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# GAME-THEORETIC $p$-LAPLACE OPERATOR INVOLVING THE GRADIENT

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*Abstract.* The game-theoretic $p$-Laplacian operator is a version of classical variational $p$-Laplacian which is in connection with stochastic games called Tug-of-War with noise. The existence of positive singular and Hölder continuous solutions of the game-theoretic $p$-Laplace operator involving the gradient in a small $C^2$ perturbation of the unit ball in $\mathbb{R}^n$ are proved. Finally, a more case problem is introduced.

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## 1. INTRODUCTION

Game theoretic stochastic or deterministic methods have recently emerged as a novel approach to study and to approximate various non-linear Partial Differential Equations (PDEs). The normalized $p$-Laplacian or game-theoretic $p$-Laplacian operator is a version of classical variational $p$-Laplacian which was introduced recently in connection with stochastic games called Tug-of-War with noise and it can be used as a unified framework for interpolation problems in signal processing on graphs, such as image processing and machine learning (see [9]).

Generally, tug-of-war, athletic contest between two teams at opposite ends of a rope, each team trying to drag the other across a center line. In some forms of the game a tape or handkerchief is tied around the center of the rope, and two others are tied six feet (1.8 meters) on either side. Three corresponding lines are marked on the ground. The game ends when one team pulls the other so that the tape on the losers' side crosses the ground mark on the winners' side. The contest is decided by the best two out of three pulls. A rural pastime in England and Scotland, the tug-of-war was an Olympic event from 1900 to 1920, with five men to a side. It has often been an outdoor contest at Scottish Highland Games and at other large social gatherings in the 20th century.

Mathematically, tug-of-war games related to the 1-Laplacian or to the $p$-Laplacian (first introduced by Peres, Schramm, Shefel, and Wilson in [22, 23]) have attracted a lot of attention and are used in many works to study the existence or the regularity of solutions for many PDEs (see [35] and references therein). Many of these games generally are formulated as well-known statistical functionals such as mean, min, or max operators. They are interpreted as a discrete approximation of the underlying PDE, and solving the latter leads to taking a suitable limit of the solution of the discrete game.

Here, we briefly review the notion of tug-of-war game and continuous PDEs problem form [8] (one can see more details in [22, 23]).

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain, $h : \Omega \to \mathbb{R}$ the running payoff function, and $g : \partial\Omega \to \mathbb{R}$ the payoff function. Fix a number $\varepsilon > 0$. The dynamics of the game are as follows: a token is placed at an initial position $x_0 \in \Omega$. At the $k$th stage of the game, Player I and Player II select points $x_k^I$ and $x_k^{II}$, respectively, each belonging to a specified set $B_\varepsilon(x_{k-1}) \subset \Omega$ (where $B_\varepsilon(x_{k-1})$ is the $\varepsilon$-ball centered in $x_{k-1}$). The token is then moved to $x_k$, where $x_k$ can be either $x_k^I$ or $x_k^{II}$ with equal probability. In other words, a fair coin is tossed to determine where the token is placed (i.e., which player won this stage).

After the $k$th stage of the game, if $x_k \in \Omega$ then the game continue to stage $k+1$. Otherwise, if $x_k \in \partial\Omega$, the game ends and Player II pays Player I the amount $g(x_k) + \varepsilon^2 \sum_{j=0}^{k-1} h(x_j)$. Player I attempts to maximize the payoff while Player II attempts to minimize it. If both player are using optimal strategy, according to the Dynamic Programming Principle, the value functions for Player I and Player II for standard $\varepsilon$-turn tug-of-war satisfy the relation:

$$\begin{cases} u^\varepsilon(x) &= \dfrac{1}{2}\left[\sup_{y\in B_\varepsilon(x)} u^\varepsilon(y) + \inf_{y\in B_\varepsilon(x)} u^\varepsilon(y)\right] + \varepsilon^2 h(x) \quad x \in \Omega, \\ u^\varepsilon(x) &= g(x) \quad x \in \partial\Omega, \end{cases}$$

The authors of [23] have shown that if the running payoff function $h$ is of constant sign, the value function $u^\varepsilon$ converges to the unique viscosity solution of the normalized $\infty$-Poisson equation:

$$\begin{cases} \Delta_\infty^N u(x) &= -h(x) \quad x \in \Omega, \\ u(x) &= g(x) \quad x \in \partial\Omega, \end{cases}$$

where $\Delta_\infty^N := \frac{1}{|\nabla u|^2}\Delta_\infty$ is the normalized $\infty$-Laplacian and

$$\Delta_\infty = |\nabla u|^2 \sum_{ij} u_{x_i} u_{x_i x_j} u_{x_j}$$

In another version of tug-of-war game with noise, the game is modified as follows: at point $x_k \in \Omega$, player I and player II play $\varepsilon$-step tug-of-war game with probability $\beta \in [0,1]$ and a random point in ball of radius $\varepsilon$ centered at $x_k$ is chosen with probability $1-\beta$. The value functions of the game satisfy the dynamic programming principle:

$$u^{\varepsilon}(x)=\frac{\beta}{2}\left[\sup_{y\in B_{\varepsilon}(x)}u^{\varepsilon}(y)+\inf_{y\in B_{\varepsilon}(x)}u^{\varepsilon}(y)\right]+\frac{1-\beta}{|B_{\varepsilon}(x)|}\int_{B_{\varepsilon}(x)}u^{\varepsilon}(y)dy+\varepsilon^{2}h(x)$$

with the boundary condition $u^{\varepsilon}(x)=g(x)$ for $x\in\partial\Omega$. A detailed proof for existence and uniqueness of these types of function was shown in [17].

Choosing the probability $\beta=\frac{p-2}{p+n}$, this dynamic programming principle gives a connection to viscosity solutions of the following $p$-Laplace equation [14, 16, 35]:

$$\begin{cases}\Delta_p^N u(x) &= -h(x) \quad x\in\partial\Omega,\\ u(x) &= g(x) \quad x\in\partial\Omega,\end{cases}$$

for $p\geq 2$ with $\Delta_p^N u := \frac{1}{p}|\nabla u|^{2-p}div(|\nabla u|^{p-2}\nabla u)$.

Recently, in [9], the authors proposed an adaptation and generalization of the normalized $p$-Laplacian operator on weighted graphs, they studied the uniqueness and existence of the solution where the Dirichlet problem associated to this operator is considered. Also there is a high interest in adapting classical signal processing tools on graphs and networks such as wavelets or PDEs (see [8] and references there in). The demand for such methods is motivated by existing and potential future applications, such as in machine learning and mathematical image processing. Indeed, any kind of data can be represented by a graph in an abstract form in which the vertices are associated to the data and the edges correspond to relationships within data.

In this paper, we are interested in obtaining positive singular and Hölder continuous solutions of

$$\begin{cases}-\Delta_p^N u(y) = & C|\nabla u(y)|^q \qquad y\in\Omega,\\ u = & 0 \qquad\qquad\quad\; y\in\partial\Omega,\end{cases} \tag{1.1}$$

where $\Omega$ is a small $C^2$ perturbation of the unit ball in $\mathbb{R}^n$ and $C>0$ is a constant.

The equation (1.1) can be rewrite as

$$0=|\nabla u|^2\Delta u+\frac{(p-2)}{2}\nabla u\cdot\nabla|\nabla u|^2+pC|\nabla u|^{q+2} \qquad y\in\Omega, \tag{1.2}$$

with $u=0$ on $y\in\partial\Omega$. We can write this in terms of the components as

$$0=\left(|\nabla u|^2\Delta u+(p-2)\sum_{i,j=1}^{n}u_{y_i}u_{y_j}u_{y_iy_j}\right)+pC|\nabla u|^{q+2}, \quad y\in\Omega. \tag{1.3}$$

Notice that we can re-write the equation as $-\Delta_p u-a(x)\cdot\nabla u=0$ in $\Omega$ with $u=0$ on $\partial\Omega$ where $a(x)=p|\nabla u|^{q+p-4}\nabla u$ and hence if $u$ sufficiently smooth we see that $a(x)$ should be sufficiently smooth so as to apply the maximum principle; hence the only solution should be $u=0$. From this informal argument we expect the only way to obtain a positive solution is for the solution to be somewhat singular. The following example gives an explicit solution on the puncture of the unit ball. Our approach will be to perturb an explicit solution on the ball.

*Example* 1. Let $B_1$ denote the unit ball centered at the origin in $\mathbb{R}^n$.

(1) Let $1 < p < n$, $1 + \frac{(p-1)}{n-1} < q < 2$ and define $w(r) := r^{-\sigma} - 1$ where $\sigma := \frac{2-q}{q-1}$ and

$$C := \frac{(n-1)(q-1)-(p-1)}{p(q-1)\sigma^{q-1}}. \tag{1.4}$$

Then $u$ is a singular weak solution of (1.1) with $\Omega = B_1$. Note the above restriction forces $\sigma > 0$ and the further restriction forces $C > 0$.

(2) Let $q > \max\{2, 1 + \frac{(p-1)}{n-1}\}$ and define $w(r) := 1 - r^{\sigma}$ where $\sigma := \frac{q-2}{q-1}$ and

$$C := \frac{(n-1)(q-1)-(p-1)}{p(q-1)\sigma^{q-1}}. \tag{1.5}$$

Then $w$ is a positive Hölder continuous weak solution of (1.1) with $\Omega = B_1$. Note the restriction forces $\sigma > 0$ and $C > 0$.

PDE's problems are important for their applications in other sciences. The standard mathematical techniques are not adequate to study these problems and they need new techniques. This may be the central development of mathematical ideas in active areas of pure mathematics which have had a decisive interaction with PDE's (see [1, 10–13, 15, 18, 20, 21, 24–34, 36–38] for more relevant problems). It is very remarkable to write that in the classical theory of the $p$-Laplace equation (as well as Laplace equation) several main parts of mathematics such as Calculus of Variations, Partial Differential Equations, Potential Theory, Function Theory are joined. The problem $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is studied for different range of $p$. It is worth to mention that (1.1) is a non variational equation and hence there are various standard tools which are not available anymore. Here, by an idea motivated form [2–7, 19] we study the problem.

Due to do this, we do a change of variables to reduce the problem to one on the unit ball; this is take from [7]. Fix $\psi : \overline{B_1} \to \mathbb{R}^n$ be a smooth map and for $\varepsilon > 0$ define

$$\Omega_\varepsilon := \{x + \varepsilon\psi(x) : x \in B_1\}.$$

This domain will be the small perturbation of the unit ball we work on. There is some small $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ one has that $\Omega_\varepsilon$ is diffeomorphic to the unit ball $B_1$. Let $y = x + \varepsilon\psi(x)$ for $x \in B_1$ and note there is some $\tilde{\psi}$ smooth such that $x = y + \varepsilon\tilde{\psi}(\varepsilon, y)$ for $y \in \Omega_\varepsilon$. Given $u(y)$ defined on $y \in \Omega_\varepsilon$ or $v(x)$ defined on $x \in B_1$ we define the other via $u(y) = v(x)$. So to find a positive singular solution $u(y)$ of (1.1) it is sufficient to find a positive singular solution $v(x)$ of some, to be determined equation, on the unit ball. To compute the equation for $v(x)$ we will use the chain rule, but we mention that the computation becomes quite involved. We know that

$$u_{y_i} = \sum_{k=1}^{n} v_{x_k}\left(\delta_{ki} + \frac{\partial\tilde{\psi}^k}{\partial y_i}\right) = v_{x_i} + \varepsilon\sum_{k=1}^{n} v_{x_k}\frac{\partial\tilde{\psi}^k}{\partial y_i}.$$

Also a computation shows (similar to [2, 3])

$$\sum_{i,j=1}^{n} u_{y_i}u_{y_j}u_{y_iy_j} = \frac{\nabla v \cdot \nabla(|\nabla v|^2)}{2} + g_0(\varepsilon)\sum_{i,j,k=1}^{n}\left\{v_{x_ix_j}v_{x_i}v_{x_j} + v_{x_i}v_{x_j}v_{x_k}\right\},$$

where $|g_0(\varepsilon)| \leq C\varepsilon$ for all $|\varepsilon|$ small. We now make some comments on this simplification. Our approach will be to look for solutions of the form $v(x) = w(x) + \phi(x)$ where $w(x) = w(r)$ is the above explicit singular radial solution. We will end up writing out fixed point argument but all these terms that were simplified will not affect the linearized operator; but will only show up in the nonlinear terms. So the exact nature of the terms is not overly important, and in fact if one checks all the dropped terms, they see they are all of the exact for of the two terms we left. Additionally we have dropped the smooth coefficients, but this won't affect anything either.

By [7] we can write $\Delta_y u(y) = \Delta_x v(x) + E_\varepsilon(v)$ where $E_\varepsilon(v)$ is defined by (1.6). So the equation for $v$ on the unit ball now becomes (after taking into account the prior mentioned simplification)

$$\begin{aligned} 0 &= |\nabla v + \varepsilon A_0 \nabla v|^2(\Delta v + E_\varepsilon(v)) + \frac{p-2}{2}\nabla v \cdot \nabla(|\nabla v|^2) \\ &\quad + g_0(\varepsilon)\sum\left\{v_{x_ix_j}v_{x_i}v_{x_j} + v_{x_i}v_{x_j}v_{x_k}\right\} + pC|\nabla v + \varepsilon A_0 \nabla v|^{q+2} \\ &= (\Delta v)|\nabla v|^2 + \frac{p-2}{2}\nabla v \cdot \nabla(|\nabla v|^2) + pC|\nabla v + \varepsilon A_0 \nabla v|^{q+2} + H_\varepsilon(v) \end{aligned}$$

where

$$\begin{aligned} H_\varepsilon(v) &:= (\Delta v)2\varepsilon(A_0\nabla v)\cdot\nabla v + \varepsilon^2(\Delta v)|A_0\nabla v|^2 + E_\varepsilon(v)|\nabla v|^2 \\ &\quad + E_\varepsilon(v)(2\varepsilon A_0 \nabla v)\cdot \nabla v + E_\varepsilon(v)\varepsilon^2|A_0\nabla v|^2 \\ &\quad + g_0(\varepsilon)\sum\left\{v_{x_ix_j}v_{x_i}v_{x_j} + v_{x_i}v_{x_j}v_{x_k}\right\} \\ E_\varepsilon(v) &:= 2\varepsilon\sum_{i,k} v_{x_ix_k}\partial_{y_i}\tilde{\psi}_k + \varepsilon\sum_{i,k} v_{x_k}\partial_{y_iy_i}\tilde{\psi}_k + \varepsilon^2\sum_{i,j,k} v_{x_jx_k}\partial_{y_i}\tilde{\psi}_j\tilde{\psi}_k. \end{aligned} \tag{1.6}$$

We now hope for small enough $\varepsilon$ we can find a solution of the form $v = w + \phi$. If we rewrite the equation putting all the linear in $\phi$ terms on the left we arrive at

$$\begin{cases} -L(\phi) = \sum_{k=1}^{7} F_k(\phi) + I_\varepsilon(\phi) + H_\varepsilon(w+\phi) & B_1, \\ \phi = 0 & \partial B_1, \end{cases} \tag{1.7}$$

where

$$\begin{aligned} F_1(\phi) &= \Delta w|\nabla\phi|^2, \\ F_2(\phi) &= (\Delta\phi)(2\nabla w\cdot\nabla\phi), \\ F_3(\phi) &= (\Delta\phi)|\nabla\phi|^2, \\ F_4(\phi) &= \frac{p-2}{2}\nabla w\cdot\nabla(|\nabla\phi|^2), \\ F_5(\phi) &= (p-2)\nabla\phi\cdot\nabla(\nabla w\cdot\nabla\phi), \end{aligned}$$

$$F_6(\phi) = \frac{(p-2)}{2}\nabla\phi\cdot\nabla|\nabla\phi|^2$$
$$I_\varepsilon(\phi) = pC|\nabla w+\nabla\phi+\varepsilon A_0(\nabla w+\nabla\phi)|^{q+2} - pC|\nabla w+\nabla\phi|^{q+2},$$
$$F_7(\phi) = pC\left\{|\nabla w+\nabla\phi|^{q+2} - |\nabla w|^{q+2} - (q+2)|\nabla w|^q\nabla w\cdot\nabla\phi\right\}.$$

The linear operator $L$ is given by

$$L(\phi) := |\nabla w|^2(\Delta\phi) + (\Delta w)(2\nabla w\cdot\nabla\phi) + (p-2)\nabla w\cdot\nabla(\nabla w\cdot\nabla\phi)$$
$$+\frac{(p-2)}{2}\nabla\phi\cdot\nabla|\nabla w|^2 + pC(q+2)|\nabla w|^q\nabla w\cdot\nabla\phi.$$

Of crucial importance will be the linear operator $L$ and what functions spaces we work in. Before we consider these issues we want to normalize $L$ by dividing by $|\nabla w|^2$. So instead of considering (1.7) we will consider

$$\begin{cases} -\tilde{L}(\phi) & := \frac{-L(\phi)}{|\nabla w|^2} = \sum_{k=1}^7 \frac{F_k(\phi)}{|\nabla w|^2} + \frac{I_\varepsilon(\phi)}{|\nabla w|^2} + \frac{H_\varepsilon(w+\phi)}{|\nabla w|^2} & B_1, \\ \phi & = 0 & \partial B_1. \end{cases} \tag{1.8}$$

To obtain a solution of this we will apply the Contraction Mapping Principle to the nonlinear mapping $J_\varepsilon(\phi) = \psi$ (for $\phi\in X$ where $X$ is yet to be determined and of course this mapping is not well defined yet)

$$\begin{cases} -\tilde{L}(\psi) & = \sum_{k=1}^7 \frac{F_k(\phi)}{|\nabla w|^2} + \frac{I_\varepsilon(\phi)}{|\nabla w|^2} + \frac{H_\varepsilon(w+\phi)}{|\nabla w|^2} & B_1, \\ \psi & = 0 & \partial B_1. \end{cases} \tag{1.9}$$

The exact form of $\tilde{L}$ will be crucial for us. A computation shows that we can write

$$\tilde{L}(\phi) = \Delta\phi + \gamma\phi_{rr} + \frac{\alpha\phi_r}{r},$$

where $\gamma := p-2$ and

$$\alpha := 2(n-1) - 2(p-1)(\sigma+1) - pC(q+2)\sigma^{q-1} \tag{1.10}$$

where $C$ is given by (1.4).

## 2. Linear theory

We study the linear theory for the problem in two different cases (1) The singular case and (2) The Hölder continuous case as follows:

For the singular case, we first define the function spaces. For $0 < s \le \frac{1}{2}$ define $A_s := \{x\in\mathbb{R}^n : s < |x| < 2s\}$ and for $\sigma\in\mathbb{R}$ and $n < t < \infty$ define the spaces $Y = Y_{t,\sigma}$ and $X = X_{t,\sigma}$ with norms given by

$$\|f\|_Y^t := \sup_{0<s\le\frac{1}{2}} s^{(2+\sigma)t-n}\int_{A_s}|f(x)|^t dx$$

$$\|\phi\|_X^t := \sup_{0<s\le\frac{1}{2}} s^{\sigma t-n}\left\{\int_{A_s}|\phi|^t dx + s^t\int_{A_s}|\nabla\phi|^t dx + s^{2t}\int_{A_s}|D^2\phi|^t dx\right\}$$

where for the space $X$ we impose the boundary condition $\phi = 0$ on $\partial B_1$. We now define the closed subspaces of $X$ and $Y$ respectively $X_1, Y_1$ where we remove the first mode. So to define this properly we need to introduce the spherical harmonics.

Consider the Laplace-Beltrami operator $\Delta_{S^{n-1}} = \Delta_\theta$ on $S^{n-1}$ and the eigenpairs

$$-\Delta_\theta \psi_k(\theta) = \lambda_k \psi_k(\theta), \quad \theta \in S^{n-1},$$

and note that $\lambda_0 = 0, \psi_0 = 1$ (multiplicity 1); $\lambda_1 = n-1$ with multiplicity $n$ and $\lambda_2 = 2n$. Given $\phi \in X, f \in Y$ we write

$$\phi(x) = \sum_{k=0}^{\infty} a_k(r)\psi_k(\theta), \qquad f(x) = \sum_{k=0}^{\infty} b_k(r)\psi_k(\theta),$$

and so we define

$$X_1 := \left\{ \phi \in X : \phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta) \right\}, \qquad \text{note there is no } k=0 \text{ mode}$$

and analogous for $Y$. Note we are abusing notation by not showing the correct multiplicity for modes which have multiplicity greater than one; but this isn't an issue for the procedures we perform. For $\gamma, \alpha \in \mathbb{R}$ we define the operator

$$L(\phi)(x) := L_{\gamma,\alpha}(\phi)(x) := \Delta\phi(x) + \gamma\phi_{rr}(x) + \frac{\alpha}{r}\phi_r(x). \tag{2.1}$$

Note we can write the operator as

$$L_{\gamma,\alpha}(\phi)(x) = \Delta\phi + \gamma \sum_{i,j=1}^{n} \frac{x_i x_j}{|x|^2}\phi_{x_i x_j} + \frac{\alpha}{|x|^2} x \cdot \nabla\phi(x).$$

In this section we will prove various results regarding this operator $L = L_{\gamma,\alpha}$. For explicit values of $\gamma, \alpha$ this operator $L_{\gamma,\alpha}$ will be exactly the operator $\tilde{L}$ from the previous section. In this section the values of $\gamma, \alpha, \sigma$ will satisfy a few constraints but are otherwise arbitrary. These constraints are: Take $1 < p < n$, $1 + \frac{(p-1)}{n-1} < q < 2$ and set

$$\begin{aligned} \gamma &:= p-2, \ \sigma := \tfrac{2-q}{q-1}, \\ \alpha &:= 2(n-1) - 2(p-1)(\sigma+1) - pC(q+2)\sigma^{q-1}, \end{aligned} \tag{2.2}$$

Note that by the above $\alpha$, one can get $n-2-\gamma+\alpha = \frac{p-1}{q-1} - (n-1)(q-1)$ and a computation shows that $n-2-\gamma+\alpha$ changes sign in the interval $1 + \frac{(p-1)}{n-1} < q < 2$. Now, similar to [3] one can prove the following theorem.

**Theorem 1.** *Suppose* $-1 < \gamma < n-2$ *and* $0 < \sigma < -\beta_1^-$. *Then there is some* $C > 0$ *such that for all* $f \in Y_1$ *there is some* $\phi \in X_1$ *such that* $L_{\gamma,\alpha}(\phi) = f$ *in* $B_1 \backslash \{0\}$ *and* $\|\phi\|_X \le C\|f\|_Y$.

For the following lemma we use the exact values of the parameters.

**Lemma 1.** *(Onto estimate for $k=0$ mode) Suppose the parameters satisfy (2.2) and set $\beta := \frac{n-1+\alpha}{1+\gamma}$ (which implies $\beta-\sigma-1<0$). There is some $C_0>0$ such that for all $b_0$ there is some $a_0$ which satisfies*

$$(1+\gamma)a_k''(r)+\frac{(n-1+\alpha)a_k'(r)}{r}-\frac{\lambda_k a_k(r)}{r^2}=b_k(r)\quad 0<r<1 \tag{2.3}$$

*with $a_k(1)=0$, for $k=0$ and $\|a_0\|_X \le C_0\|b_0\|_Y$.*

**Corollary 1.** *Suppose the parameters satisfy (2.2). Then there is some $C>0$ such that for all $f\in Y$ there is some $\phi\in X$ which satisfies $L_{\gamma,\alpha}(\phi)=f$ in $B_1\backslash\{0\}$ with $\phi=0$ on $\partial B_1$. Moreover, one has $\|\phi\|_X\le C\|f\|_Y$.*

For the Hölder continuous case, we examine the needed linear theory to linearize around the radial Hölder continuous solution from Example 1 Case 2 where $w(r)=1-r^\sigma$ where $\sigma := \frac{q-2}{q-1}$. If one takes the same approach as in the singular case, they see we need to examine the operator $L=L_{\gamma,\alpha}$ (defined by (2.1)), where $\gamma := p-2, \sigma := \frac{q-2}{q-1}$ and $\alpha := 2(n-1)+2(p-1)(\sigma-1)-pC(q+2)\sigma^{q-1}$, where $C$ defined in (1.5).

The spaces we work in are the same as before (again we have $n<t<\infty$) except now note the change of sign in front of $\sigma$; define the spaces $Y=Y_{t,\sigma}$ and $X=X_{t,\sigma}$ with norms given by

$$\|f\|_Y^t := \sup_{0<s\le\frac{1}{2}} s^{(2-\sigma)t-n}\int_{A_s}|f(x)|^t dx$$

$$\|\phi\|_X^t := \sup_{0<s\le\frac{1}{2}} s^{-\sigma t-n}\left\{\int_{A_s}|\phi|^t dx+s^t\int_{A_s}|\nabla\phi|^t dx+s^{2t}\int_{A_s}|D^2\phi|^t dx\right\}$$

where for the space $X$ we impose the boundary condition $\phi=0$ on $\partial B_1$.

By a similar argument in [3], we have the following theorem.

**Theorem 2.** *Let $n\ge 2$, $p>1$ and $q>\max\left\{2,1+\frac{p-1}{n-1}\right\}$ and $\sigma,\gamma,\alpha$ be as above. Then there is some $C>0$ such that for all $f\in Y$ there is some $\phi\in X$ which satisfies*

$$L(\phi)=L_{\gamma,\alpha}(\phi)=f \quad \text{in } B_1\backslash\{0\}, \qquad \phi=0 \quad \text{on } \partial B_1, \tag{2.4}$$

*and $\|\phi\|_X\le C\|f\|_Y$.*

## 3. Fixed point theory

In this section we study the fixed point argument to show that in the singular and Hölder continuous cases $J_\varepsilon$ has a fixed point. Due to do, we show that $J_\varepsilon(B_R)\subset B_R$ and $J_\varepsilon(B_R)$ is a contraction on $B_r$, where $B_r$ is the closed ball of radius $r$ centered at the origin in $X$.

For the singular case, we know that $\sigma=\frac{2-q}{q-1}$, $1<q\le 2$ and the norms given by

$$\|f\|_Y^t := \sup_{0<s\le\frac{1}{2}} s^{(2+\sigma)t-n}\int_{A_s}|f(x)|^t dx$$

and

$$\|\phi\|_X^t := \sup_{0<s\le\frac{1}{2}} s^{\sigma t-n}\left\{\int_{A_s}|\phi|^t dx + s^t\int_{A_s}|\nabla\phi|^t dx + s^{2t}\int_{A_s}|D^2\phi|^t dx\right\}.$$

Also a computation shows that

$$\frac{\Delta w}{|\nabla w|^2} = \sigma(\sigma+2-n)r^{\sigma}.$$

Recall we have defined $J_\varepsilon(\phi)=\psi$, where $\psi$ satisfies (1.9). In order to obtain a solution $\phi$ of (1.7) we will show that $J_\varepsilon$ is a contraction on $B_r$ where $B_r$ is the closed ball of radius $r$ centered at the origin in $X$. First of all note that $J_\varepsilon$ is into $X$.

**Theorem 3.** *Assume $\phi \in B_R \subset X$. Then there exists $C$ such that*

*(1)* $\left\|\frac{F_1(\phi)}{|\nabla w|^2}\right\|_Y^t \le C\|\phi\|_X^{2t}$. *(2)* $\left\|\frac{F_2(\phi)}{|\nabla w|^2}\right\|_Y^t \le C\|\phi\|_X^{2t}$.

*(3)* $\left\|\frac{F_3(\phi)}{|\nabla w|^2}\right\|_Y^t \le C\|\phi\|_X^{3t}$. *(4)* $\left\|\frac{F_4(\phi)}{|\nabla w|^2}\right\|_Y^t \le C\|\phi\|_X^{2t}$.

*(5)* $\left\|\frac{F_5(\phi)}{|\nabla w|^2}\right\|_Y^t \le C\|\phi\|_X^{2t}$. *(6)* $\left\|\frac{F_6(\phi)}{|\nabla w|^2}\right\|_Y^t \le C\|\phi\|_X^{3t}$.

*(7)* $\left\|\frac{F_7(\phi)}{|\nabla w|^2}\right\|_Y^t \le C\left(\|\phi\|_X^{2t} + \|\phi\|_X^{(q-p+4)t}\right)$.

*(8)* $\left\|\frac{I_\varepsilon(\phi)}{|\nabla w|^2}\right\|_Y^t \le C\varepsilon^t\left(1 + \|\phi\|_X^{(q-p+4)t}\right)$.

*(9)* $\left\|\frac{H_\varepsilon(w+\phi)}{|\nabla w|^2}\right\|_Y^t \le C\varepsilon^t$ *for all $\phi \in B_R$ with $R \le 1$.*

*Proof.* The proof is straightforward. □

Combining the above results we see that for $0<R<1$ chosen sufficiently small and then $\varepsilon>0$ chosen sufficiently small we have $J_\varepsilon(B_R)\subset B_R$.

**Contraction:** We want to show that for small enough $\varepsilon>0$ that $J_\varepsilon$ is a contraction on $B_R \subset X$ for suitably (small) $R$. Let $J_\varepsilon(\phi)=\psi$ and $J_\varepsilon(\phi_0)=\psi_0$ with $\phi,\phi_0 \in B_r$. Note that

$$\tilde{L}(\psi)-\tilde{L}(\psi_0) = \sum_{k=1}^{7}\frac{F_k(\phi)-F_k(\phi_0)}{|\nabla w|^2} + \frac{I_\varepsilon(\phi)-I_\varepsilon(\phi_0)}{|\nabla w|^2} + \frac{H_\varepsilon(w+\phi)-H_\varepsilon(w+\phi_0)}{|\nabla w|^2} \tag{3.1}$$

**Theorem 4.** *$J_\varepsilon : B_R \to B_R$ is a contraction, where $\varepsilon$ and $R$ are small enough.*

*Proof.* We have to show that for sufficiently small $\varepsilon$ and $R$, $J_\varepsilon : B_R \to B_R$ is a contraction. In other words we need to show there exists a $k_{R,\varepsilon}<1$ such that

$$\|J_t(\phi)-J_t(\phi_0)\|_Y \le k_{R,\varepsilon}\|\phi-\phi_0\|_X.$$

We need to prove there exist $k_{R,\varepsilon}$ such that

$$\begin{aligned} \left\| \frac{F_k(\phi)-F_k(\phi_0)}{|\nabla w|^2} \right\|_Y &\leq k_{R,\varepsilon} \left\|\phi-\phi_0\right\|_X \quad \text{for } k=1,2,\cdots,7 \\ \left\| \frac{I_\varepsilon(\phi)-I_\varepsilon(\phi_0)}{|\nabla w|^2} \right\|_Y &\leq k_{R,\varepsilon} \left\|\phi-\phi_0\right\|_X \quad \text{and} \\ \left\| \frac{H_\varepsilon(w+\phi)-H_\varepsilon(w+\phi_0)}{|\nabla w|^2} \right\|_Y &\leq k_{R,\varepsilon} \left\|\phi-\phi_0\right\|_X . \end{aligned} \tag{3.2}$$

A computation shows that each of the above inequalities are hold. □

*Remark* 1. Notice that $v>0$. Firstly, $v \geq 0$ and since $v(x)=w(x)+\phi(x)$, the pointwise estimates on $\phi$ and it's gradient for $R>0$ small enough implies $v \neq 0$ in $B_1$.

By the same argument in the singular continuous case, for the Hölder continuous case one can show that for $0<R<1$ chosen sufficiently small and then $\varepsilon>0$ chosen sufficiently small we have $J_\varepsilon(B_R) \subset B_R$, and for small enough $\varepsilon>0$, $J_\varepsilon$ is a contraction on $B_R \subset X$ for suitably (small) $R$. Thus the fixed point argument is working for the the Hölder continuous case.

Thus, by combining the fixed point argument and the linear theory, we can state that the main result as follows:

**Theorem 5.** *Assume $n \geq 2$.*

(1) *Suppose $p,q,n,\sigma,C$ are as in Example 1 part 1. Then for sufficiently small $C^2$ perturbations of the unit ball, say $\Omega_\varepsilon$, there exists a positive singular weak solution $u$ of (1.1) (with $\Omega=\Omega_\varepsilon$) which blows up at exactly one point $x_\varepsilon$ (near the origin) and behaves like $u(x) \approx |x-x_\varepsilon|^{-\sigma}$ near $x_\varepsilon$. The proof gives the exact behaviour near $x_\varepsilon$.*

(2) *Suppose $p,q,n,\sigma,C$ are as in Example 1 part 2. Then for sufficiently small $C^2$ perturbations of the unit ball, say $\Omega_\varepsilon$, there exists a positive weak solution $u$ of (1.1) (with $\Omega=\Omega_\varepsilon$) with $u \in C^\infty(\Omega_\varepsilon \backslash \{x_\varepsilon\})$ and with $u \in C^{0,\sigma}(\overline{\Omega_\varepsilon})$. In addition $u$ is not in $C^{0,\sigma+\delta}(\overline{\Omega_\varepsilon})$ for any $\delta>0$.*

## 4. More general case

By a similar argument in the above sections, one can study the existence of positive singular solutions of

$$\begin{cases} -div(|x|^\alpha |\nabla u|^\beta \nabla u)+\varepsilon h(x)\cdot \nabla u = |x|^\gamma |\nabla u|^q + \varepsilon_1 u^\eta + \varepsilon_2 g(x) & in\ \Omega, \\ u=0 & on\ \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega$ is a small $C^2$ perturbation of the unit ball in $\mathbb{R}^n$, $\eta>0$ is a positive constant, and $\varepsilon,\varepsilon_1,\varepsilon_2$ are small enough. Assume there exist $C_0$ such that $|h(x)| \leq C_0|x|^{\sigma+1}$ and $|g_i(x)| \leq C_0|x|^{\sigma+2}$, $i=1,2$. Under suitable conditions on $\alpha,\beta,\gamma$ and $q$, one can prove that if $\Omega$ is a sufficiently small $C^2$ perturbation of the unit ball there exists a singular positive weak solution $u$ of (4.1). For other ranges of $\alpha, \beta, \gamma$ and $q$, one may prove the existence of Hölder continuous positive solution (with optimal regularity) on a $C^2$ perturbation of the unit ball.

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