# VARIANCE BOUNDS FOR DISC-POLYGONS 

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#### Abstract

We prove asymptotic lower bounds on the variance of the number of vertices and missed area of random disc-polygons in convex discs whose boundary is $C_{+}^{2}$ smooth. The established lower bounds are of the same order as the upper bounds proved previously in 10 .


## 1. Results

We work in the Euclidean plane $\mathbb{R}^{2}$ with origin centred closed unit ball $B=B^{2}$ whose boundary is the unit circle $S^{1}=\partial B$. We denote the (origin centred) open unit ball by $B^{\circ}$. We denote by $A(\cdot)$ the area of measurable sets in $\mathbb{R}^{2}$. For general information about convex sets we refer to the books by Gruber [11 and Schneider 20.

For asymptotic inequalities, we use the following common notation: for two real sequences $f, g$, we write $f \ll g$ if there is a positive constant $\gamma$ such that $|f(n)| \leq \gamma g(n)$ for every $n \in \mathbb{N}$.

Let $K \subset \mathbb{R}^{2}$ be a convex disc with $C_{+}^{2}$ smooth boundary (twice continuously differentiable with positive curvature $\kappa(x)>0$ at every point $x \in \partial K)$. Let $r_{M}=$ $1 / \kappa_{m}$, where $\kappa_{m}=\min _{x \in \partial K} \kappa(x)>0$. It is known (see 20, Theorem 3.2.12 on p. 164]) that $K$ slides freely in a circle of radius $r_{M}$, that is, for every $x \in \partial K$,
 For $r \geq r_{M}$ and $X \subset K$, let $\operatorname{conv}_{r}(X)$ denote the intersection of all radius $r$ closed circular discs that contain $X$, that is,

$$
\operatorname{conv}_{r}(X):=\bigcap_{\substack{v \in \mathbb{R}^{2}, X \subset r B+v}}(r B+v)
$$

The set $\operatorname{conv}_{r}(X)$ is called the closed $r$-spindle convex hull or $r$-hull of $X$. It is known that $\operatorname{conv}_{r}(X) \subset K$, see Bezdek et al 6. For more information about the geometric properties of the $r$-spindle convex hull see, for example, [6] and [8] and the references therein.

We investigate the following probability model. Let $X_{n}=\left\{x_{1}, \ldots, x_{n}\right\} \subset K$ be a sample of $n$ i.i.d. random points selected according to the uniform probability distribution. Let $K_{n}^{r}:=\operatorname{conv}_{r}\left(X_{n}\right)$. The set $K_{n}^{r}$ is a (uniform) random convex $r$-disc-polygon in $K$ whose sides are arcs of radius $r$ circles. The edges, vertices and angles of $K_{n}^{r}$ are defined the usual way. Let $f_{0}\left(K_{n}^{r}\right)$ denote the number of vertices of $K_{n}^{r}$. We call $A\left(K \backslash K_{n}^{r}\right)$ the missed area of $K_{n}^{r}$.

[^0]If $\bar{K}_{n}$ denotes the (usual) convex hull of $X_{n}$, then $\bar{K}_{n} \subset K_{n}^{r} \subset K$ for all $r \geq r_{M}$. Since the containment $\bar{K}_{n} \subset K_{n}^{r}$ is strict, $K_{n}^{r}$ approximates $K$ better than $K_{n}$ from the point of view of area and perimeter. It is also clear that for fixed $X_{n}$, the $r$-disc-polygons $K_{n}^{r}$ tend to $\bar{K}_{n}$ in the Hausdorff-metric as $r \rightarrow \infty$. Furthermore, $f_{0}\left(K_{n}^{r}\right) \leq f_{0}\left(\bar{K}_{n}\right)$ for all $r \geq r_{M}$.

The geometric properties of the (classical) random polygon $\bar{K}_{n}$ have been investigated extensively. Starting with the seminal papers of Rényi and Sulanke $17-19$ the asymptotic behaviour of the expected number of vertices, area and perimeter have been determined. For a detailed overview of known results about the classical model, see for example, the surveys [1, 2, 13] and 21].

Fodor, Kevei and Vígh 8, Theorem 1.1 on p. 901] proved the following asymptotic formulas: for $r>r_{M}$, it holds that

$$
\begin{align*}
\lim _{n \rightarrow \infty} \mathbb{E} f_{0}\left(K_{n}^{r}\right) \cdot n^{-1 / 3} & =\sqrt[3]{\frac{2}{3 A(K)}} \Gamma\left(\frac{5}{3}\right) c_{1}(K, r)  \tag{1}\\
\lim _{n \rightarrow \infty} \mathbb{E} A\left(K \backslash K_{n}^{r}\right) \cdot n^{2 / 3} & =\sqrt[3]{\frac{2 A^{2}(K)}{3}} \Gamma\left(\frac{5}{3}\right) c_{1}(K, r), \tag{2}
\end{align*}
$$

where the constant

$$
c_{1}(K, r)=\int_{\partial K}\left(\kappa(x)-\frac{1}{r}\right)^{1 / 3} d x
$$

seems to resemble to the affine arc-length although it is unclear what is its exact geometric meaning. The formulas (1) and (2) are generalizations of the corresponding results of Rényi and Sulanke for the classical case, see Section 3 of 8 ]. We also note that in (1) and (2) the condition that $r>r_{M}$ is important. If $K$ is a discpolygon of radius $r$ itself, then the order of magnitude of $\mathbb{E} f_{0}\left(K_{n}^{r}\right)$ and $\mathbb{E} A\left(K \backslash K_{n}^{r}\right)$ are different, see 9 .

In the case when $K=B$, it is proved in [8, Theorem 1.3 on p. 902] that for $r=1$, it holds that

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \mathbb{E} f_{0}\left(B_{n}^{1}\right) & =\frac{\pi^{2}}{2} \\
\lim _{n \rightarrow \infty} \mathbb{E} A\left(B \backslash B_{n}^{1}\right) \cdot n & =\frac{\pi^{3}}{2}
\end{aligned}
$$

Significantly less is known about the higher moments of these quantities.
In the classical case, Reitzner $[14,15$ proved the following upper bounds for the variance of the number of $j$-dimensional faces and the volume of the random polytopes in smooth convex bodies in dimension $d$ :

$$
\begin{aligned}
\operatorname{Var}\left(\operatorname{Vol}\left(\bar{K}_{n}\right)\right) & \ll n^{-(d+3) /(d+1)}, \\
\operatorname{Var}\left(f_{j}\left(\bar{K}_{n}\right)\right) & \ll n^{(d-1) /(d+1)} .
\end{aligned}
$$

Here $\bar{K}_{n}$ denotes the (classical) convex hull of $n$ i.i.d. random points selected from the $d$-dimensional convex body $K \subset \mathbb{R}^{d}$ with $C_{+}^{2}$ smooth boundary. The symbol $\operatorname{Vol}(\cdot)$ denotes the volume of Lebesgue measurable sets in $\mathbb{R}^{d}$ and $f_{j}(\cdot)$ is the number of $j$-dimensional faces. The implied constants depend only on $K$ and the dimension. The upper bounds also imply strong laws of large numbers for these quantities.

Reitzner [16] gave matching lower bounds for the variance in case $K$ is smooth:

$$
\operatorname{Var}\left(\operatorname{Vol}\left(\bar{K}_{n}\right)\right) \gg n^{-(d+3) /(d+1)},
$$

$$
\operatorname{Var}\left(f_{j}\left(\bar{K}_{n}\right)\right) \gg n^{(d-1) /(d+1)}
$$

Using these lower bounds Reitzner [16] established central limit theorems for the number of $j$-dimensional faces and the volume for smooth convex bodies.

Upper and lower bounds, laws of large numbers and central limit theorems were extended for the case when $K \subset \mathbb{R}^{d}$ is a polytope by Bárány and Reitzner (4). Bárány and Steiger 3 proved asymptotic upper bounds and strong laws for the missed area and the number of vertices in the classical random model for arbitrary convex discs in $\mathbb{R}^{2}$ without any smoothness condition.

Fodor and Vígh [10 proved asymptotic upper bounds for the variance of the vertex number and the missed area of uniform random disc-polygons in $C_{+}^{2}$ smooth convex discs: For any $r>r_{M}$ it holds that

$$
\begin{align*}
& \operatorname{Var}\left(f_{0}\left(K_{n}^{r}\right)\right) \ll n^{1 / 3}  \tag{3}\\
& \operatorname{Var}\left(A\left(K_{n}^{r}\right)\right) \ll n^{-5 / 3} \tag{4}
\end{align*}
$$

where the implied constants depend only on $K$ and $r$. In the case when $K=B^{2}$, they proved 10 that

$$
\begin{align*}
& \operatorname{Var}\left(f_{0}\left(K_{n}^{r}\right)\right) \approx 1  \tag{5}\\
& \operatorname{Var}\left(A\left(K_{n}^{r}\right)\right) \ll n^{-2} \tag{6}
\end{align*}
$$

where the implied constants are universal. Note that the lower bound in (5) follows from the fact that the expected number of vertices $\mathbb{E} f_{0}\left(K_{n}\right)$ is a non-integer constant. Formulas (3), (4) and (5) imply laws of large number for the corresponding quantities, see 10].

In this paper we prove matching lower bounds for the variance of the vertex number and the missed area for the case when $r>r_{M}$. Our main results are summarized in the following theorem.

Theorem 1. Let $K$ be a convex disc whose boundary is of class $C_{+}^{2}$. For any $r>r_{M}$ it holds that

$$
\begin{align*}
& \operatorname{Var}\left(f_{0}\left(K_{n}^{r}\right)\right) \gg n^{\frac{1}{3}}  \tag{7}\\
& \operatorname{Var}\left(A\left(K_{n}^{r}\right)\right) \gg n^{-\frac{5}{3}} \tag{8}
\end{align*}
$$

where the implied constants depend only on $K$ and $r$.
We note that Theorem 1 is contained in the Master thesis of B. Grünfelder 12], and also in his Hungarian OTDK student competition paper.

We achieve these results using a modified version of the method of Reitzner 15 which had already been used in adapted forms in several different settings, see, for example, $7,22,23$. In the disc-polygonal setting, the difficulty lies in the intrinsic geometry of the model.

The lower bounds in Theorem 1 may open the road towards quantitative central limit theorems, similarly as in [5, 22, 23 using normal approximation bounds from Stein's method.

The layout of the paper is the following: in Section 2 we collect some necessary preparatory material. Section 3 contains the proof of (8) and Section 4 contains the outlines of the changes necessary for the proof of $(7)$.

## 2. Preparations

Without loss of generality, we may assume that $r=1$ and prove Theorem 1 for the case when $r_{M}<1$, since the general statements follow by a scaling argument. For simplicity, we write $K_{n}$ for $K_{n}^{1}$.

For $p \in \mathbb{R}^{2}$, the set $K \backslash\left(B^{\circ}+p\right)$ is called a disc-cap (of radius 1 ) of $K$. We use the notations from $[8]$. Let $x$ and $y$ be two points in $K$. The two unit circles that pass through these points, determine two disc-caps of $K$, denoted by $D_{-}(x, y)$ and $D_{+}(x, y)$, respectively, such that $A\left(D_{-}(x, y)\right) \leq A\left(D_{+}(x, y)\right)$. Briefly, we write $A_{-}(x, y)=A\left(D_{-}(x, y)\right)$ and $A_{+}(x, y)=A\left(D_{+}(x, y)\right)$. Lemma 4.3 in 8 states that if the boundary of $K$ is of class $C_{+}^{2}\left(r_{M}<1\right)$, then there exists a $\delta>0$ (depending only on $K$ ) with the property that for any $x, y \in \operatorname{int} K$ it holds that $A_{+}(x, y)>\delta$.

We need some further technical statements about general disc-caps. Denote the (unique) outer unit normal to $K$ at the boundary point $x$ by $u_{x} \in S^{1}$, and the unique boundary point with outer unit normal $u \in S^{1}$ by $x_{u} \in \partial K$. It is proved in Lemma 4.1 of 8 that if $K$ is a convex disc with $C_{+}^{2}$ boundary and $\kappa_{m}>1$, and $D=K \backslash\left(B^{\circ}+p\right)$ is a non-empty disc-cap, then there exists a unique point $x_{0} \in \partial K \cap \partial D$ such that $p=x_{0}-(1+t) u_{x_{0}}$ for some $t \geq 0$. The point $x_{0}$ is the vertex and the number $t$ is the height of $D$.

Let us denote the disc-cap with vertex $x_{u} \in \partial K$ and height $t$ by $D(u, t)$. To simplify the notation, we write $A(u, t)=A(D(u, t))$, and let $\ell(u, t)$ denote the arclength of $\partial D(u, t) \cap\left(\partial B+x_{u}-(1+t) u\right)$. The latter also exists since for each $u \in S^{1}$, there exists a maximal positive constant $t^{*}(u)$ such that $\left(B+x_{u}-(1+t) u\right) \cap K \neq \emptyset$ for all $t \in\left[0, t^{*}(u)\right]$.

The following limit relations for $A(u, t)$ and $\ell(u, t)$ are proved (in a more precise form) in [8, p. 905, Lemma 4.2]:

$$
\begin{equation*}
\ell\left(u_{x}, t\right) \approx t^{1 / 2}, \quad A\left(u_{x}, t\right) \approx t^{3 / 2} \tag{9}
\end{equation*}
$$

as $t \rightarrow 0^{+}$, where the implied constants depend only on $K$.
Let $D$ be a disc-cap of $K$ with vertex $x$. For a line $e \subset \mathbb{R}^{2}$ perpendicular to $u_{x}$, let $e_{+}$denote the closed half plane that contains $x$. Then there exists a maximal cap $C_{-}(D)=K \cap e_{+}$that is fully contained in $D$, and a minimal cap $C_{+}(D)=e_{+}^{\prime} \cap K$ containing $D$. We recall [10, Claim 1 on p. 1146], that gives a relation between classical caps and disc-caps, as follows: There exists a constant $\hat{c}$ depending only on $K$ such that if the height of the disc-cap $D$ is sufficiently small, then

$$
\begin{equation*}
C_{-}(D)-x \supset \hat{c}\left(C_{+}(D)-x\right) . \tag{10}
\end{equation*}
$$

The relation (10) means that the area of a disc-cap can be bounded by two classical caps such that one of them is an enlarged image of the other one by a constant.

Let $x_{i}, x_{j}(i \neq j)$ be two points of $X_{n}$, and let $B\left(x_{i}, x_{j}\right)$ be one of the unit discs containing $x_{i}$ and $x_{j}$ on its boundary. The shorter arc of $\partial B\left(x_{i}, x_{j}\right)$ forms an edge of $K_{n}$ if the entire set $X_{n}$ is contained in $B\left(x_{i}, x_{j}\right)$. It may happen that the pair $x_{i}, x_{j}$ determines two edges of $K_{n}$ if the above condition holds for both unit discs that contain $x_{i}$ and $x_{j}$ on its boundary.

## 3. Proof of (8) in Theorem 3

The proof is based on the ideas of Reitzner 15: we give small (disc-)caps which contribute to the variance geometrically independently and show that the variance in these caps is already sufficiently large.

For every $x \in \partial K$ and $t \in(0,1)$ consider the disc-cap $D(x, t)$ of vertex $x$ and height $t$. Let the Euclidean cap of vertex $x$ and height $t$ be $C(x, t)$. Let the line cutting off the cap $C(x, t)$ be $H(x, t)$. Clearly, $D(x, t) \supset C(x, t)$.

In the following we use large values of $n$, thus by an inequality of type $\ll$ we always assume that $t$ is sufficiently small.

Denote the intersections of $\partial K$ and the line $H(x, t)$ by $w_{1}$ and $w_{2}$, and let $w_{0}=x$. For the triangle $\Delta=\left[w_{0}, w_{1}, w_{2}\right]$ we have

$$
\Delta \subset C(x, t) \subset D(x, t)
$$

Let us define for $j=0,1,2$ the small triangles

$$
\Delta_{j}=\Delta_{j}(x, t)=w_{j}+\frac{1}{20}\left(\left[w_{0}, w_{1}, w_{2}\right]-w_{j}\right)
$$

i.e. we shrink the $\Delta$ from each of its vertices by a factor of $1 / 20$. It follows from (10) that $A\left(\Delta_{j}(x, t)\right) \approx t^{\frac{3}{2}}$, since the order of magnitude of the height of triangle $\Delta$ is $t$ and of its base is $\sqrt{t}$.

Let $x, t$ and the points $z_{1} \in \Delta_{1}(x, t)$ and $z_{2} \in \Delta_{2}(x, t)$ be fixed. For $z_{0} \in \Delta_{0}(x, t)$ let $\hat{A}\left(z_{0}\right)$ denote the area of the non-convex triangular region $\widetilde{\Delta}\left(z_{0}\right)$ we obtain by joining $z_{0}$ with $z_{1}$ and $z_{2}$ by circular arcs of radius 1 that are outside of the triangle $z_{0} z_{1} z_{2}$, and also joining $z_{1}$ and $z_{2}$ such that the arc intersects the interior of $z_{0} z_{1} z_{2}$.


Figure 1. Splitting $\Delta^{m}$

Lemma 1. Let $Z$ be a uniform random point in $\Delta_{0}(x, t)$. Then

$$
\operatorname{Var}(\hat{A}(Z)) \gg t^{3}
$$

Proof. Let $w$ denote the midpoint of the side opposite to $x$ in the triangle $\Delta_{0}(x, t)$. Let

$$
\Delta_{0}^{(1)}(x, t)=x+\frac{1}{3}\left(\Delta_{0}(x, t)-x\right)
$$

and

$$
\Delta_{0}^{(2)}(x, t)=w+\frac{1}{3}\left(\Delta_{0}(x, t)-w\right)
$$



Figure 2. Angle of circular arcs and chords
i.e. in the triangle $\Delta_{0}(x, t)$ we take two smaller triangles which are the shrunk images of $\Delta_{0}(x, t)$ by a factor of $1 / 3$ from $x$ and $w$. The area of $\Delta_{0}^{(1)}$ and $\Delta_{0}^{(2)}$ is one-ninth of that of $\Delta_{0}$, respectively.

For every $Z_{1} \in \Delta_{0}^{(1)}$ and $Z_{2} \in \Delta_{0}^{(2)}$, it holds that $\widetilde{\Delta}\left(Z_{1}\right) \supset \widetilde{\Delta}\left(Z_{2}\right)$, therefore $\hat{A}\left(Z_{1}\right)>\hat{A}\left(Z_{2}\right)$. Let $\Delta^{m}=\widetilde{\Delta}\left(Z_{1}\right) \backslash \widetilde{\Delta}\left(Z_{2}\right)$. We need $A\left(\Delta^{m}\right)$.

Cut $\Delta^{m}$ by a segment trough $Z_{1}$ perpendicular to the line $H$, and denote the other intersection point of this segment with $\partial \Delta^{m}$ by $a$. Then $d\left(Z_{1}, a\right) \approx t$. Suppose that after the cut $Z_{2}$ is contained in the set that has $z_{2}$ on its boundary, see Figure 1.

Consider the Euclidean triangle $\left[Z_{1}, a, z_{1}\right]$, and let $\gamma$ denote the angle at the vertex $z_{1}$. The radius of the circumscribed circle of this triangle is of order $\sqrt{t}$. Thus, since the side opposite to the angle $\gamma$ has is of order $t$, the law of sines gives that $\sin \gamma \approx \sqrt{t}$. By the smallness of $t$, the angle $\gamma$ has the same order of magnitude as $\sin \gamma$.

After that, we translate $a$ along $\partial \Delta^{m}$ into a point $a^{\prime}$ such that $d\left(z_{1}, a^{\prime}\right)=$ $d\left(z_{1}, Z_{1}\right)$ holds. (In case $a^{\prime}$ has reached $Z_{2}$ and the distances are still not equal, we translate $Z_{1}$ closer to $z_{1}$.) By this, for the angle at $z_{1}$ we have $\gamma^{\prime} \geq \gamma$.

Since $d\left(z_{1}, a^{\prime}\right)=d\left(z_{1}, Z_{1}\right)$, the angle between the two circular arcs of radius 1 corresponding to the two segments is $\gamma^{\prime}$ as well, see Figure 2 By the angle between two circular arc we mean the angle between their tangent lines.

Consider the sector-like shape determined by $z_{1}, a^{\prime}$ and $Z_{1}$, whose legs are circular arcs of radius 1 . This is a part of a circular disc of radius $\varrho=d\left(z_{1}, Z_{1}\right)$, by which, rotating around the centre, we can cover the whole disc. The area of a shape of this property is proportional to the central angle and the square of the radius. Here we have $\varrho \approx \sqrt{t}$ and the order of magnitude of the angle is at least $\sqrt{t}$, therefore the area of this shape is at least of order $t^{3 / 2}$.

We have estimated the area $\hat{A}\left(Z_{1}\right)-\hat{A}\left(Z_{2}\right)$ we are looking for from below, examining its subset. This gives us the following lower estimate:

$$
\begin{equation*}
\hat{A}\left(Z_{1}\right)-\hat{A}\left(Z_{2}\right) \gg t^{3 / 2} \tag{11}
\end{equation*}
$$

Let $Z, Z_{1}^{\prime}$ and $Z_{2}^{\prime}$ be i.i.d uniform random points in $\Delta_{0}(x, t)$. Using (11), we obtain the desired lower bound:

$$
\begin{aligned}
\operatorname{Var}(\hat{A}(Z)) & =\mathbb{E}\left[\hat{A}(Z)^{2}\right]-\mathbb{E}[\hat{A}(Z)]^{2}= \\
& =\mathbb{E}\left[\hat{A}(Z)^{2}\right]-\mathbb{E}\left[\hat{A}\left(Z_{1}^{\prime}\right) \hat{A}\left(Z_{2}^{\prime}\right)\right]= \\
& =\frac{1}{2} \mathbb{E}\left[\hat{A}\left(Z_{1}^{\prime}\right)^{2}-2 \hat{A}\left(Z_{1}^{\prime}\right) \hat{A}\left(Z_{2}\right)+\hat{A}\left(Z_{2}^{\prime}\right)^{2}\right]= \\
& =\frac{1}{2} \mathbb{E}\left[\left(\hat{A}\left(Z_{1}^{\prime}\right)-\hat{A}\left(Z_{2}^{\prime}\right)\right)^{2}\right] \geq \\
& \geq \frac{1}{2} \mathbb{E}\left[\left(\hat{A}\left(Z_{1}^{\prime}\right)-\hat{A}\left(Z_{2}^{\prime}\right)\right)^{2} \mathbb{1}\left(Z_{1}^{\prime} \in \Delta_{1}, Z_{2}^{\prime} \in \Delta_{2}\right)\right] \gg \\
& \gg t^{3} \mathbb{E}\left[\mathbb{1}\left(Z_{1}^{\prime} \in \Delta_{1}, Z_{2}^{\prime} \in \Delta_{2}\right)\right] \gg t^{3} .
\end{aligned}
$$

We may assume that $n>n_{0}$ for some suitable $n_{0}$, since it is sufficient to prove the lower bound of variance for large $n$. In the following, we consider disc-caps of height $t_{n}$ for

$$
\begin{equation*}
t_{n}=n^{-\frac{2}{3}} \tag{12}
\end{equation*}
$$

Choose a maximal set of points $y_{1}, \ldots, y_{m}$ on $\partial K$ such that $\left|x_{i}-x_{j}\right| \geq 2 \sqrt{c_{2}} \sqrt{t_{n}}$ for any $i, j \in\{1, \ldots, m\}$ for some constant $c_{2}$ that we specify later. Then

$$
\begin{equation*}
m \gg n^{\frac{1}{3}} \tag{13}
\end{equation*}
$$

Consider the pairwise disjoint disc-caps $D\left(y_{j}, t_{n}\right)$ and the previously defined triangles $\Delta\left(y_{j}, t_{n}\right)$ in them, for $j \in[m]$. For each $\Delta\left(y_{j}, t_{n}\right)$, also construct the small triangles $\Delta_{i}\left(y_{j}, t_{n}\right)$, for $i=0,1,2$. Let $E_{j}$ be the event that each of the small triangles $\Delta_{i}\left(y_{j}, t_{n}\right)$ contains exactly one of the random points $x_{1}, \ldots, x_{n}$ and that $D\left(y_{j}, c_{2} t_{n}\right)$ contains no other random point. By $\sqrt{9}$ ) we have

$$
A\left(D\left(y_{j}, c_{2} t_{n}\right)\right) \ll \frac{1}{n}
$$

and for $i=0,1,2$

$$
A\left(\Delta_{i}\left(y_{j}, t_{n}\right)\right) \gg \frac{1}{n}
$$

We have for every $j \in[m]$

$$
\mathbb{P}\left(E_{j}\right) \gg\binom{n}{3}\left(\frac{1}{n}\right)^{3}\left(1-\frac{1}{n}\right)^{n-3} \gg 1
$$

thus

$$
\begin{equation*}
\mathbb{E}\left(\sum_{j=1}^{m} \mathbb{1}\left(E_{j}\right)\right)=\sum_{j=1}^{m} \mathbb{P}\left(E_{j}\right) \gg m \tag{14}
\end{equation*}
$$

In the case the event $E_{j}$ occurs, let the random point in $\Delta_{0}\left(y_{j}, t_{n}\right)$ be denoted by $Z_{j}$.

Lemma 2. Assume that $J \subset[m]$ and $E_{j}$ occurs for every $j \in J$. Then

$$
A\left(K_{n}\right)=A\left(\operatorname{conv}_{r}\left(X_{n} \backslash\left\{Z_{1}, \ldots Z_{j}\right\}\right)\right)+\sum_{j \in J} \hat{A}\left(Z_{j}\right)
$$

Proof. Our goal is to show that if for indices $j, k \in J, j \neq k, Z_{j}$ and $Z_{k}$ are the random points in triangles $\Delta_{0}\left(y_{j}, t_{n}\right)$ and $\Delta_{0}\left(y_{k}, t_{n}\right)$, then $Z_{j}$ and $Z_{k}$ are vertices of $K_{n}$ and there is no edge between them. This means that the contributions of $Z_{i}$ and $Z_{j}$ to the area of $K_{n}$ are geometrically "independent".

For this, we need that for every $j$, the circular arc of radius 1 determined by the two random points in triangles $\Delta_{0}\left(y_{j}, t_{n}\right)$ and $\Delta_{i}\left(y_{j}, t_{n}\right)(i \in\{1,2\})$, meets the boundary of $K$ without intersecting any other disc-cap $D\left(y_{k}, t_{n}\right)$.

For simplicity take a fixed disc-cap of height $t$ and vertex $x$, and the triangles $\Delta_{0}$ and $\Delta_{1}$ in it. Orient the cap in such a way that the outer normal at $x$ points in the positive direction of the y-axis. The intersection with $\partial K$ has minimal $y$-coordinate in case the random point in $\Delta_{0}$ lies at the bottom corner, nearest to $\Delta_{1}$, and the random point in $\Delta_{1}$ is in the point farthest from the boundary, see Figure 3. Let us denote these points by $a_{0}$ and $a_{1}$, respectively.

In the case when $K$ is a circle of radius $r$, we can exactly compute the $y$ coordinate of the intersection for a fixed $r$. The depth will be smaller than $c_{3} t$ for a suitable constant $c_{3}$ depending only on $K$. This $c_{3}$ is bounded as $r$ is strictly smaller than 1 . Now we can specify the constant $c_{2}$ to be as large such that the disc-caps $D\left(y_{k}, t_{n}\right)$ are far apart enough and the observed intersection point is not contained in any other disc-cap. The constant $c_{2}$ depends only on $K$. Therefore the statement of the Lemma is true for circles.

For a general convex disc $K$ with $C_{+}^{2}$ boundary, we estimate the $y$-coordinate of the intersection point as follows. Consider the osculating circle of $K$ at the point $x$ with radius $\left(R_{0}(x)=\right) R_{0}<1$. There exists an $\varepsilon>0$, such that in any neighbourhood of radius less than $\varepsilon$ of $x$, for the circles of radii $R_{0}+\varepsilon$ and $R_{0}-\varepsilon$ having the same tangent line as $K$ at $x$, it is true that $K$ is locally inside of the larger circle and the smaller circle is inside of $K$.

The line $H(x, t)$ meets these circles in $p_{1}, p_{2}$ and $q_{1}, q_{2}$, where the points with the same indices are close to each other. Then for $i=1,2$,

$$
\begin{align*}
d\left(p_{i}, q_{i}\right) & =\sqrt{\left(2 R_{0}+2 \varepsilon-t\right) t}-\sqrt{\left(2 R_{0}-2 \varepsilon-t\right) t} \\
& =\sqrt{t}\left(\sqrt{2 R_{0}+2 \varepsilon-t}-\sqrt{2 R_{0}-2 \varepsilon-t}\right) \\
& =\sqrt{t}\left(\frac{4 \varepsilon}{\sqrt{2 R_{0}+2 \varepsilon-t}+\sqrt{2 R_{0}-2 \varepsilon-t}}\right) . \tag{15}
\end{align*}
$$

Since $t \rightarrow 0$, we may assume that $t<\varepsilon$, thus we can estimate from above as follows.

$$
\sqrt{t}\left(\frac{4 \varepsilon}{\sqrt{2 R_{0}+2 \varepsilon-t}+\sqrt{2 R_{0}-2 \varepsilon-t}}\right) \leq \frac{4}{\sqrt{2 R_{0}}} \varepsilon \sqrt{t} \leq c_{4} \varepsilon \sqrt{t}
$$

where $c_{4}=\max _{x \in \partial K} 4 / \sqrt{2 R_{0}(x)}$.
Let $\beta$ denote the length of the side of $\Delta_{0}$ opposite to $x$. Then the distance between the intersection of $\partial K$ with the line $H$ and $a_{1}$ is also $\beta$. The order of magnitude of $\beta$ is $\sqrt{t}$. We may assume that $\beta=c_{5} \sqrt{t}$ for some constant $c_{5}$.

Consider the unique point of $H \cap K$ which is at distance $3 / 2 \beta$ from $p_{1}$. The circular arc of radius 1 determined by this point and $a_{0}$, meets the outer circle at a computable, bounded depth depending on $\left(R_{0}+\varepsilon\right)$. In the case $c_{4} \cdot \varepsilon<c_{5} / 2$, the circular arc incident with $a_{0}$ and $a_{1}$ meets the outer circle less deeply. For this, we


Figure 3. $K$ bounded between circles
need that

$$
\varepsilon<\frac{c_{5}}{2 c_{4}} .
$$

Note that constant on the right-hand side of the inequality depends only on $K$. Since $t \rightarrow 0, \varepsilon$ can be chosen smaller than that. The only restriction on $\varepsilon$ is that $K$ has to be locally between the two circles, that is, the intersection point $c_{6} t$ deep is in this range. Therefore, we can choose $\varepsilon$ arbitrary small.

Thus for every $K$, the circular arc determined by the points $a_{0}$ and $a_{1}$ meets the boundary of $K$ in a depth of order $t$. Therefore this arc will not intersect any other disc-cap $D\left(y_{j}, t_{n}\right)$.

Let $\mathcal{F}$ denote the $\sigma$-algebra generated by the events $E_{j}$. Consider the conditional variance on $\mathcal{F}$. By the Law of Total Variance,

$$
\begin{align*}
\operatorname{Var} A\left(K_{n}\right) & =\mathbb{E} \operatorname{Var}\left(A\left(K_{n}\right) \mid \mathcal{F}\right)+\operatorname{Var} \mathbb{E}\left(A\left(K_{n}\right) \mid \mathcal{F}\right) \\
& \geq \mathbb{E} \operatorname{Var}\left(A\left(K_{n}\right) \mid \mathcal{F}\right) \tag{16}
\end{align*}
$$

For the set of indices $J \subseteq[m]$, let $E(J)$ be the event that the event $E_{j}$ occurs for exactly the indices $j \in J$, and does not occur for the other indices in $[m]$. Then the conditional variance on $\mathcal{F}$ can be expanded in the following form:

$$
\begin{equation*}
\operatorname{Var}\left(A\left(K_{n}\right) \mid \mathcal{F}\right)=\sum_{J \subseteq[m]} \operatorname{Var}\left(A\left(K_{n}\right) \mid E(J)\right) \cdot \mathbb{1}(E(J)) \tag{17}
\end{equation*}
$$

By Lemma 2, in case the event $E(J)$ occurs, the area of $K_{n}$ can be written as the sum of $\hat{A}\left(Z_{j}\right)$ and the area of the hull of the remaining points:

$$
\begin{equation*}
\operatorname{Var}\left(A\left(K_{n}\right) \mid E(J)\right)=\operatorname{Var}\left[A\left(\operatorname{conv}_{r}\left(X_{n} \backslash\left\{Z_{1}, \ldots Z_{j}\right\}\right)\right)+\sum_{j \in J} \hat{A}\left(Z_{j}\right) \mid E(J)\right] \tag{18}
\end{equation*}
$$

Let the points of $X_{n} \backslash\left\{Z_{1}, \ldots Z_{j}\right\}$ be fixed. Then the first term on the right-hand side of $\sqrt{18}$, the area of the $r$-hull of the remaining points is constant, thus we can omit it. The other terms are independent since the contributions of the points $Z_{j}$ are geometrically independent, therefore we can take the variance term-by-term.

This holds for every fixed set of points $X_{n} \backslash\left\{Z_{1}, \ldots Z_{j}\right\}$, thus we have the following inequality. The variance $\operatorname{Var}_{Z_{j}}$ means that the variance is taken only for the corresponding variable.

$$
\operatorname{Var}\left(A\left(K_{n}\right) \mid E(J)\right) \geq \sum_{j \in J} \operatorname{Var}_{Z_{j}}\left(\hat{A}\left(Z_{j}\right) \mid E_{j}\right)
$$

Substituting it in 17), we obtain that

$$
\begin{equation*}
\sum_{J \subseteq[m]} \operatorname{Var}\left(A\left(K_{n}\right) \mid E(J)\right) \cdot \mathbb{1}(E(J)) \geq \sum_{J \subseteq[m]} \mathbb{1}(E(J)) \sum_{j \in J} \operatorname{Var}_{Z_{j}}\left(\hat{A}\left(Z_{j}\right) \mid E_{j}\right) \tag{19}
\end{equation*}
$$

Rearrange the right-hand side of 19 such that the sum is according to the index of $\operatorname{Var}_{Z_{j}}\left(\hat{A}\left(Z_{j}\right) \mid E_{j}\right)$ :

$$
\begin{aligned}
\sum_{J \subseteq[m]} \mathbb{1}(E(J)) \sum_{j \in J} \operatorname{Var}_{Z_{j}}\left(\hat{A}\left(Z_{j}\right) \mid E_{j}\right) & =\sum_{j=1}^{m} \operatorname{Var}_{Z_{j}}\left(\hat{A}\left(Z_{j}\right) \mid E_{j}\right) \sum_{\{J: j \in J\}} \mathbb{1}(E(J))= \\
& =\sum_{j=1}^{m} \operatorname{Var}_{Z_{j}}\left(\hat{A}\left(Z_{j}\right) \mid E_{j}\right) \mathbb{1}\left(E_{j}\right)
\end{aligned}
$$

Thus, we have

$$
\begin{equation*}
\operatorname{Var}\left(A\left(K_{n}\right) \mid \mathcal{F}\right) \geq \sum_{j=1}^{m} \operatorname{Var}_{Z_{j}}\left(\hat{A}\left(Z_{j}\right) \mid E_{j}\right) \mathbb{1}\left(E_{j}\right) \tag{20}
\end{equation*}
$$

By Lemma 1 together with $(12)-(14),(16)$, and 20$)$,

$$
\begin{aligned}
\operatorname{Var} A\left(K_{n}\right) & \geq \mathbb{E}\left(\sum_{j=1}^{m} \operatorname{Var}_{Z_{j}}\left(\hat{A}\left(Z_{j}\right) \mid E_{j}\right) \mathbb{1}\left(E_{j}\right)\right) \gg \mathbb{E}\left(\sum_{j=1}^{m} t_{n}^{3} \mathbb{1}\left(E_{j}\right)\right) \gg \\
& \gg n^{-2}\left(\mathbb{E} \sum_{j=1}^{m} \mathbb{1}\left(E_{j}\right)\right) \gg n^{-2} m \gg n^{-\frac{5}{3}}
\end{aligned}
$$

which is (8) of Theorem 1, thus we have finished the proof.

## 4. The variance of the number of vertices

We give the outline of the necessary changes in the previous argument to prove (7) of Theorem 1] the lower bound of the variance of the number of vertices, see also Reitzner 16.

Let the disc-caps $D\left(y_{j}, t_{n}\right)$ be defined as before, and also the triangles $\Delta\left(y_{n}, t_{n}\right)$ together with the small triangles $\Delta_{i}\left(y_{n}, t_{n}\right)$, where $j \in[m]$ and $i \in\{0,1,2\}$. Let $F_{j}$ denote the event that $\Delta_{1}\left(y_{n}, t_{n}\right)$ and $\Delta_{2}\left(y_{n}, t_{n}\right)$ each contain exactly one of the random points $x_{1}, \ldots, x_{n}, \Delta_{0}\left(y_{n}, t_{n}\right)$ contains exactly two of $x_{1}, \ldots, x_{n}$, and there is no other random point in the disc-cap $D\left(y_{j}, c_{2} t_{n}\right)$. The probability of $F_{j}$ is

$$
\mathbb{P}\left(F_{j}\right) \gg\binom{n}{4}\left(\frac{1}{n}\right)^{4}\left(1-\frac{1}{n}\right)^{n-4} \gg 1
$$

thus, similar to (14), we have

$$
\begin{equation*}
\mathbb{E}\left(\sum_{j=1}^{m} \mathbb{1}\left(F_{j}\right)\right) \gg m \tag{21}
\end{equation*}
$$

If the event $F_{j}$ occurs, denote the random points in $\Delta_{0}\left(y_{j}, t_{n}\right)$ by $Y_{j}$ and $Z_{j}$. It follows from the proof of Lemma 2, that in this case each one of the triangles $\Delta_{i}\left(y_{n}, t_{n}\right)(i \in\{0,1,2\})$ contains a vertex of $K_{n}$. Either only one of the points $Y_{j}$ and $Z_{j}$ is a vertex of the $K_{n}$, or both of them are. Therefore the disc-cap $D\left(y_{j}, t_{n}\right)$ contains either 3 or 4 vertices, both of these events have positive probability. It follows, taking the variance for only these two points, that

$$
\begin{equation*}
\operatorname{Var}_{Y_{j}, Z_{j}}\left(\hat{f}_{0}\left(\left[Y_{j}, Z_{j}, z_{j, 1}, z_{j, 2}\right]\right) \mid F_{j}\right) \gg 1 \tag{22}
\end{equation*}
$$

where $\hat{f}_{0}\left(\left[Y_{j}, Z_{j}, z_{j, 1}, z_{j, 2}\right]\right.$ denotes the number of vertices contained in the $j$-th disccap.

Let $\mathcal{G}$ be the $\sigma$-algebra generated by the events $F_{j}$. Similar to the proof of Lemma 2, it holds here as well, that the number of vertices in one of the disc-caps does not affect how many vertices are there in an other disc-cap $D\left(y_{j}, t_{n}\right)$. Using this fact, together with $\sqrt[13]{2},(16)$ and $(21)-(22)$, we get

$$
\begin{aligned}
\operatorname{Var} f_{0}\left(K_{n}\right) & \geq \mathbb{E} \operatorname{Var}\left(f_{0}\left(K_{n}\right) \mid \mathcal{G}\right) \\
& \geq \mathbb{E}\left(\sum_{j=1}^{m} \operatorname{Var}_{Y_{j}, Z_{j}}\left(\hat{f}_{0}\left(\left[Y_{j}, Z_{j}, z_{j, 1}, z_{j, 2}\right]\right) \mid F_{j}\right) \mathbb{1}\left(F_{j}\right)\right) \\
& \gg \mathbb{E}\left(\sum_{j=1}^{m} \mathbb{1}\left(F_{j}\right)\right) \gg m>n^{\frac{1}{3}},
\end{aligned}
$$

which is the statement of (7) of Theorem 1, thus we have finished the proof.

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