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SOME RESULTS ON *I*-BALLS, RADICALS, SEQUENCES AND TOPOLOGY IN *BL*-ALGEBRAS

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Abstract. In this paper, we obtain some new results on radicals of an ideal in *BL*-algebras. Further, we introduce *I*-balls in *BL*-algebras and prove that *I*-balls constitutes a basis for a topology on *BL*-algebras. We also derive some new relations of sequences in *BL*-algebras.

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1. INTRODUCTION

In 1958, C. C. Chang [2] devised the notion of *MV*-algebra in order to provide an algebraic proof of the completeness theorem of Łukasiewicz axioms for infinite valued propositional logic. In 1998, P. Hájek [7] introduced a very general many-valued logic, called Basic Logic (or BL), with the idea to formalize the many-valued semantics induced by a continuous t-norm on the unit real interval [0, 1]. This Basic Logic turns to be a fragment common to three important many-valued logics: $\aleph_0$-valued Łukasiewicz logic, Gödel logic and Product logic. The Lindenbaum-Tarski algebras for Basic Logic are called *BL*-algebras. Apart from their logical attention, *BL*-algebras have important algebraic properties and they have been hard studied from an algebraic point of view. Some well-known examples of a *BL*-algebra are Łukasiewicz, Gödel and Product structures. These examples are defined by the unit interval [0,1] endowed with the structure induced by a continuous t-norm [7].

In 2013, C. Lele and J. B. Nganou [9], introduced the concept of ideals in *BL*-algebras by generalizing the notion of an ideal in *MV*-algebras and showed that, unlike what is in *MV*-algebras, in *BL*-algebras, the notions of filter and ideal are not dual.

A. Paad in [14], defined the notion of $rad(I)$ on a *BL*-algebra *L*, where $rad(I)$ is the radical of an ideal *I* of *L*. G. Georgescu and A. Popescu [6] defined the sequences in *BL*-algebras. C. Luan and Y. Yang in [10] defined the *I*-balls on *MV*-algebras. Also, the same authors, introduce a topology on *MV*-algebras by filters and studied some properties of the filters in *MV*-algebras.

Proved by Höhle [8], a *BL*-algebra becomes an *MV*-algebra if, we adjoin to the axioms the double negation law, $x = x^{**}$. Thus, a *BL*-algebra is in some intuitive way, a non-double negation *MV*-algebra. Hence the theory of *MV*-algebras, becomes one of the guides to the development of the theory of *BL*-algebras. Therefore, we define the notion of an I-ball on a *BL*-algebra L and derive a topology with respect to I-balls on L. We also obtain some new results of radicals and sequences in *BL*-algebras.

This paper is organized as follows:

In Section 2, we recall some definitions and results related to the *BL*-algebra and operations, which we need for the rest of the paper. In Section 3, we derive some relations on ideals, quotient ideals and $rad(I)$ on *BL*-algebras. We further define open I-balls and prove that open I-balls constitutes a basis for a topology. In Section 4, we obtain some new results on sequences in *BL*-algebras.

2. Preliminaries

In this section, we recall some definitions and properties of *BL*-algebras which will be used throughout of the paper.

Definition 1 ([7]). An algebraic structure $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ is called a *BL*-algebra, if it satisfies the following conditions for all $x, y, z \in L$:

BL_1: $(L, \wedge, \vee, 0, 1)$ is a bounded lattice relative to the order $\leq$;
BL_2: $(L, \odot, 1)$ is a commutative monoid;
BL_3: $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$;
BL_4: $x \wedge y = x \odot (x \rightarrow y)$;
BL_5: $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

By L, we denote the universe of a *BL*-algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ for any $x \in L$ and a natural number n, we define $x^* = x \rightarrow 0$, $x^n = x^{n-1} \odot x$, for $n \geq 1$, $x^0 = 1$. Let $x \in L$, if there is the least integer $n \in \mathbb{N}$ such that $x^n = 0$. We set $ord(x) = n$, if there is no such an integer, we set $ord(x) = \infty$.

A *BL*-algebra L is called *MV*-algebra, if $x^{**} = x$, for all $x \in L$. Also an element $x \in L$ is called an nilpotent element of L, if $x^n = 0$, for some $n \in \mathbb{N}$.

A *BL*-algebra L is linear, if for every $x, y \in L$, $x \leq y$ or $y \leq x$ [3].

An *MV*-algebra L is locally finite iff every element $0 \neq x \in L$ has a finite order, or equivalently, for every $0 \neq x \in L$, $nx = 1$, for some $n \in \mathbb{N}$ [2].

The following properties are well known in *BL*-algebras.

Proposition 1 ([7, 15]). *Let L be a BL-algebra. For all $x, y, z \in L$, and $n \in \mathbb{N}$, the following statements hold:*

(1) $x \odot y \leq z$ *iff* $x \leq y \rightarrow z$;
(2) $x \odot y \leq x \wedge y \leq x, y$;
(3) $x \leq y$ *implies that* $x \odot z \leq y \odot z$;
(4) $x \leq y$ *iff* $x \rightarrow y = 1$;

(5) $1 \to x = x$, $x \to x = 1$, $x \leq y \to x$ *and* $x \to 1 = 1$*;*
(6) $x \odot x^* = 0$ *and* $x \odot 0 = 0$*;*
(7) $x \odot y = 0$ *iff* $x \leq y^*$*;*
(8) $1^* = 0$, $0^* = 1$, $x \leq x^{**}$, $x^* = x^{***}$*;*
(9) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$*;*
(10) $x \leq y$ *implies that* $z \to x \leq z \to y$, $y \to z \leq x \to z$ *and* $y^* \leq x^*$*;*
(11) $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$*;*
(12) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$, $(x \vee y)^{**} = x^{**} \vee y^{**}$, $(x \odot y)^{**} = x^{**} \odot y^{**}$ *and* $(x \to y)^{**} = x^{**} \to y^{**}$;
(13) $\bigvee_{i \in I} (y_i \to x) \leq (\bigwedge_{i \in I} y_i) \to x$, *where* L *is complete and* $\{y_i\}_{i \in I} \subseteq L$*;* $x \leftrightarrow y = (x \to y) \wedge (y \to x)$.

Definition 2 ([4])**.** Let L be a *BL*-algebra. We define the following operations which is known for any $x, y \in L$:

(i) $x \oplus y = (x^* \odot y^*)^*$;
(ii) $x \ominus y = x \odot y^*$.

From [9], for every $x, y \in L$, $x \oslash y = x^* \to y$.

Proposition 2 ([9])**.** *In every BL-algebra L, the following statements hold for any* $x, y, z, t \in L$*:*

(i) The operation $\oslash$ *is associative;*
(ii) $x \leq y$ *and* $z \leq t$ *imply* $x \oslash z \leq y \oslash t$.

Lemma 1 ([15])**.** *Let L be a BL-algebra. For all* $x, y, z \in L$*, if* $x \leq y$*, then the following hold:*

(i) $x \odot z \leq y \odot z$*;*
(ii) $x \oplus z \leq y \oplus z$*;*
(iii) $z \ominus y \leq z \ominus x$*;*
(iv) $z \oslash x \leq z \oslash y$ *and* $y \oslash z \leq x \oslash z$.

Definition 3 ([9])**.** Let L be a *BL*-algebra and I be a nonempty subset of L, then I is an ideal of L if it satisfies the following conditions:

(I_1) For every $x, y \in I$, $x \oslash y \in I$;
(I_2) For every $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$.

It is trivial to see that for any ideal I, $0 \in I$ and for every $x \in L$, $x \in I$ if and only if $x^{**} \in I$ [9]. A proper ideal I is called a maximal ideal of L, if it is not properly contained in any other ideal of L.

From [5, 15], *BL*-algebras are distributive lattices and a distributive lattice $\langle L, \leq, \wedge, \vee \rangle$ in which for every element $x \in L$ there is an associated element $x^* \in L$ such that for every $y \in L$, $(x \wedge x^*) \vee y = y$ and $(x \vee x^*) \wedge y = y$ is called a Boolean algebra. The element x^* is called the lattice complement of x. The set of all complemented elements of the corresponding distributive lattice to the BL-algebra L, is a Boolean algebra and denoted by $B(L)$.

Theorem 1 ([5, 7]). *Let L be a BL-algebra. Then for $x \in L$, the following statements are equivalent:*

(i) $x \in B(L)$;
(ii) $x \odot x = x$ *and* $x^{**} = x$;
(iii) $x \odot x = x, \ \ x^* \to x = x$;
(iv) $x^* \vee x = 1$;
(v) $(x \to y) \to x = 0$, *for any* $y \in L$.

Theorem 2 ([13]). *Let M be a proper ideal of a BL-algebra L. Then the following conditions are equivalent:*

(i) *M is a maximal ideal of L;*
(ii) *For all $x \notin M$, there exists $n \in \mathbb{N}, (x^*)^n \in M$;*
(iii) *$\frac{L}{M}$ is a locally finite MV-algebra.*

Definition 4 ([9]). Let I be a proper ideal of a BL-algebra L. Then the intersection of all maximal ideals of L that contain I is called the radical of I and is denoted by $rad(I)$.

From [14], $rad(I)$ is an ideal of L and $I \subseteq rad(I)$.

Theorem 3 ([14]). *Let L be a BL-algebra and I be a proper ideal of L. Then* $rad(I) = \{x \in L| \ \ (x \to (x^*)^n)^* \in I, \ \ for \ \ all \ \ n \in \mathbb{N}\}$.

Definition 5 ([14]). An element a of a BL-algebra L is called unity if, for all $n \in \mathbb{N}$, $((\mathrm{a}^n)^*)^k = 0$, for some $k \in \mathbb{N}$, i.e., $(\mathrm{a}^n)^*$ is a nilpotent element of L.

Definition 6 ([7]). Let X and Y be two BL-algebras. A map $f : X \longrightarrow Y$ is called a BL-homomorphism if, for all $x, y \in X$:

(i) $f(x \odot y) = f(x) \odot f(y)$;
(ii) $f(x \to y) = f(x) \to f(y)$;
(iii) $f(0_X) = 0_Y$.

If $f : X \longrightarrow Y$ is a BL-homomorphism, then the kernel of f is the set

$$\ker(f) = \{x \in X| \ \ f(x) = 0_Y\}.$$

From [9] the following, as immediate consequent of Definition 6, are hold, for all $x, y \in X$:

(i) $f(x \wedge y) = f(x) \wedge f(y)$;
(ii) $f(x \vee y) = f(x) \vee f(y)$;
(iii) $f(x^*) = (f(x))^*$;
(iv) $f(1_X) = 1_Y$;

(v) If $x \leq y$, then $f(x) \leq f(y)$;
(vi) $f(x \oslash y) = f(x) \oslash f(y)$.

Definition 7 ([3]). In a *BL*-algebra L, the distance function $d : L \times L \longrightarrow L$ is defined by $d(x,y) = (x \to y) \wedge (y \to x)$, for all $x, y \in L$.

Proposition 3 ([3,6]). *Let L be a BL-algebra. Then the following statements hold:*

(i) $d(x,y) = d(y,x)$;
(ii) $d(x,y) = 1$ *if and only if* $x = y$;
(iii) $d(x,1) = x, d(x,0) = x^*$;
(iv) $d(x,z) \odot d(z,y) \leq d(x,y)$;
(v) $d(x,y) \leq d(x \odot u, y \odot u)$;
(vi) $d(x,u) \odot d(y,v) \leq d(y \to x, v \to u)$;
(vii) $d(x,u) \wedge d(y,v) \leq d(x \wedge y, u \wedge v)$;
(viii) $d(x,u) \wedge d(y,v) \leq d(x \vee y, u \vee v)$;
(ix) $d(x,y) \leq d(x^*, y^*)$.

Definition 8 ([3]). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a *BL*-algebra L. If $(x_n)_{n\in\mathbb{N}}$ is increasing, we denote $(x_n)_{n\in\mathbb{N}} \uparrow$. Similarly, if $(x_n)_{n\in\mathbb{N}}$ is decreasing, we denote $(x_n)_{n\in\mathbb{N}} \downarrow$. If $(x_n)_{n\in\mathbb{N}}$ is increasing, $\bigwedge_n x_n$ exists and $\bigwedge_n x_n = x$, we denote $(x_n)_{n\in\mathbb{N}} \uparrow x$. Similarly, if $(x_n)_{n\in\mathbb{N}}$ is decreasing, $\bigvee_n x_n$ exists and $\bigvee_n x_n = x$, we denote $(x_n)_{n\in\mathbb{N}} \downarrow x$.

Definition 9 ([3]). Let L be a *BL*-algebra and $(x_n)_{n\in\mathbb{N}}$ be a sequence in L. Then $(x_n)_{n\in\mathbb{N}}$ converges to $x \in L$, if there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in L such that $(s_n)_{n\in\mathbb{N}} \uparrow 1$ and $d(x_n, x) \geq s_n$ for all $n \in \mathbb{N}$, which we denote by $x_n \to_s x$.

Definition 10 ([3]). A *BL*-algebra L is called special if, $(x \to y)^* = (y \to x)^*$, for all $x, y \in L$.

We denote a special *BL*-algebra L by L^*.

Proposition 4 ([11]). *For a BL-algebra L, the following conditions are equivalent:*

(i) L *is a special BL-algebra;*
(ii) $x^* = 0$, *for any* $0 \neq x \in L$.

Note that if I is an ideal of a special *BL*-algebra L, then $\frac{L}{I}$ is a special *BL*-algebra.

3. Some results on radicals and topology on *BL*-algebras

In this section, we obtain some new results on radical of an ideal of a *BL*-algebra L. We also define the open I-balls on L and prove that these open I-balls constitute a basis for a topology on L.

Lemma 2. *Let L be a BL-algebra. Then for all* $x, y, z \in L$, *the following hold:*

(i) $x \oslash (y \oslash z) = (x \oplus y) \oslash z$;
(ii) $x \ominus (y \oplus z) = (x \ominus y) \ominus z$;

(iii) $x \oslash (y \oslash z) = y \oslash (x \oslash z)$;
(iv) $x \oslash (y \odot z) \leq (x \oslash y) \oslash z$.

Proof. Let $x, y, z \in L$, then by Definition 2 and Proposition 1, we conclude:

(i) $x \oslash (y \oslash z) = x^* \to (y \oslash z) = x^* \to (y^* \to z) = (x^* \odot y^*) \to z = (x^{***} \odot y^{***}) \to z$. By Proposition 1, (12), we have, $(x^{***} \odot y^{***}) \to z = (x^* \odot y^*)^{**} \to z = (x^* \odot y^*)^* \oslash z = (x \oplus y) \oslash z$.

(ii) $x \ominus (y \oplus z) = x \odot (y \oplus z)^* = x \odot (y^* \odot z^*) = (x \odot y^*) \odot z^* = (x \odot y^*) \ominus z = (x \ominus y) \ominus z$.

(iii) $x \oslash (y \oslash z) = x \oslash (y^* \to z) = x^* \to (y^* \to z)$. By Proposition 1, (9), $x^* \to (y^* \to z) = y^* \to (x^* \to z) = y \oslash (x^* \to z) = y \oslash (x \oslash z)$.

(iv) By applying BL_3, we have, $0 = z \odot 0 = z \odot (y \odot y^*) = (y \odot z) \odot y^* \leq z$ if and only if $y \odot z \leq y^* \to z = y \oslash z$, therefore, $x \oslash (y \odot z) \leq x \oslash (y \oslash z) = (x \oslash y) \oslash z$. □

Lemma 3. *Let L be a BL-algebra and $x, y \in L$. Then, the following conditions are equivalent:*

(i) $x^* \oplus y = 1$;
(ii) $x \ominus y = 0$;
(iii) $x \leq y^{**}$.

Proof. $(i) \Rightarrow (ii)$ By Definition 2, $1 = x^* \oplus y = (x^{**} \odot y^*)^* = (x^{**} \odot y^{***})^* = (x \odot y^*)^{***} = (x \odot y^*)^*$. This means that $x \odot y^* = 0$, i.e., $x \ominus y = 0$.
$(ii) \Rightarrow (iii)$ Suppose $x \ominus y = 0$. Thus $x \odot y^* = 0$, i.e., $x \leq y^{**}$.
$(iii) \Rightarrow (i)$ Since $x \leq y^{**}$, so $y^{***} \leq x^*$, i.e., $y^* \leq x^*$. This means that $y^* \odot x = 0$. By Proposition 1, $(y^* \odot x)^* = (y^* \odot x)^{***} = (y^{***} \odot x^{**})^* = (y^* \odot x^{**})^* = 1$, therefore $x^* \oplus y = 1$. □

Corollary 1. *Let L be a BL-algebra and $x, y, z \in L$. If $x \ominus y \leq z$, then $x \leq y \oslash z$.*

Proof. $x \ominus y \leq z$ iff $x \odot y^* \leq z$ iff $x \leq y^* \to z$, which is equivalent to $x \leq y \oslash z$. □

Theorem 4. Let I and K be two ideals of a BL-algebra L such that $K \subseteq I$. Then $\frac{I}{K}$ is a proper ideal of $\frac{L}{K}$ if and only if $rad(\frac{I}{K})$ is a proper ideal of $\frac{L}{K}$.

Proof. Let $\frac{I}{K}$ be a proper ideal of $\frac{L}{K}$ and $rad(\frac{I}{K})$ is not a proper ideal, then $rad(\frac{I}{K}) = \frac{L}{K}$, i.e., $\bigcap_{\frac{M}{K} \in Max(\frac{L}{K})} \frac{M}{K} = \frac{L}{K}$, such that $\frac{I}{K} \subseteq \frac{M}{K}$ and $Max(\frac{L}{K})$ is the set of all maximal ideals of $\frac{L}{K}$, this means that $\frac{M}{K} = \frac{L}{K}$ and $rad(\frac{I}{K}) = \frac{I}{K}$, therefore $\frac{L}{K} = \frac{I}{K}$, which is a contradiction.

Conversely, let $rad(\frac{I}{K})$ be a proper ideal of $\frac{L}{K}$. Then $\frac{y}{K} \notin rad(\frac{I}{K})$, for some $\frac{y}{K} \in \frac{L}{K}$. There exists a maximal ideal $\frac{M}{K}$ such that $\frac{I}{K} \subseteq \frac{M}{K}$ with $\frac{y}{K} \notin \frac{M}{K}$. We suppose that $\frac{I}{K}$ is not proper, then $\frac{I}{K} = \frac{L}{K}$, i.e., for all $\frac{x}{K} \in \frac{L}{K}$. Since $\frac{I}{K} \subseteq \frac{M}{K}$, so $\frac{x}{K} \in \frac{M}{K}$ for all $\frac{x}{K} \in \frac{L}{K}$. Therefore, $\frac{y}{K} \in \frac{M}{K}$, which is a contradiction. □

Theorem 5. *Let I, J and K be proper ideals of a BL-algebra L such that $K \subseteq I$, $K \subseteq J$. Then the following assertions hold:*

(i) If for all $x \in L$, $x^* = 1$, then $rad(I) = L$;
(ii) If $\frac{I}{K} \subseteq \frac{J}{K}$, then $rad\left(\frac{I}{K}\right) \subseteq rad\left(\frac{J}{K}\right)$;
(iii) $rad\left(\frac{I}{K}\right) = \frac{L}{K}$ if and only if $\frac{I}{K} = \frac{L}{K}$;
(iv) $rad(rad\left(\frac{I}{K}\right)) = rad\left(\frac{I}{K}\right)$.

Proof. (i) Let for all $x \in L$, $x^* = 1$, then $(x \to (x^*)^n)^* = (x \to 1)^* = 1^* = 0 \in I$. So by Theorem 3, $x \in rad(I)$. Then $L \subseteq rad(I)$. Thus $rad(I) = L$.

(ii) If $\frac{I}{K} \subseteq \frac{J}{K}$ and $\frac{x}{K} \in rad(\frac{I}{K})$, then by Theorem 3, $(\frac{x}{K} \to (\frac{x}{K}^*)^n)^* \in \frac{I}{K} \subseteq \frac{J}{K}$, for all $n \in \mathbb{N}$, i.e., $\frac{x}{K} \in rad(\frac{J}{K})$. Hence $rad(\frac{I}{K}) \subseteq rad(\frac{J}{K})$.

(iii) Suppose $rad(\frac{I}{K}) = \frac{L}{K}$. Since $\frac{0}{K} \in \frac{L}{K}$, so $\frac{0}{K} \in rad(\frac{I}{K})$ and by Theorem 3, $\frac{0}{K} \to (((\frac{0}{K})^*)^n)^* \in \frac{I}{K}$ for all $n \in \mathbb{N}$. We conclude $(\frac{0}{K} \to \frac{1}{K})^* \in \frac{I}{K}$, i.e., $(\frac{1}{K})^* \in \frac{I}{K}$ and $\frac{0}{K} \in \frac{I}{K}$, therefore, $\frac{I}{K} = \frac{L}{K}$.

Conversely, let $\frac{I}{K} = \frac{L}{K}$. Since $\frac{I}{K} \subseteq \frac{M}{K}$, so we have, $rad(\frac{I}{K}) = \bigcap_{\frac{M}{K} \in Max(\frac{L}{K})} \frac{M}{K} = \frac{L}{K}$.

(iv) By (ii) and the fact $\frac{I}{K} \subseteq rad(\frac{I}{K})$, we conclude, $rad(\frac{I}{K}) \subseteq rad(rad(\frac{I}{K}))$.

Conversely, let $\frac{x}{K} \in rad(rad(\frac{I}{K}))$, then $\frac{x}{K} \in \frac{M}{K}$ for any $\frac{M}{K} \in Max(\frac{L}{K})$ with $rad(\frac{I}{K}) \subseteq \frac{M}{K}$. Now, let $\frac{N}{K}$ be a arbitrary maximal ideal of $\frac{L}{K}$ such that $\frac{I}{K} \subseteq \frac{N}{K}$. Then by (ii), $rad(\frac{I}{K}) \subseteq rad(\frac{N}{K}) = \frac{N}{K}$, so $\frac{x}{K} \in \frac{N}{K}$ and $\frac{x}{K} \in rad(\frac{I}{K})$. Thus $rad(rad(\frac{I}{K})) \subseteq rad(\frac{I}{K})$ and hence $rad(rad(\frac{I}{K})) = rad(\frac{I}{K})$. □

Theorem 6. *Let I and J be two ideals of a BL-algebra L. If $a \in I$ and $a \leq b$, for some $b \in J$, then $rad(I) \subseteq rad(J)$.*

Proof. Let $a \leq b$ for some $b \in J$. Since J is an ideal, so $a \in J$, i.e., $I \subseteq J$. Thus by Theorem 5, $rad(I) \subseteq rad(J)$. □

Theorem 7. *Let I, J and K be ideals of a BL-algebra L such that $K \subseteq I, J$. Then $rad(\frac{I \bigcap J}{K}) = rad(\frac{I}{K}) \bigcap rad(\frac{J}{K})$.*

Proof. We know that $\frac{I}{K} \subseteq rad(\frac{I}{K})$ and $\frac{J}{K} \subseteq rad(\frac{J}{K})$. Then $\frac{I \bigcap J}{K} \subseteq \frac{I}{K} \subseteq rad(\frac{I}{K})$. So by Theorem 5, $rad(\frac{I \bigcap J}{K}) \subseteq rad(rad(\frac{I}{K})) = rad(\frac{I}{K})$. Similarly, $rad(\frac{I \bigcap J}{K}) \subseteq rad(\frac{J}{K})$. Thus $rad(\frac{I \bigcap J}{K}) \subseteq rad(\frac{J}{K}) \bigcap rad(\frac{J}{K})$.

Conversely, let $\frac{x}{K} \in rad(\frac{I}{K}) \bigcap rad(\frac{J}{K})$, then $\frac{x}{K} \in rad(\frac{I}{K})$ and $\frac{x}{K} \in rad(\frac{J}{K})$. By Theorem 3, $(\frac{x}{K} \to (\frac{x}{K}^*)^n)^* \in \frac{I}{K}$ and $(\frac{x}{K} \to (\frac{x}{K}^*)^n)^* \in \frac{J}{K}$ for all $n \in \mathbb{N}$. Therefore $(\frac{x}{K} \to (\frac{x}{K}^*)^n)^* \in \frac{I}{K} \bigcap \frac{J}{K} = \frac{I \bigcap J}{K}$, for all $n \in \mathbb{N}$, i.e., $\frac{x}{K} \in rad(\frac{I \bigcap J}{K})$ and hence $rad(\frac{I}{K}) \bigcap rad(\frac{J}{K}) \subseteq rad(\frac{I \bigcap J}{K})$. □

Theorem 8. *Let I and K be two ideals of a BL-algebra L such that $K \subseteq I$. Then $rad(\frac{I}{K}) \bigcap B(\frac{L}{K}) \subseteq \frac{I}{K}$, where $B(\frac{L}{K})$ is the Boolean center of BL-algebra $\frac{L}{K}$.*

Proof. Let $\frac{x}{K} \in rad(\frac{I}{K}) \bigcap B(\frac{L}{K})$, then $\frac{x}{K} \in rad(\frac{I}{K})$ and $\frac{x}{K} \in B(\frac{L}{K})$. By Theorem 1 and Theorem 3, we conclude that $(\frac{x}{K} \to (\frac{x^*}{K})^n)^* \in \frac{I}{K}$, for all $n \in \mathbb{N}$, $\frac{x}{K} \odot \frac{x}{K} = \frac{x}{K}$ and

$\frac{x^{**}}{K} = \frac{x}{K}$. This means that $(\frac{x}{K} \to (\frac{x^*}{K})^n)^* \in \frac{I}{K}$, for all $n \in \mathbb{N}$, i.e., $\frac{x^n}{K} = \frac{x}{K}$ and $\frac{x^{**}}{K} = \frac{x}{K}$. Therefore $(\frac{x}{K} \to \frac{x^*}{K})^* \in \frac{I}{K}$. Since $\frac{x^*}{K} \to \frac{x}{K} = \frac{x}{K}$, so $\frac{x^{**}}{K} \to \frac{x^*}{K} = \frac{x^*}{K}$. We have, $\frac{x^{**}}{K} = \frac{x}{K}$, i.e., $\frac{x}{K} \to \frac{x^*}{K} = \frac{x^*}{K}$, therefore $(\frac{x^*}{K})^* \in I$ and hence $\frac{x}{K} \in I$. □

From Theorem 8, we conclude the following results:

Corollary 2. $rad\{1\} \bigcap B(L) = \{1\}$.

Corollary 3. $rad\{0\} \subseteq \{x \in L |\ x \text{ is a nilpotent element}\}$.

Proof. Let $x \in rad\{0\}$. Then by Theorem 3, $(x \to (x^*)^n)^* = 0$, for all $n \in \mathbb{N}$, i.e., $x \to (x^*)^n = 1$, and $x \to (x^n)^* = 1$, thus by Proposition 1, (4), $x \leq (x^n)^*$ and hence $x \odot x^n = 0$. This means that $x^{n+1} = 0$ and x is a nilpotent element. □

Theorem 9. *Let I be an ideal of a BL-algebra L. Then the following assertions hold:*

(i) $D(I) \subseteq rad(I)$, where $D(I) = \{x \in I |\ x^* = 1\}$;

(ii) $rad\left(\frac{I}{D(I)}\right) = \frac{rad(I)}{D(I)}$.

Proof. (i) Let $x \in D(I)$. Then $x^* = 1$ and $(x \to (x^*)^n)^* = (x \to 1)^* = 1^* = 0 \in I$, i.e., $x \in rad(I)$.

(ii) We know that $rad\left(\frac{I}{D(I)}\right) = \bigcap \frac{N}{D(I)}$. Since N is a maximal ideal and $D(I) \subseteq I \subseteq N$, so $rad\left(\frac{I}{D(I)}\right) = \frac{\bigcap N}{D(I)} = \frac{rad(I)}{D(I)}$. □

Theorem 10. *Let I be an ideal of a BL-algebra L. If $D(L) \subseteq I$, then $\frac{L}{I}$ is an MV-algebra.*

Proof. Suppose $D(L) \subseteq I$ and $\frac{L}{I}$ is not an MV-algebra. Then there exists $x \in L$ such that $\left(\frac{x}{I}\right)^{**} \neq \frac{x}{I}$. We have $\frac{x}{I} \leq \left(\frac{x}{I}\right)^{**}$ and $\left(\frac{x}{I}\right)^{**} \nleq \frac{x}{I}$. Thus $\left(\frac{x}{I}\right)^{**} \to \frac{x}{I} \neq 1$, i.e., $x^{**} \to x \notin I$ and $x^{**} \to x \notin D(I)$. Therefore $(x^{**} \to x)^* \neq 1$, i.e., $x^{**} \to x \neq 0$ which is a contradiction. □

Theorem 11. *Let L be a linear BL-algebra and I be a proper ideal of BL-algebra L. If x is a unity element of L, then $x^* < x$.*

Proof. Let x be a unity element in L and $x < x^*$, then $(x^n)^* = 0$, for all $n \in \mathbb{N}$. We have $x \leq x^*$, i.e., $x^2 = x \odot x = 0$. Therefore, $(x^2)^* = 1$, which is a contradiction with $(x^n)^* = 0$. Since L is a linear BL-algebra and $x \nleq x^*$, thus $x^* < x$. □

Lemma 4. *Let X and Y be two BL-algebras and $f : X \longrightarrow Y$ be a BL-homomorphism. Then $f(d(x,y)) = d(f(x), f(y))$, for all $x, y \in X$.*

Proof. By Definitions 6 and 7, we have:

$$\begin{aligned} f(d(x,y)) &= f((x \to y) \wedge (y \to x)) \\ &= f(x \to y) \wedge f(y \to x) \end{aligned}$$

$$= (f(x) \to f(y)) \wedge (f(y) \to f(x))$$
$$= d(f(x), f(y)).$$

□

Theorem 12. *Let X and Y be two BL-algebras, and $f : X \longrightarrow Y$ be a BL-homomorphism. If K and J are two ideals of Y such that $K \subseteq J$, then the following assertions hold:*

(i) $rad(f^{-1}(\frac{J}{K})) = f^{-1}(rad(\frac{J}{K}))$;

(ii) $rad(\ker(f)) = f^{-1}(rad\{1\})$.

Proof. (i) Let $\frac{x}{K} \in rad(f^{-1}(\frac{J}{K}))$, then by Theorem 3, $(\frac{x}{K} \to ((\frac{x}{K})^*)^n)^* \in f^{-1}(\frac{J}{K})$, for all $n \in \mathbb{N}$. So $f(\frac{x}{K} \to ((\frac{x}{K})^*)^n)^* \in \frac{J}{K}$ and by Definition 6, $f((\frac{x}{K}) \to ((\frac{x}{K})^*)^n)^* = (\frac{f(x)}{K} \to (\frac{f(x^*)}{K})^n)^* \in \frac{J}{K}$. Therefore $f(\frac{x}{K}) \in rad(\frac{J}{K})$ and hence $\frac{x}{K} \in f^{-1}(rad(\frac{J}{K}))$. Conversely, it is clear by the similar way.

(ii) Let $x \in rad(\ker(f))$, then by Theorem 3, $(x \to (x^*)^n)^* \in \ker(f)$, for all $n \in \mathbb{N}$. So by Definition 6, $(f(x) \to f(x^*))^n)^* = 0$, for all $n \in \mathbb{N}$, i.e., $(f(x) \to f(x^*))^n = 1$, for all $n \in \mathbb{N}$. This means that $f(x) \in rad\{1\}$ and $x \in f^{-1}(rad\{1\})$. Conversely, it is clear by the same way. □

Definition 11. Let L be a BL-algebra and I be an ideal of L. By $U^*_{x_0,r}$, we define the open I-ball of radius $r \in I$, with center x_0 (around x_0), by

$$U^*_{x_0,r} = \{x \in L| \ \ (r \oslash d(x, x_0))^* \in I\}.$$

Proposition 5. *Let L be a BL-algebra and I be an ideal of L. Then the following assertions hold, for all $x, y \in L$ and $r, s \in I$:*

(i) $U^*_{1,0} = \{x \in L| \ \ x^* \in I\}$;

(ii) If $y \in U^*_{x,r}$ *then* $y^* \in U^*_{x^*,r}$;

(iii) If $s \leq r$ *then* $U^*_{x,s} \subseteq U^*_{x,r}$.

Proof. (i) By Definition 11,

$$U^*_{1,0} = \{x \in L| \ \ (0 \oslash d(x,1))^* \in I\} = \{x \in L| \ \ (0^* \to d(x,1))^* \in I\}$$
$$= \{x \in L| \ \ (1 \to x)^* \in I\} = \{x \in L| \ \ x^* \in I\}.$$

(ii) Suppose $y \in U^*_{x,r}$, then $(r \oslash d(x,y))^* \in I$. By Proposition 3, since $d(x,y) \leq d(x^*, y^*)$, so $r \oslash d(x,y) \leq r \oslash d(x^*, y^*)$. Therefore $(r \oslash d(x^*, y^*))^* \leq (r \oslash d(x,y))^*$. I is an ideal, thus $(r \oslash d(x^*, y^*))^* \in I$. Therefore $y^* \in U^*_{x^*,r}$.

(iii) Suppose $y \in U^*_{x,s}$, then $(s \oslash d(x,y))^* \in I$. By assumption $s \leq r$, then $s \oslash d(x,y) \leq r \oslash d(x,y)$. Thus $(r \oslash d(x,y))^* \leq (s \oslash d(x,y))^*$. Since I is an ideal, so $(r \oslash d(x,y))^* \in I$. Therefore $y \in U^*_{x,r}$. □

From [12], we recall that if X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis element) such that the following hold:

(i) For each $x \in X$, there is at least one basis element B containing x.
(ii) If x belongs to the intersection of two basis element B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

Proposition 6. *Let L be a BL-algebra and I be an ideal of L. Then the open I-balls constitute a basis for a topology on L (we call this topology, ideal topology).*

Proof. Let $x \in L$ and $r \in I$. By Propositions 1 and 3, $0 = 1^* = (r^* \to 1)^* = (r^* \to d(x,x))^* = (r \oslash d(x,x))^*$. Since I is an ideal and $0 \in I$, so $(r \oslash d(x,x))^* \in I$, i.e., $x \in U^*_{x,r}$. Thus there exists an element of the I-balls of topology, which is contains x, for all $x \in L$. Now, let $t \in U^*_{x,r} \cap U^*_{y,s}$, then $(r \oslash d(x,t))^* \in I$ and $(s \oslash d(t,y))^* \in I$. This means that there exist $c^*, d^* \in I$, such that $c = r \oslash d(x,t)$, $d = s \oslash d(t,y)$. We put $e = c^* \vee d^*$ and claim that $U^*_{t,e} \subseteq U^*_{x,r} \cap U^*_{y,s}$. Let $z \in U^*_{t,e}$, then $(e \oslash d(z,t))^* = k^* \in I$, for some $k^* \in I$. By Proposition 3, since $d(z,t) \odot d(t,x) \leq d(z,x)$, so $r \oslash (d(z,t) \odot d(t,x)) \leq r \oslash d(z,x)$. Also, by Lemma 2, $(r \oslash d(x,t))^* \oslash d(z,t)) \leq r \oslash (d(x,t) \odot d(z,t))$. Since $c^* \leq c^* \vee d^*$, we conclude $(c^* \vee d^*) \oslash d(z,t) \leq c^* \oslash d(z,t)$, hence $e \oslash d(z,t) \leq r \oslash d(z,t)$. Therefore $(r \oslash d(z,t))^* \leq (e \oslash d(z,t))^*$. Since I is an ideal, so $(r \oslash d(z,x))^* \in I$ and it follows that $z \in U^*_{x,r}$. By the similar way, we conclude that, if $z \in U^*_{t,e}$, then $z \in U^*_{y,s}$. Therefore, $t \in U^*_{x,r} \cap U^*_{y,s}$. □

Proposition 7. *Every ideal topology on a BL-algebra of L, makes L into a topological BL-algebra.*

Proof. By [1, 16], it is enough to show that the operations $\odot$ and $*$ are continuous. First, we consider the mapping $\odot : L \times L \longrightarrow L$ by $(x,y) \longmapsto x \odot y$. For $e \in I$, let U be an open I-ball of radius e around $t \odot s$ and V be an open I-ball of radius e around t. Then, $V = \{x \in L |\ (e \oslash d(x,t))^* \in I\}$ and $U = \{x \in L |\ (e \oslash d(x,t \odot s))^* \in I\}$. Take $x \in V$ and we assume that $e \oslash d(x,t) = e_x \in I$. Let W be an open I-ball of radius e_x around s, then $V \times W$ is an open neighborhood around (t,s). So, by Proposition 3 and Lemma 2, we have

$$\begin{aligned}(e \oslash d(x \odot y, t \odot s))^* &\leq (e \oslash (d(x,t) \odot d(y,s)))^* \\ &\leq ((e \oslash d(x,t)) \oslash d(y,s))^* \\ &= (e_x \oslash d(y,s))^* \in I.\end{aligned}$$

Since I is an ideal of L, so $(e \oslash d(x \odot y, t \odot s))^* \in I$ and hence $\odot(V \times W) \subseteq U$.

Now, we prove that the mapping $* : L \to L, x \mapsto x^*$ is continuous. Let $e \in L$, $t \in I$ and U be an open I-ball of radius e around t^* and V be an open I-ball of radius e around t. By Propositions 2 and 3, since $d(x,t) \leq d(x^*,t^*)$, so $e \oslash d(x,t) \leq e \oslash d(x^*,t^*) \in I$. Then $(e \oslash d(x^*,t^*))^* \leq (e \oslash d(x,t))^* \in I$. Since I is an ideal of L, so $(e \oslash d(x^*,t^*))^* \in I$ and $x^* \in U$. Therefore $V^* \subseteq U$ and hence the mapping $*$ is continuous. □

4. SEQUENCES IN *BL*-ALGEBRAS

In this section we derive some new results on sequences in *BL*-algebras.

Theorem 13. *Let I be an ideal of a BL-algebra L and $(\frac{x_n}{I})_{n\in\mathbb{N}}$, $(\frac{y_n}{I})_{n\in\mathbb{N}}$ be two sequences in $\frac{L}{I}$ such that $(\frac{x_n}{I})_{n\in\mathbb{N}} \uparrow 1_I$ and $(\frac{y_n}{I})_{n\in\mathbb{N}} \uparrow 1_I$, then $(\frac{x_n\odot y_n}{I})_{n\in\mathbb{N}} \uparrow 1_I$.*

Proof. Let $(\frac{x_n}{I})_{n\in\mathbb{N}} \uparrow 1_I$, $(\frac{y_n}{I})_{n\in\mathbb{N}} \uparrow 1_I$ and $\frac{t}{I} \in \frac{L}{I}$ such that for each $n\in\mathbb{N}$, $(\frac{x_n\odot y_n}{I}) \leq \frac{t}{I}$. We show that $\frac{t}{I} = 1_I$. Since $(\frac{y_n}{I})_{n\in\mathbb{N}} \uparrow 1_I$, so by Definition 8, there exists $m\in\mathbb{N}$ such that $\bigvee_n(\frac{y_n}{I}) = 1_I$, for all $n\in\mathbb{N}$ with $n\geq m$. By the assumption $\frac{x_n}{I}\odot\frac{y_n}{I} = \frac{x_n\odot y_n}{I} \leq \frac{t}{I}$, we have $\frac{y_n}{I} \leq \frac{x_n}{I} \to \frac{t}{I}$. So $\frac{y_n}{I} \leq \bigvee_{n\geq m}(\frac{x_n}{I}\to\frac{t}{I})$ and $\bigvee\frac{y_n}{I} \leq \bigvee_{n\geq m}(\frac{x_n}{I}\to\frac{t}{I})$. Therefore $\bigvee_{n\geq m}(\frac{x_n}{I}\to\frac{t}{I}) = 1_I$. By Proposition 1, (13), $\vee(\frac{x_n}{I}\to\frac{t}{I}) \leq (\wedge\frac{x_n}{I})\to\frac{t}{I}$ for $n\geq m$ and $1_I \leq (\frac{x_m}{I}\wedge\frac{x_{m+1}}{I}\wedge\ldots)\to\frac{t}{I}$. So $1_I \leq \frac{x_m}{I}\to\frac{t}{I}$ and $\frac{x_m}{I}\to\frac{t}{I} = 1_I$, i.e., $\frac{x_m}{I}\leq\frac{t}{I}$. This means that $\bigvee_{n\geq m}\frac{x_m}{I} \leq \frac{t}{I}$ and $1_I \leq \frac{t}{I}$, thus $\frac{t}{I} = 1_I$. □

Proposition 8. *Let I be an ideal of a BL-algebra L and $(\frac{x_n}{I})_{n\in\mathbb{N}}$ be a sequence in $\frac{L}{I}$. If $\frac{x_n}{I} \to_s \frac{x_1}{I}$ and $\frac{x_n}{I}\to_s\frac{x_2}{I}$, then $\frac{x_1}{I} = \frac{x_2}{I}$.*

Proof. By the assumption, since $\frac{x_n}{I}\to\frac{x_1}{I}$, $\frac{x_n}{I}\to\frac{x_2}{I}$, so by Definition 9, $(\frac{s_n}{I})_{n\in\mathbb{N}} \uparrow 1_I$, $(\frac{t_n}{I})_{n\in\mathbb{N}}\uparrow 1_I$ with $d(\frac{x_n}{I},\frac{x_1}{I}) \geq \frac{s_n}{I}$, $d(\frac{x_n}{I},\frac{x_2}{I}) \geq \frac{t_n}{I}$. By Proposition 3, $d(\frac{x_1}{I},\frac{x_2}{I}) \geq d(\frac{x_1}{I},\frac{x_n}{I})\odot d(\frac{x_n}{I},\frac{x_2}{I})$, then $d(\frac{x_1}{I},\frac{x_2}{I}) \geq \frac{s_n}{I}\odot\frac{t_n}{I}$. By Theorem 13, $(\frac{s_n}{I}\odot\frac{t_n}{I})_{n\in\mathbb{N}}\uparrow 1_I$, therefore $d(\frac{x_1}{I},\frac{x_2}{I}) = 1_I$ and hence $\frac{x_1}{I} = \frac{x_2}{I}$. □

Proposition 9. *Let L be a BL-algebra and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be two sequences in L such that $x_n \to_s x$, $y_n \to y$. Then $(x_n \leftrightarrow y_n) \to_s (x\leftrightarrow y)$.*

Proof. First we show that if $x_n \to_s x$, $y_n\to_s y$, then, (i) $x_n\wedge y_n \to_s x\wedge y$, (ii) $(x_n\to y_n)\to_s(x\to y)$. Since $x_n\to_s x$ and $y_n\to_s y$, by Definition 9, there exist $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that $(s_n)_{n\in\mathbb{N}}\uparrow 1$, $(t_n)_{n\in\mathbb{N}}\uparrow 1$ and $d(x_n,x)\geq s_n$, $d(y_n,y)\geq t_n$. By Proposition 3, $d(x_n\wedge y_n,\ x\wedge y) \geq d(x_n,x)\wedge d(y_n,y) \geq s_n\wedge t_n$. Since $(s_n\wedge t_n)_{n\in\mathbb{N}}\uparrow 1$, so $x_n\wedge y_n\to_s x\wedge y$.

From Proposition 3, $d(x_n\to y_n,\ x\to y) \geq d(x_n,x)\odot d(y_n\to y)\geq s_n\odot t_n$. Since $(s_n\odot t_n)_{n\in\mathbb{N}}\uparrow 1$, so $(x_n\to y_n)\to_s(x\to y)$. By Proposition 1, (14), $d(x_n\leftrightarrow y_n,\ x\leftrightarrow y) = d((x_n\to y_n)\wedge(y_n\to x_n),\ (x\to y)\wedge(y\to x)) \geq d(x_n\to y_n,\ x\to y)\wedge d(y_n\to x_n,\ y\to x) \geq (s_n\odot t_n)\wedge(t_n\odot s_n) = s_n\odot t_n\uparrow 1$, therefore $(x_n\leftrightarrow y_n)\to_s(x\leftrightarrow y)$. □

Theorem 14. *Let X and Y be two BL-algebras and $f: X\longrightarrow Y$ be a BL-homomorphism. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X, then $(f(x_n))_{n\in\mathbb{N}}$ is a sequence in Y such that $f(x_n)\to_s f(x)$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. There exists a sequence $(s_n)_{n\in\mathbb{N}}$ of X such that $(s_n)_{n\in\mathbb{N}}\uparrow 1, d(x_n,x)\geq s_n$. Since $f(1) = 1$, so $(f(s_n))\uparrow f(1) = 1$. We also have $d(x_n,x)\geq s_n$, then by Definition 6, $d(f(x_n),f(x)) = f(d(x_n),d(x)) \geq f(s_n)$, i.e., $f(x_n)\to_s f(x)$. □

Corollary 4. *Let X and Y be two BL-algebras, $f : X \to Y$ be a BL-homomorphism and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be sequences in X, in which $x_n \to_s x$, $y_n \to_s y$, then $f(x_n \leftrightarrow y_n) \to_s f(x \leftrightarrow y)$.*

Proof. By Proposition 9 and Theorem 14, it is clear. □

Proposition 10. *Let I be a proper ideal of special BL-algebra L^*. Then $\frac{L^*}{I}$ is not an MV-algebra.*

Proof. Suppose that there exists a proper ideal I of L^* such that $\frac{L^*}{I}$ is an MV-algebra. Then we have $\frac{x}{I} = \frac{x^{**}}{I}$, for all $x \in L^*$ and $x \to x^{**} \in I$. Since $x \in L^*$, so $x^* = 0$ and $x^{**} = 1$. Thus $x \to 1 \in I$ and $1 \in I$. This means that $I = L^*$, which is a contradiction. □

Theorem 15. *Let I be an ideal of MV-algebra L. Then $\frac{L}{I}$ is a special BL-algebra if and only if I is a maximal ideal of L.*

Proof. We know that $\frac{L}{I}$ is special BL-algebra iff $\frac{x^*}{I} = (\frac{x}{I})^* = \frac{0}{I}$, for all $0 \neq x \in L$. It is equal to $x^* \to 0 \in I$, for all $0 \neq x \in L$ iff $x^{**} = x \in L$, for all $0 \neq x \in L$ which in turn equals to $I = L - \{0\}$. □

Definition 12. Let L be a BL-algebra and I be an ideal of L. Then I is a special ideal if, for all $x, y \in I$, $(x \to y)^* = (y \to x)^*$.

Example 1. Let $L = \{0, a, b, 1\}$. Define "$\odot$" and "$\to$" as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	0	a	1	1
1	0	a	b	1

It is easy to see that L is a BL-algebra and $I = \{0, a\}$ is a special ideal of L.

Theorem 16. *Let I be an ideal of BL-algebra L. Then I is a special ideal of L if and only if $D^*(I) = I$, where $D^*(I) = \{x \in I \mid x^{**} = 0\}$.*

Proof. Since $0 \in D^*(I)$, it is clear that $\varnothing \neq D^*(I) \subseteq I$. Let I be a special ideal of L and $t \in I$, then for every $x, y \in I$, $(x \to y)^* = (y \to x)^*$. We put $x = t$ and $y = 0$, then, by Proposition 1, $(t \to 0)^* = (0 \to t)^*$, i.e., $t^{**} = 1^* = 0$. This means that $t \in D^*(I)$ and hence, $I \subseteq D^*(I)$.

Conversely, let $D^*(I) = I$, and $x, y \in I$, then $x^{**} = y^{**} = 0$. By Proposition 1, we have $(x \to y)^* = (x \to y)^{***} = ((x \to y)^{**})^* = (x^{**} \to y^{**})^* = (0 \to 0)^* = (y^{**} \to x^{**})^* = ((y \to x)^{**})^* = (y \to x)^{***} = (y \to x)^*$. Therefore, I is a special ideal of L. □

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