d}v_F \geq \mathsf{S}_{\mu,F}(\Omega) \int_\Omega u(x)^2 \mathrm{d}v_F, \tag{4.12}$$

where

$$\mathsf{S}_{\mu,F}(\Omega) = \mathsf{AVR}_F^{\frac{2}{n}} j_{\overline{\mu}}^2 \left(\frac{\omega_n}{\mathrm{Vol}_F(\Omega)}\right)^{\frac{2}{n}} \quad \textit{and} \quad \overline{\mu} = \sqrt{\frac{(n-2)^2}{4} - \mu\, \mathsf{AVR}_F^{-\frac{2}{n}}}, \tag{4.13}$$

$j_{\overline{\mu}}$ being the first positive zero of the Bessel function of the first kind $J_{\overline{\mu}}$.

Proof. Let $B := \Omega^\star_{|\cdot|}$ and $u^\star_{|\cdot|} : \mathbb{R}^n \to [0, \infty)$ be the symmetric rearrangements of Ω and u w.r.t. the Euclidean norm $|\cdot|$, i.e., B is an Euclidean open ball with center in the origin such that $\mathrm{Vol}(B) = \mathrm{Vol}_F(\Omega)$.

On the one hand, by inequality (1.3), the left hand side of (4.12) turns out to be nonnegative whenever $\mu \leq \frac{(n-2)^2}{4}\mathsf{AVR}_F^{\frac{2}{n}}$. On the other hand, relations (3.9), (3.15) and (3.8) together with the result of Kristály and Szakál [22, Theorem 1.1] imply that if $\mu \in \left[0, \frac{(n-2)^2}{4}\mathsf{AVR}_F^{\frac{2}{n}}\right]$, one has

$$\begin{aligned} \int_\Omega F^*(x, Du(x))^2 \mathrm{d}v_F - \mu \int_\Omega \frac{u(x)^2}{d_F(x_0, x)^2} \mathrm{d}v_F &\geq \mathsf{AVR}_F^{\frac{2}{n}} \int_B |\nabla u^\star_{|\cdot|}(x)|^2 \mathrm{d}x - \mu \int_B \frac{u^\star_{|\cdot|}(x)^2}{|x|^2} \mathrm{d}x \\ &\geq \mathsf{AVR}_F^{\frac{2}{n}}\, j_{\overline{\mu}}^2 \omega_n^{\frac{2}{n}}\, \mathrm{Vol}(B)^{-\frac{2}{n}} \int_B u^\star_{|\cdot|}(x)^2 \mathrm{d}x \\ &= \mathsf{AVR}_F^{\frac{2}{n}}\, j_{\overline{\mu}}^2 \omega_n^{\frac{2}{n}}\, \mathrm{Vol}_F(\Omega)^{-\frac{2}{n}} \int_\Omega u(x)^2 \mathrm{d}v_F, \end{aligned}$$

which ends the proof. □

We conclude this subsection with some comments concerning the sharpness and attainability of the best constant in the Brezis-Poincaré-Vázquez inequality. Namely, we have that:

- if (M, F) is isometric to a Minkowski space $(\mathbb{R}^n, H)$ (thus, $\mathsf{AVR}_F = 1$ in particular), the constant $\mathsf{S}_{\mu,F}(\Omega)$ is sharp and attained (for sufficiently small μ) if and only if the set Ω has a Wulff-shape, see Kristály and Szakál [22, Theorem 1.1];
- if $(M, F) = (M, g)$ is a Riemannian manifold, the constant $\mathsf{S}_{0,F}(\Omega)$ is sharp and it is attained whenever (M, g) is isometric to the usual Euclidean space $(\mathbb{R}^n, g_0)$ and $\Omega \subset M$ is isometric to a ball in $\mathbb{R}^n$, see Balogh and Kristály [4, Theorem 3.5].

The latter statements can be reformulated in terms of eigenvalues for a model problem. Indeed, putting ourselves into the setting of Theorem 4.2, we consider the *eigenvalue problem*

$$\begin{cases} -\Delta_F u(x) - \mu \dfrac{u(x)}{d_F(x_0, x)^2} = \lambda u(x), \quad x \in \Omega, \\ u \in W_{0,F}^{1,2}(\Omega). \end{cases} \tag{$(EP)_{\mu,\lambda}$}$$

Then, we can prove that

- if (M, F) is a Minkowski space, $\lambda = \mathsf{S}_{\mu,F}(\Omega)$ is the first eigenvalue (with sufficiently small μ) for the problem $(EP)_{\mu,\lambda}$ if and only if Ω has a Wulff-shape;
- if $(M, F) = (M, g)$ is a Riemannian manifold, $\lambda = \mathsf{S}_{0,F}(\Omega)$ is the first eigenvalue for the problem $(EP)_{0,\lambda}$ if and only if (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, g_0)$ and $\Omega \subset M$ is isometric to a ball in $\mathbb{R}^n$.

In general, however, the sharpness of $\mathsf{S}_{\mu,F}(\Omega)$ in (4.12) (and its attainability) remains an open question.

4.4. Example. Riemannian manifolds with nonnegative Ricci curvature and positive asymptotic volume ratio are provided e.g. in Balogh and Kristály [4]. In the sequel, we construct a family of non-Riemannian Finsler manifolds where our results apply, i.e., noncompact, complete n-dimensional reversible Finsler manifolds (M,F) with $\mathsf{Ric}_n \geq 0$ and $0 < \mathsf{AVR}_F \leq 1$.

To do this, we endow the space $\mathbb{R}^{n-1}$ $(n \geq 3)$ with a Riemannian metric g such that $(\mathbb{R}^{n-1}, g)$ is complete with nonnegative Ricci curvature and assume that the induced warped metric $\tilde{g}$ on $\mathbb{R}^{n-1} \times \mathbb{R}$, defined by

$$\tilde{g}_{(x,t)}(v,w) = \sqrt{g_x(v,v) + w^2}, \quad (x,t) \in \mathbb{R}^n, (v,w) \in T_x\mathbb{R}^{n-1} \times T_t\mathbb{R},$$

has the property that $0 < \mathsf{AVR}_{\tilde{g}} \leq 1$; the family of such metrics is rich, see [4].

For any fixed $\varepsilon > 0$, we consider on $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ the Finsler metric $F_\varepsilon : T\mathbb{R}^n \longrightarrow [0,\infty)$ given by

$$F_\varepsilon((x,t),(v,w)) = \sqrt{g_x(v,v) + w^2 + \varepsilon\sqrt{g_x(v,v)^2 + w^4}}, \quad (x,t) \in \mathbb{R}^n, (v,w) \in T_x\mathbb{R}^{n-1} \times T_t\mathbb{R},$$

endowed with its natural Busemann-Hausdorff measure. By Kristály and Ohta [21], we know that $(\mathbb{R}^n, F_\varepsilon)$ is a noncompact, complete, reversible non-Riemannian Berwald space with nonnegative Ricci curvature. In particular, $(\mathbb{R}^n, F_\varepsilon)$ being a Berwald space, it has vanishing mean covariation, thus $\mathsf{Ric}_n \geq 0$. By Bishop-Gromov comparison principle we clearly have that $\mathsf{AVR}_{F_\varepsilon} \leq 1$. It remains to show that (M, F_ε) has Euclidean volume growth, i.e., $\mathsf{AVR}_{F_\varepsilon} > 0$. To this end, we observe that

$$\tilde{g}_{(x,t)}(v,w) \leq F_\varepsilon((x,t),(v,w)) \leq \sqrt{1+\varepsilon}\,\tilde{g}_{(x,t)}(v,w), \quad (x,t) \in \mathbb{R}^n, (v,w) \in T_x\mathbb{R}^{n-1} \times T_t\mathbb{R},$$

thus the density functions from (2.1) verify $\sigma_{F_\varepsilon} \geq \sigma_{\tilde{g}}$. Moreover, based on the assumption that $0 < \mathsf{AVR}_{\tilde{g}} \leq 1$, simple estimates show that

$$\mathsf{AVR}_{F_\varepsilon} \geq \frac{\mathsf{AVR}_{\tilde{g}}}{(1+\varepsilon)^{\frac{n}{2}}} > 0,$$

which concludes our claim.

5. Applications to PDEs

5.1. Multiple solutions for a Dirichlet problem involving the p-Finsler-Laplacian. In this section, we provide an application of Theorem 1.1 by considering the Dirichlet problem

$$\begin{cases} -\Delta_{F,p} u(x) = \lambda h(u(x)), & x \in \Omega \\ u \in W_{0,F}^{1,p}(\Omega), \end{cases} \qquad (\mathcal{D}_\lambda)$$

where (M,F) is an n-dimensional Finsler manifold, $\Omega \subset M$ is a bounded open set with C^1 boundary, $\Delta_{F,p}$ is the p-Finsler-Laplace operator with $p > n$, $\lambda > 0$, and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function with $h(0) = 0$. Furthermore, for each $s \in \mathbb{R}$, let $H(s) = \int_0^s h(t)\mathrm{d}t$, and suppose that

(A_1): $H(s) \geq 0$ for all $s \geq 0$;
(A_2): $0 < \limsup\limits_{s\to+\infty} \dfrac{H(s)}{s^p} < +\infty$.

In the spirit of Cammaroto, Chinnì and Di Bella [10], one can prove the existence of infinitely many weak solutions of problem $(\mathcal{D}_\lambda)$, as follows:

Theorem 5.1. *Let (M,F) be a noncompact, complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n \geq 0$, $0 < \mathsf{AVR}_F \leq 1$, and $2 \leq n < p < \infty$. Let $\Omega \subset M$ be a bounded open set with C^1 boundary, and let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function with $h(0) = 0$, such that H verifies conditions (A_1) – (A_2). Furthermore, assume that there exist two sequences $\{a_k\}$ and $\{b_k\}$ in $(0,+\infty)$, such that $a_k < b_k$, $\lim_{k\to\infty} b_k = +\infty$, $\lim_{k\to\infty} \frac{b_k}{a_k} = +\infty$, and $\max_{[a_k,b_k]} h \leq 0$, for all*

$k \in \mathbb{N}$. Then, there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, problem $(\mathcal{D}_\lambda)$ admits an unbounded sequence of weak solutions in $W^{1,p}_{0,F}(\Omega)$.

The proof is based on the following critical point result of Ricceri [29, Theorem 2.5]:

Theorem 5.2. *Let $(X, \|\cdot\|)$ be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals, such that Ψ is (strongly) continuous and satisfies $\lim_{\|x\|\to+\infty} \Psi(x) = +\infty$. For every $r > \inf_X \Psi$, put*

$$\varphi(r) = \inf_{u\in\Psi^{-1}(-\infty,r)} \frac{\Phi(u) - \inf_{v\in\overline{(\Psi^{-1}(-\infty,r))_w}} \Phi(v)}{r - \Psi(u)}, \tag{5.1}$$

where $\overline{(\Psi^{-1}(-\infty,r))_w}$ is the closure of $\Psi^{-1}(-\infty,r)$ in the weak topology. Let $\lambda \in \mathbb{R}$ be fixed. If $\{r_k\}$ is a real sequence such that $\lim_{k\to\infty} r_k = +\infty$ and $\varphi(r_k) < \lambda$ for all $k \in \mathbb{N}$, then either $\Phi + \lambda\Psi$ has a global minimum, or there exists a sequence $\{u_k\}$ of critical points of $\Phi + \lambda\Psi$ such that $\lim_{k\to\infty} \Psi(u_k) = +\infty$.

Proof of Theorem 5.1. We shall apply Theorem 5.2 by choosing $X = W^{1,p}_{0,F}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega F^*(x, Du(x))^p \mathrm{d}v_F\right)^{1/p}.$$

As Ω is bounded, by Theorem 1.1 it follows that there exists a constant $c > 0$ such that for any $u \in W^{1,p}_{0,F}(\Omega)$,

$$\sup_{x\in\Omega} |u(x)| \le c\|u\|. \tag{5.2}$$

Moreover, the embedding

$$W^{1,p}_{0,F}(\Omega) \hookrightarrow L^\infty(\Omega) \tag{5.3}$$

is compact, which follows from Hebey [15] and the equivalence of the Finsler metric F to any complete Riemannian metric on the bounded Ω.

Let $\Phi, \Psi : W^{1,p}_{0,F}(\Omega) \to \mathbb{R}$ be defined by

$$\Phi(u) = -\int_\Omega H(u(x))\mathrm{d}v_F \quad \text{and} \quad \Psi(u) = \int_\Omega F^*(x, Du(x))^p \mathrm{d}v_F,$$

and for $\lambda > 0$ we consider the energy functional associated with problem $(\mathcal{D}_\lambda)$ as

$$\mathcal{E}_\lambda : W^{1,p}_{0,F}(\Omega) \to \mathbb{R}, \quad \mathcal{E}_\lambda(u) = \lambda\Phi(u) + \frac{1}{p}\Psi(u).$$

Then, the critical points of $\mathcal{E}_\lambda$ are precisely the weak solutions of problem $(\mathcal{D}_\lambda)$. Standard arguments based on the compact embedding (5.3) imply that the functionals Φ and Ψ are sequentially weakly lower semicontinuous and Gâteaux differentiable. Also, as a norm-type function, Ψ is (strongly) continuous and coercive. Furthermore, for each $r > 0$, the function φ in (5.1) takes the form

$$\varphi(r) = \inf_{\|u\|^p<r} \frac{\sup_{\|v\|^p\le r} \int_\Omega H(v(x))\mathrm{d}v_F - \int_\Omega H(u(x))\mathrm{d}v_F}{r - \|u\|^p}.$$

Following the arguments of Cammaroto, Chinnì and Di Bella [10, Theorem 1.1], by taking $r_k = \left(\frac{b_k}{c}\right)^p$ (where $c > 0$ comes from (5.2)), it can be shown that $\varphi(r_k) < \frac{1}{p}$ for every $k \in \mathbb{N}$, where the crucial step is the point-wise estimate (5.2). Finally, one can prove that there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, the functional $\mathcal{E}_\lambda$ is not bounded from below in $W^{1,p}_{0,F}(\Omega)$. Therefore, Theorem 5.2 yields the existence of a sequence $\{u_k\} \subset W^{1,p}_{0,F}(\Omega)$ of critical points of $\mathcal{E}_\lambda$ such that $\lim_{k\to\infty} \|u_k\| = +\infty$. □

5.2. Existence of a nonzero solution for a Dirichlet problem involving a singular term. As an application of Theorem 4.2, we consider on (M,F) the following semilinear Dirichlet problem

$$\begin{cases} -\Delta_F u(x) - \mu \dfrac{u(x)}{d_F(x_0,x)^2} + \lambda u(x) = |u(x)|^{p-2}u(x), \quad x\in\Omega \\ u\geq 0,\ u\in W^{1,2}_{0,F}(\Omega), \end{cases} \tag{$\mathcal{P}_{\mu,\lambda}$}$$

where Δ_F is the 2-Finsler-Laplace operator on (M,F), $\Omega\subset M$ is a bounded, open set with C^1 boundary, and $x_0\in\Omega$ is arbitrarily fixed. In addition, suppose that $p\in(2,2^*)$, 2^* being the critical Sobolev exponent, i.e., $2^*=2n/(n-2)$ if $n\geq 3$ and $2^*=+\infty$ if $n=2$.

If μ and $\lambda\in\mathbb{R}$ belong to a suitable range of parameters, one can show the existence of a nonzero solution of problem $(\mathcal{P}_{\mu,\lambda})$, namely:

Theorem 5.3. *Let (M,F) be a noncompact, complete n-dimensional reversible Finsler manifold with $\mathsf{Ric}_n\geq 0$, $0<\mathsf{AVR}_F\leq 1$, and $n\geq 2$. Let $\Omega\subset M$ be a bounded open set with C^1 boundary, $x_0\in\Omega$ and $p\in(2,2^*)$. If either $\mu=0$ when $n=2$, or $\mu\in\left[0,\frac{(n-2)^2}{4}\mathsf{AVR}_F^{\frac{2}{n}}\right)$ when $n\geq 3$, and $\lambda>-S_{\mu,F}(\Omega)$, $S_{\mu,F}(\Omega)$ being the constant given by (4.13), then problem $(\mathcal{P}_{\mu,\lambda})$ has a nontrivial and nonnegative weak solution.*

Proof. Suppose that $\mu=0$ when $n=2$, or $\mu\in\left[0,\frac{(n-2)^2}{4}\mathsf{AVR}_F^{\frac{2}{n}}\right)$ when $n\geq 3$. Let $\lambda>-S_{\mu,F}(\Omega)$, and let us define the number $c_{\mu,\lambda}\in(0,1]$ by

$$c_{\mu,\lambda}:=\begin{cases} \min\left(1,1+\dfrac{\lambda}{S_{\mu,F}(\Omega)}\right), & \text{if } n=2 \\ \dfrac{4}{(n-2)^2}\overline{\mu}^2\min\left(1,1+\dfrac{\lambda}{S_{\mu,F}(\Omega)}\right), & \text{if } n\geq 3 \end{cases},$$

where $S_{\mu,F}(\Omega)$ and $\overline{\mu}$ are given by (4.13).

Then, using the *Hardy* inequality (1.3) and the *Brezis-Poincaré-Vázquez* inequality (4.12), it turns out that for every $u\in W^{1,2}_{0,F}(\Omega)$, we have

$$\mathcal{K}^2_{\mu,\lambda}(u):=\int_\Omega\left\{F^*(x,Du(x))^2-\mu\frac{u(x)^2}{d_F(x_0,x)^2}+\lambda u(x)^2\right\}\mathrm{d}v_F\geq c_{\mu,\lambda}\int_\Omega F^*(x,Du(x))^2\mathrm{d}v_F.$$

Therefore, the functional $u\mapsto\mathcal{K}_{\mu,\lambda}(u)$ defines a norm on $W^{1,2}_{0,F}(\Omega)$, which is equivalent to the usual Dirichlet-norm $\|\cdot\|_{D^1_0}$.

We associate with problem $(\mathcal{P}_{\mu,\lambda})$ its energy functional $\mathcal{E}_{\mu,\lambda}:W^{1,2}_{0,F}(\Omega)\to\mathbb{R}$ defined by

$$\mathcal{E}_{\mu,\lambda}(u):=\frac{1}{2}\mathcal{K}^2_{\mu,\lambda}(u)-\int_\Omega G(u(x))\mathrm{d}v_F,$$

where $G:\mathbb{R}\to[0,\infty)$, $G(s)=\frac{s_+^p}{p}$ and $s_+=\max(0,s)$. In a standard manner one can prove that $\mathcal{E}_{\mu,\lambda}\in C^1(W^{1,2}_{0,F}(\Omega);\mathbb{R})$. Moreover, $\mathcal{E}_{\mu,\lambda}$ verifies the conditions of the mountain pass theorem. Indeed, for any $p\in(2,2^*)$, by the Sobolev embedding theorem the continuous embedding $W^{1,2}_{0,F}(\Omega)\subset L^p(\Omega)$ holds, thus there exists a constant $c_1>0$ such that $\|u\|_{L^p}\leq c_1\|u\|_{W^{1,2}_0}$ for every $u\in W^{1,2}_{0,F}(\Omega)$. Furthermore, as Ω is a bounded domain, the norms $\|\cdot\|_{W^{1,2}_0}$ and $\|\cdot\|_{D^1_0}$ are equivalent, therefore we obtain that there exists a constant $c_2=c_2(c_1,p)>0$ such that

$$\mathcal{E}_{\mu,\lambda}(u)\geq\frac{1}{2}\mathcal{K}^2_{\mu,\lambda}(u)-\frac{c_1^p}{p}\|u\|^p_{W^{1,2}_0}\geq\mathcal{K}^2_{\mu,\lambda}(u)\left(\frac{1}{2}-c_2\cdot\mathcal{K}^{p-2}_{\mu,\lambda}(u)\right),$$

thus there exists a sufficiently small $\rho>0$ such that

$$\inf_{\mathcal{K}_{\mu,\lambda}(u)=\rho}\mathcal{E}_{\mu,\lambda}(u)>0=\mathcal{E}_{\mu,\lambda}(0).$$

Furthermore, for any $t > 0$ and $u \in W^{1,2}_{0,F}(\Omega)$ with $u \geq 0$, we have

$$\mathcal{E}_{\mu,\lambda}(tu) = \frac{t^2}{2}\mathcal{K}^2_{\mu,\lambda}(u) - t^p \int_\Omega G(u(x))\mathrm{d}v_F \leq \frac{t^2}{2}\int_\Omega F^*(x, Du(x))^2 \mathrm{d}v_F - \frac{t^p}{p}\int_\Omega u(x)^p \mathrm{d}v_F,$$

thus there exists a sufficiently large $t > 0$ and $\overline{u} \in W^{1,2}_{0,F}(\Omega) \setminus \{0\}$, $\overline{u} \geq 0$ such that $\mathcal{K}_{\mu,\lambda}(t\overline{u}) > \rho$ and $\mathcal{E}_{\mu,\lambda}(t\overline{u}) \leq 0$.

Since $W^{1,2}_{0,F}(\Omega)$ is compactly embedded into $L^p(\Omega)$ for any $p \in [2, 2^*)$, it can be proven that $\mathcal{E}_{\mu,\lambda}$ satisfies the Palais-Smale condition at each level. Therefore, by the Ambrosetti-Rabinowitz theorem (see Willem [33, Lemma 1.20]), it follows that $\mathcal{E}_{\mu,\lambda}$ has a positive critical value corresponding to a nontrivial weak solution $u \in W^{1,2}_{0,F}(\Omega)$ of the problem

$$\begin{cases} -\Delta_F u(x) - \mu \dfrac{u(x)}{d_F(x_0, x)^2} + \lambda u(x) = u(x)_+^{p-1}, & x \in \Omega \\ u \in W^{1,2}_{0,F}(\Omega). \end{cases}$$

Multiplying the first equation by $u_-(x) = \min(0, u(x)) \in W^{1,2}_{0,F}(\Omega)$ and integrating over Ω, we obtain that $\mathcal{K}^2_{\mu,\lambda}(u_-) = 0$, which in turn yields that $u_- = 0$. Thus $u \geq 0$ is a nontrivial solution to problem ($\mathcal{P}_{\mu,\lambda}$). □

References

[1] A. Alvino, V. Ferone, G. Trombetti, P.-L. Lions, Convex symmetrization and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 14 (1997), no. 2, 275–293.

[2] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev. *J. Differential Geometry* 11 (1976), no. 4, 573–598.

[3] A. A. Balinsky, W. D. Evans, R. T. Lewis, *The Analysis and Geometry of Hardy's Inequality*. Universitext, Springer-Verlag, 2015.

[4] Z. M. Balogh, A. Kristály, Sharp isoperimetric and Sobolev inequalities in spaces with nonnegative Ricci curvature. *Math. Ann.*, in press. DOI: https://doi.org/10.1007/s00208-022-02380-1.

[5] D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*. Graduate Texts in Mathematics Vol. 200, Springer-Verlag, 2000.

[6] E. Berchio, D. Ganguly, G. Grillo, Y. Pinchover, An optimal improvement for the Hardy inequality on the hyperbolic space and related manifolds. *Proc. Roy. Soc. Edinburgh Sect. A* 150 (2020), no. 4, 1699–1736.

[7] S. Brendle, Sobolev inequalities in manifolds with nonnegative curvature. *Comm. Pure. Appl. Math.*, to appear.

[8] J. E. Brothers, W. P. Ziemer, Minimal rearrangements of Sobolev functions. *J. Reine Angew. Math.* 384 (1988), 153–179.

[9] X. Cabré, X. Ros-Oton, J. Serra, Sharp isoperimetric inequalities via the ABP method. *J. Eur. Math. Soc.* 18 (2016), 2971–2998.

[10] F. Cammaroto, A. Chinnì, B. Di Bella, Infinitely many solutions for the Dirichlet problem involving the p-Laplacian. *Nonlinear Anal.* 61 (2005), 41–49.

[11] L. D'Ambrosio, S. Dipierro, Hardy inequalities on Riemannian manifolds and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 31 (2014), no. 3, 449–475.

[12] O. Druet, E. Hebey, M. Vaugon, Optimal Nash's inequalities on Riemannian manifolds: the influence of geometry. *Int. Math. Res. Not.* 14 (1999), 735–779.

[13] M. P. do Carmo, C. Xia, Complete manifolds with nonnegative Ricci curvature and the Caffarelli-Kohn-Nirenberg inequalities. *Compos. Math.* 140 (2004), 818–826.

[14] M. Ghomi, J. Spruck, Total curvature and the isoperimetric inequality in Cartan-Hadamard manifolds. *J. Geom. Anal.* 32 (2022), no. 2, Paper No. 50, 54 pp.

[15] E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*. Courant Lecture Notes in Mathematics, 5. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[16] L. Huang, A. Kristály, W. Zhao, Sharp uncertainty principles on general Finsler manifolds. *Trans. Amer. Math. Soc.* 373 (2020), no. 11, 8127-8161.

[17] B.R. Kloeckner, G. Kuperberg, The Cartan-Hadamard conjecture and the Little Prince. *Rev. Mat. Iberoam.* 35 (2019), no. 4, 1195–1258.

[18] A. Kristály, Sharp Morrey-Sobolev inequalities on complete Riemannian manifolds. *Potential Anal.* 42 (2015), no. 1, 141–154.

[19] A. Kristály, Sharp uncertainty principles on Riemannian manifolds: the influence of curvature, *J. Math. Pures. Appl. (Liouville Journal)*, 119 (2018), 326–346.

[20] A. Kristály, Metric measure spaces supporting Gagliardo-Nirenberg inequalities: volume non-collapsing and rigidities. *Calc. Var. Partial Differential Equations* 55 (2016), no. 5, Art. 112, 27 pp.
[21] A. Kristály, S. Ohta, Caffarelli-Kohn-Nirenberg inequality on metric measure spaces with applications. *Math. Ann.* 357 (2013), no. 2, 711–726.
[22] A. Kristály, A. Szakál, Interpolation between Brezis-Vázquez and Poincaré inequalities on nonnegatively curved spaces: sharpness and rigidities. *J. Differential Equations* 266 (2019), no. 10, 6621–6646.
[23] M. Ledoux, On manifolds with nonnegative Ricci curvature and Sobolev inequalities. *Comm. Anal. Geom.* 7 (1999), no. 2, 347–353.
[24] E. H. Lieb, M. Loss, *Analysis. Second edition.* Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
[25] M. Muratori, N. Soave, Some rigidity results for Sobolev inequalities and related PDEs on Cartan-Hadamard manifolds. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, to appear.
[26] S. Ohta, Finsler interpolation inequalities. *Calc. Var. Partial Differential Equations* 36 (2009), 211–249.
[27] S. Ohta, K.-T. Sturm, Heat flow on Finsler manifolds. *Comm. Pure Appl. Math.* 62, no. 10, (2009), 1386–1433.
[28] P. Petersen, *Riemannian geometry*, Third edition, Graduate Texts in Mathematics, Springer, 2016.
[29] B. Ricceri, A general variational principle and some of its applications. *J. Comput. Appl. Math.* 113 (2000), 401–410.
[30] Z. Shen, *Lectures on Finsler geometry.* World Scientific Publishing Co., Singapore, 2001.
[31] Z. Shen, Volume comparison and its applications in Riemann-Finsler geometry. *Adv. Math.* 128 (1997), 306–328.
[32] G. Talenti, Inequalities in rearrangement invariant function spaces. Nonlinear Analysis, Function Spaces and Applications, *Proceedings of the Spring School held in Prague, May 23-28, 1994.* Vol. 5. (1994), 177–230.
[33] M. Willem, *Minimax theorems.* Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.
[34] W. Zhao, Hardy inequalities with best constants on Finsler metric measure manifolds. *J. Geom. Anal.* 31 (2021), no. 2, 1992–2032.

A. Kristály: Institute of Applied Mathematics, Óbuda University, Bécsi út 96/B, 1034 Budapest, Hungary & Department of Economics, Babeş-Bolyai University, Str. Teodor Mihali 58-60, 400591 Cluj-Napoca, Romania

Email address: kristaly.alexandru@nik.uni-obuda.hu; alexandru.kristaly@ubbcluj.ro

Á. Mester: Department of Mathematics and Computer Sciences, Babeş-Bolyai University, Str. Mihail Kogălniceanu 1, 400084 Cluj-Napoca, Romania & Institute of Applied Mathematics, Óbuda University, Bécsi út 96/B, 1034 Budapest, Hungary

Email address: agnes.mester@ubbcluj.ro

I. Mezei: Department of Mathematics and Computer Sciences, Babeş-Bolyai University, Str. Mihail Kogălniceanu 1, 400084 Cluj-Napoca, Romania

Email address: ildiko.mezei@ubbcluj.ro