AN EXTENSION OF THE ABEL–LIOUVILLE IDENTITY

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Abstract. In this note, we present an extension of the celebrated Abel–Liouville identity in terms of noncommutative complete Bell polynomials for generalized Wronskians. We also characterize the range equivalence of n-dimensional vector-valued functions in the subclass of n-times differentiable functions with a nonvanishing Wronskian.

1. Introduction

Throughout this paper let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive and nonnegative integers, respectively, and let I stand for a nonempty open real interval.

For an n-dimensional vector valued $(n-1)$-times continuously differentiable function $f : I \to \mathbb{R}^n$, its *Wronskian* $W_f : I \to \mathbb{R}$ is defined by

$$W_f := \begin{vmatrix} f^{(n-1)} & \dots & f' & f \end{vmatrix}.$$

Consider now the nth-order homogeneous linear differential equation

$$y^{(n)} = a_1 y^{(n-1)} + \dots + a_n y, \tag{1}$$

where $a_1, \dots, a_n : I \to \mathbb{R}$ are continuous functions. By the classical *Abel–Liouville identity* (cf. [4]), if $f : I \to \mathbb{R}^n$ is a fundamental system of solutions of (1), then W_f does not vanish on I and

$$W_f' = a_1 W_f.$$

For a sufficiently smooth function $f : I \to \mathbb{R}^n$ and $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, we introduce now the *generalized Wronskian* $W_f^k : I \to \mathbb{R}$ by

$$W_f^k := \begin{vmatrix} f^{(k_1)} & \dots & f^{(k_n)} \end{vmatrix}.$$

One can easily see that, with this notation, we have

$$W_f = W_f^{(n-1,n-2,\dots,0)} \qquad \text{and} \qquad W_f' = W_f^{(n,n-2,\dots,0)}.$$

Therefore, the Abel–Liouville identity can be rewritten as

$$W_f^{(n,n-2,\dots,0)} = a_1 W_f^{(n-1,n-2,\dots,0)}. \tag{2}$$

One of the main goal of this short paper is to establish a formula for W_f^k in terms of the coefficients of differential equation (1). Another goal is to introduce the range equivalence for n-dimensional vector-valued functions and to characterize this equivalence relation in the subclass of n-times differentiable functions with a nonvanishing Wronskian.

Date: September 4, 2020.

2010 *Mathematics Subject Classification.* 34A30, 34A06.

Key words and phrases. Abel–Liouville identity; generalized Wronskians; complete Bell polynomials.

The research of the first author was supported by the K-134191 NKFIH Grant and the 2019-2.1.11-TÉT-2019-00049, the EFOP-3.6.1-16-2016-00022 and the EFOP-3.6.2-16-2017-00015 projects. The last two projects are co-financed by the European Union and the European Social Fund.

2. Main results

For the description of our main result, we recall the notion of *noncommutative complete Bell polynomials*, which was introduced by Schimming and Rida [3]. Let $\mathbb{R}^{n\times n}$ denote the ring of $n\times n$ matrices with real entries and let $\mathbb{I}_n$ denote the $n\times n$ unit matrix. Now define $B_m:(\mathbb{R}^{n\times n})^m\to\mathbb{R}^{n\times n}$ by the following recursive formula

$$B_0:=\mathbb{I}_n,\qquad B_{m+1}(X_1,\dots,X_{m+1}):=\sum_{j=0}^{m}\binom{m}{j}B_j(X_1,\dots,X_j)X_{m+1-j}. \tag{3}$$

The notion of *complete Bell polynomials* in the commutative setting (i.e., when $n=1$) was introduced by Bell [1], [2]. One can easily compute the first few Bell polynomials as follows:

$$\begin{aligned}
B_1(X_1)&=X_1,\\
B_2(X_1,X_2)&=X_1^2+X_2,\\
B_3(X_1,X_2,X_3)&=X_1^3+2X_1X_2+X_2X_1+X_3,\\
B_4(X_1,X_2,X_3,X_4)&=X_1^4+3X_1^2X_2+2X_1X_2X_1+3X_1X_3+3X_2^2+X_2X_1^2+X_3X_1+X_4,\\
B_5(X_1,X_2,X_3,X_4,X_5)&=X_1^5+4X_1^3X_2+3X_1^2X_2X_1+6X_1^2X_3+8X_1X_2^2+2X_1X_2X_1^2\\
&\quad+3X_1X_3X_1+4X_1X_4+3X_2^2X_1+X_2X_1^3+4X_2X_1X_2+6X_2X_3\\
&\quad+X_3X_1^2+4X_3X_2+X_4X_1+X_5.
\end{aligned}$$

The statement of the next basic lemma was proved in the paper [3].

Lemma 1. *For every $j\in\mathbb{N}_0$, and j-times differentiable matrix-valued function $X:I\to\mathbb{R}^{n\times n}$,*

$$B_{j+1}\big(X,\dots,X^{(j)}\big)=XB_j\big(X,\dots,X^{(j-1)}\big)+\Big(B_j\big(X,\dots,X^{(j-1)}\big)\Big)'. \tag{4}$$

Lemma 2. *Let $n,m\in\mathbb{N}$, let $X:I\to\mathbb{R}^{n\times n}$ be an $(m-1)$-times continuously differentiable function and $Y:I\to\mathbb{R}^{n\times n}$ be a differentiable function such that*

$$Y'=YX \tag{5}$$

holds on I. Then Y is m-times continuously differentiable and

$$Y^{(j)}=YB_j\big(X,\dots,X^{(j-1)}\big)\qquad (j\in\{0,\dots,m\}). \tag{6}$$

Proof. If $m=1$, then X is continuous, hence the continuity of Y and (5) imply that Y is continuously differentiable. If $m>1$, then using (5), a simple inductive argument shows that Y is m-times continuously differentiable.

The equality (6) is trivial if $j=0$, because $B_0=\mathbb{I}_n$. For $j=1$, the equality (6) is equivalent to (5). Now assume that (6) has been verified for some $1\le j<m$. Then, using (5) and Lemma 1, we get

$$\begin{aligned}
Y^{(j+1)}=\big(Y^{(j)}\big)'=\Big(YB_j\big(X,\dots,X^{(j-1)}\big)\Big)'&=Y'B_j\big(X,\dots,X^{(j-1)}\big)+Y\Big(B_j\big(X,\dots,X^{(j-1)}\big)\Big)'\\
&=Y\Big[XB_j\big(X,\dots,X^{(j-1)}\big)+\Big(B_j\big(X,\dots,X^{(j-1)}\big)\Big)'\Big]=YB_{j+1}\big(X,\dots,X^{(j)}\big).
\end{aligned}$$

This proves the assertion for $j+1$. □

In what follows, let $e_1,\dots,e_n$ denote the elements of the standard basis in $\mathbb{R}^n$.

Corollary 3. *Let $n,m\in\mathbb{N}$, let $a:I\to\mathbb{R}^n$ be an $(m-1)$-times continuously differentiable function and let $f:I\to\mathbb{R}^n$ be a fundamental system of solutions of the differential equation (1). Let the matrix-valued functions $X_a:I\to\mathbb{R}^{n\times n}$ and $Y_f:I\to\mathbb{R}^{n\times n}$ be defined by*

$$X_a:=\begin{pmatrix}a & e_1 & \dots & e_{n-1}\end{pmatrix}\qquad\text{and}\qquad Y_f:=\begin{pmatrix}f^{(n-1)} & \dots & f' & f\end{pmatrix}. \tag{7}$$

Then Y_f is m-times continuously differentiable and

$$Y_f^{(j)}=Y_fB_j\big(X_a,\dots,X_a^{(j-1)}\big)\qquad (j\in\{0,\dots,m\}). \tag{8}$$

Proof. The function f satisfies the differential equation (1), therefore $f^{(n)} = Y_f \cdot a$. On the other hand, $f^{(n-i)} = Y_f \cdot e_i$ holds for $i \in \{1, \dots, n-1\}$. These equalities imply that

$$Y_f' = \begin{pmatrix} f^{(n)} & f^{(n-1)} & \dots & f' \end{pmatrix} = \begin{pmatrix} Y_f \cdot a & Y_f \cdot e_1 & \dots & Y_f \cdot e_{n-1} \end{pmatrix} = Y_f X_a. \tag{9}$$

Therefore, equation (5) holds with $Y := Y_f$ and $X := X_a$, consequently, the statement is a consequence of Lemma 2. □

Using the above corollary, we can easily establish a formula for the computation of the generalized Wronskian W_f^k.

Theorem 4. *Let $n, m \in \mathbb{N}$, let $a : I \to \mathbb{R}^n$ be an $(m-1)$-times continuously differentiable function and let $f : I \to \mathbb{R}^n$ be a fundamental system of solutions of the differential equation (1). Let the matrix-valued functions $X_a : I \to \mathbb{R}^{n\times n}$ be defined by (7). Then, for $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$ with $\max(k_1, \dots, k_n) \le m + n - 1$,*

$$W_f^k = W_f \left| B_{\ell_1}\big(X_a, \dots, X_a^{(\ell_1-1)}\big) e_{n+\ell_1-k_1} \quad \dots \quad B_{\ell_n}\big(X_a, \dots, X_a^{(\ell_n-1)}\big) e_{n+\ell_n-k_n} \right|, \tag{10}$$

where, for $i \in \{1, \dots, n\}$, $\ell_i := (k_i - n + 1)^+$.

Proof. Define the matrix valued function $Y_f : I \to \mathbb{R}^{n\times n}$ by (7) and observe that, by Corollary 3, for all $\ell \in \{0, \dots, m+n-1\}$, we have that

$$f^{(\ell)} = Y_f^{(i)} e_{n+i-\ell} = Y_f B_i\big(X_a, \dots, X_a^{(i-1)}\big) e_{n+i-\ell} \qquad (i \in \{(\ell - n + 1)^+, \dots, \min(\ell, m)\}).$$

By taking the smallest possible value for i in the above formula, we get

$$f^{(\ell)} = Y_f B_{(\ell-n+1)^+}\big(X_a, \dots, X_a^{((\ell-n+1)^+-1)}\big) e_{n+(\ell-n+1)^+-\ell}.$$

Applying this equality for $\ell \in \{k_1, \dots, k_n\}$, we obtain

$$\begin{pmatrix} f^{(k_1)} & \dots & f^{(k_1)} \end{pmatrix} = Y_f \left(B_{\ell_1}\big(X_a, \dots, X_a^{(\ell_1-1)}\big) e_{n+\ell_1-k_1} \quad \dots \quad B_{\ell_n}\big(X_a, \dots, X_a^{(\ell_n-1)}\big) e_{n+\ell_n-k_n} \right).$$

Now taking the determinant side by side and using the product rule for determinants, the equality (10) follows. □

In the subsequent corollary, we consider the case when $\ell_i = 0$ for $i \in \{2, \dots, n\}$. In this particular setting, the determinant on the left hand side of (10) can easily be computed.

Corollary 5. *Let $n, m \in \mathbb{N}$, let $a : I \to \mathbb{R}^n$ be an $(m-1)$-times continuously differentiable function and let $f : I \to \mathbb{R}^n$ be a fundamental system of solutions of the differential equation (1). Let the matrix-valued functions $X_a : I \to \mathbb{R}^{n\times n}$ be defined by (7) and let $d \in \{0, ..., m-1\}$ and $j \in \{0, \dots, n-1\}$. Then*

$$W_f^{(n+d,n-1,\dots,j+1,j-1,\dots,0)} = (-1)^{n-j-1} W_f \langle B_{d+1}\big(X_a, \dots, X_a^{(d)}\big) e_1, e_{n-j} \rangle. \tag{11}$$

If $d = 0$ and $j = n-1$, then this equality reduces to the Abel–Liouville identity (2). More generally, for $d = 0, 1, 2$, we get the following formulas:

$$\begin{aligned} W_f^{(n,n-1,\dots,j+1,j-1,\dots,0)} &= (-1)^{n-j-1} W_f a_{n-j}, \\ W_f^{(n+1,n-1,\dots,j+1,j-1,\dots,0)} &= (-1)^{n-j-1} W_f (a_1 a_{n-j} + a_{n-j+1} + a_{n-j}'), \\ W_f^{(n+2,n-1,\dots,j+1,j-1,\dots,0)} &= (-1)^{n-j-1} W_f (a_1^2 a_{n-j} + a_1 a_{n-j+1} + a_2 a_{n-j} + a_{n-j+2} \\ &\qquad + a_1 a_{n-j}' + 2a_1' a_{n-j} + 2a_{n-j+1}' + a_{n-j}''). \end{aligned} \tag{12}$$

(Here we define $a_{n+1} := a_{n+2} := 0$.)

Proof. We apply the previous theorem for $k := (n+d, n-1, \dots, j+1, j-1, \dots, 0)$, where $d \in \{0, ..., m-1\}$ and $j \in \{0, \dots, n-1\}$. Then we get that $\ell_1 = d+1$, and $\ell_i = 0$ for $i \in \{2, \dots, n\}$. Therefore,

$$\begin{aligned} W_f^{(n+d,n-1,\dots,j+1,j-1,\dots,0)} &= W_f \left| B_{d+1}\big(X_a, \dots, X_a^{(d)}\big)e_1 \quad \mathbb{I}_n e_1 \quad \dots \quad \mathbb{I}_n e_{n-j-1} \quad \mathbb{I}_n e_{n-j+1} \quad \dots \quad \mathbb{I}_n e_n \right| \\ &= (-1)^{n-j-1} W_f \langle B_{d+1}\big(X_a, \dots, X_a^{(d)}\big)e_1, e_{n-j} \rangle. \end{aligned}$$

Thus, equality (11) has been shown. In the case $d = 0$, we have that

$$\langle B_1(X_a)e_1, e_{n-j} \rangle = \langle X_a e_1, e_{n-j} \rangle = a_{n-j}$$

because the $(n-j)$th entry of X_a equals a_{n-j}. This implies the first equality in (12) for $j \in \{0, \dots, n-1\}$. In particular, for $j = n-1$, this equality is equivalent to the Abel–Liouville identity (2).

In the case $d = 1$, a simple computation gives that

$$\langle B_2(X_a, X_a')e_1, e_{n-j} \rangle = \langle (X_a^2 + X_a')e_1, e_{n-j} \rangle = a_1 a_{n-j} + a_{n-j+1} + a_{n-j}',$$

which yields the second equality in (12) for $j \in \{0, \dots, n-1\}$.

In the case $d = 2$, a somewhat more difficult computation gives that

$$\begin{aligned} \langle B_3(X_a, X_a', X_a'')e_1, e_{n-j} \rangle &= \langle (X_a^3 + 2X_a X_a' + X_a' X_a + X_a'')e_1, e_{n-j} \rangle \\ &= a_1^2 a_{n-j} + a_1 a_{n-j+1} + a_2 a_{n-j} + a_{n-j+2} \\ &\qquad + a_1 a_{n-j}' + 2a_1' a_{n-j} + 2a_{n-j+1}' + a_{n-j}'', \end{aligned}$$

which then yields the third equality in (12). □

For the sake of convenience and brevity, we introduce the following notation: for an n-times continuously differentiable function $f : I \to \mathbb{R}^n$ such that W_f is nonvanishing and $j \in \{0, \dots, n-1\}$, the function $\Phi_f^{[j]} : I \to \mathbb{R}$ is defined by

$$\Phi_f^{[j]} := (-1)^{n-j-1} \frac{W_f^{(n,\dots,j+1,j-1,\dots,0)}}{W_f}.$$

For instance, if f is n-times continuously differentiable function whose components form a fundamental system of solutions for (1), then the Abel–Liouville identity (2) can be rewritten as

$$\Phi_f^{[n-1]} = a_1.$$

More generally, the first equality in (12) gives that

$$\Phi_f^{[j]} = a_{n-j} \qquad (j \in \{0, \dots, n-1\})$$

or, equivalently,

$$a_j = \Phi_f^{[n-j]} \qquad (j \in \{1, \dots, n\}). \tag{13}$$

Lemma 6. *Let $f : I \to \mathbb{R}^n$ be an n-times continuously differentiable function such that W_f is nonvanishing. Then the components of f form a fundamental system of solutions of the nth-order homogeneous linear differential equation*

$$y^{(n)} = \sum_{j=0}^{n-1} \Phi_f^{[j]} y^{(j)}. \tag{14}$$

Proof. This equation is equivalent to the following identity

$$\left| f^{(n-1)} \quad \dots \quad f^{(0)} \right| y^{(n)} = \sum_{j=0}^{n-1} (-1)^{n-j-1} \left| f^{(n)} \quad \dots \quad f^{(j+1)} \quad f^{(j-1)} \quad \dots \quad f^{(0)} \right| y^{(j)}.$$

We can now rearrange this equation to obtain

$$\begin{vmatrix} y^{(n)} & y^{(n-1)} & \dots & y \\ f_1^{(n)} & f_1^{(n-1)} & \dots & f_1 \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(n)} & f_n^{(n-1)} & \dots & f_n \end{vmatrix} = 0.$$

It is easily seen that if $y \in \{f_1, \dots, f_n\}$, then the determinant vanishes. Therefore, $f_1, \dots, f_n$ are solutions of (14). Due to the condition that W_f is nonvanishing, the components of f are linearly independent, therefore they form a fundamental solution system for (14). □

Corollary 7. *Let $n, m \in \mathbb{N}$ with $m \geq n$ and let $f : I \to \mathbb{R}^n$ be an m-times continuously differentiable function such that W_f is nonvanishing. Define $a : I \to \mathbb{R}^n$ by* (13) *and $X_a : I \to \mathbb{R}^{n\times n}$ by* (7)*. Then the equality* (10) *holds for $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, if $k_i \leq m$ and $\ell_i := (k_i - n + 1)^+$ for $i \in \{1, \dots, n\}$.*

Proof. It follows from the definition of a, that it is $(m - n)$-times continuously differentiable. On the other hand, by Lemma 6, we have that f satisfies the n-th order homogeneous linear differential equation (1). Thus, the statement is a consequence of Theorem 4. □

We say that two continuous functions $f, g : I \to \mathbb{R}^n$ are *range equivalent*, denoted by $f \sim g$, if there exists a nonsingular $n \times n$-matrix A such that

$$f = Ag. \tag{15}$$

Theorem 8. *Let $f, g : I \to \mathbb{R}^n$ be an n-times continuously differentiable functions such that W_f and W_g are nonvanishing. Then $f \sim g$ holds if and only if*

$$\Phi_f^{[j]} = \Phi_g^{[j]} \qquad (j \in \{0, \dots, n-1\}). \tag{16}$$

Proof. If $f \sim g$, then there exists a nonsingular $n \times n$-matrix A such that $f = Ag$. The product rule for determinants shows that $W_f^k = |A| W_g^k$ for every $k \in \mathbb{N}_0^n$. Using this identity and the definition of $\Phi_f^{[j]}$ and $\Phi_g^{[j]}$, we obtain the equalities in (16).

On the other hand, if the identities (16) are valid on I, then the nth-order homogeneous linear differential equation (14) is equivalent to the following one

$$y^{(n)} = \sum_{j=0}^{n-1} \Phi_g^{[j]} y^{(j)}.$$

Therefore, the (n-dimensional) solution spaces of these differential equations are identical, which in view of Lemma 6 yields that the components of f are linear combinations of the components of g. Thus identity (15) holds for some nonsingular $n \times n$-matrix A. □

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