Miskolc Mathematical Notes
Vol. 23 (2022), No. 2, pp. 889–896
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3625

SOME APPLICATIONS OF FIRST-ORDER DIFFERENTIAL SUBORDINATIONS FOR HOLOMORPHIC FUNCTIONS IN COMPLEX NORMED SPACES

HARI MOHAN SRIVASTAVA AND ABBAS KAREEM WANAS

Received 29 December, 2020

Abstract. In Geometric Function Theory of Complex Analysis, there have been many interesting and fruitful usages of a wide variety of differential subordinations for holomorphic functions in the unit disk $\mathbb{U}$:

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Here, in this article, we derive some properties of the first-order differential subordinations for holomorphic functions which are defined in the unit ball $\mathbb{B}$:

$$\mathbb{B} = \{z : z \in \mathbb{C}^n \quad \text{and} \quad \|z\| < 1\}$$

by using a certain class of admissible functions. We also make use of the theory of biholomorphic functions in our investigation here.

2010 *Mathematics Subject Classification:* 30C45; 30C50

Keywords: holomorphic functions, differential subordinations, admissible functions, normed spaces, biholomorphic functions, unit disk, unit ball

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

We first introduce some of the notation that will be used in this paper. The symbol ν' represents the transpose of the vector ν. We denote by $\mathbb{C}^n$ the space of n complex variables $z = (z_1, z_2, \cdots, z_n)'$ with an arbitrary norm $\|\cdot\|$. The origin $(0, 0, \cdots, 0)$ is denoted by 0.

Let $\mathbb{B}$ given by

$$\mathbb{B} = \{z : z \in \mathbb{C}^n \quad \text{and} \quad \|z\| < 1\}$$

be the unit ball in $\mathbb{C}^n$. and Also let $\mathcal{H}(\mathbb{B})$ denote the class of holomorphic functions $f : \mathbb{B} \longrightarrow \mathbb{C}^n$ which have the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We now review the principle of subordination between holomorphic functions in the normed space $(\mathbb{C}^n, \|\cdot\|)$. Let the functions f and g be members of the class $\mathcal{H}(\mathbb{B})$.

Then the function f is said to be subordinate to g, if there exists a function $\varphi \in \mathcal{H}(\mathbb{B})$ with

$$\varphi(0) = 0 \quad \text{and} \quad \|\varphi(z)\| < 1 \quad (z \in \mathbb{B}),$$

such that

$$f(z) = g\big(\varphi(z)\big).$$

This subordination is denoted by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{B}).$$

Further, if the function g is *biholomorphic* in $\mathbb{B}$, then

$$f \prec g \quad (z \in \mathbb{B}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{B}) \subseteq g(\mathbb{B}).$$

Formally, a biholomorphic function is a function ϕ defined on an open subset U of the n-dimensional complex space $\mathbb{C}^n$ with values in $\mathbb{C}^n$, which is holomorphic and one-to-one, such that its image is an open set V in $\mathbb{C}^n$ and the inverse $\phi^{-1} : V \longrightarrow U$ is also holomorphic (see, for details, [6]).

We next define the classes of starlike and convex functions in the unit ball $\mathbb{B}$ as follows.

Definition 1. A function $f \in \mathcal{H}(\mathbb{B})$ with $f(z) \neq 0$ is said to be starlike in $\mathbb{B}$, if it satisfies the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{B}).$$

Furthermore, a function $f \in \mathcal{H}(\mathbb{B})$ with $f'(z) \neq 0$ is said to be convex in $\mathbb{B}$, if it satisfies the following condition:

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0 \quad (z \in \mathbb{B}).$$

Our next definitions (Definitions 2, 3 and 4 below) make use of biholomorphic functions.

Definition 2 ([6])**.** Let $\psi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{B} \longrightarrow \mathbb{C}$. Also let h be a biholomorphic function in $\mathbb{B}$. If p is holomorphic in $\mathbb{B}$ and satisfies the first-order differential subordination given by

$$\psi\left(p(z), zp'(z); z\right) \prec h(z) \quad (z \in \mathbb{B}), \tag{1.2}$$

then p is called a solution of the differential subordination (1.2). The function q is called a dominant of the solutions of the differential subordination or, more simply, a dominant if $p(z) \prec q(z)$ for all solution p of (1.2). A dominant $\check{q}$ that satisfies $\check{q}(z) \prec q(z)$ for all dominants q of (1.2) is said to be the best dominant.

Definition 3 ([6]). Let $k \geqq 1$ and $\xi \in \partial\mathbb{B}$. Also let q be a biholomorphic function in $\mathbb{B}_\rho$ with $\rho > 1$. Then the subset $Q_k(q,\xi)$ of $\mathbb{C}^n \times \mathbb{C}^n$ is defined by

$$Q_k(q,\xi) = \left\{(u,v) : u = q(\xi) \quad \text{and} \quad k\left\|\left(q'(\xi)\right)^{-1}\right\|^{-1} \leqq \|v\| \leqq k\left\|q'(\xi)\right\|\right\}.$$

And the subset $Q(q)$ of $\mathbb{C}^n \times \mathbb{C}^n$ is defined by

$$Q(q) = \bigcup \{Q_k(q,\xi) : k \geqq 1 \quad (\xi \in \partial\mathbb{B})\}.$$

Definition 4 ([6]). Let $\Omega \subset \mathbb{C}^n$ and q be a biholomorphic function in $\mathbb{B}_\rho$ with $\rho > 1$. The class of admissible functions $\Psi[\Omega,q]$ consists of those functions $\psi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{B} \longrightarrow \mathbb{C}$ that satisfy the conditions $\psi(q(0),0;0) \in \Omega$ and $\psi(u,v;z) \notin \Omega$ when $(u,v) \in Q(q)$ and $z \in \mathbb{B}$.

The following lemma will be used in proving our main results.

Lemma 1 ([6]). *Let $\psi \in \Psi[\Omega,q]$ and suppose that $p \in \mathcal{H}(\mathbb{B})$ with $p(0) = q(0)$. If*

$$\psi\left(p(z), zp'(z); z\right) \in \Omega,$$

then $p \prec q$.

The existing literature in Geometric Function Theory of Complex Analysis contains a considerably large number of interesting investigations dealing with differential subordination problems for analytic functions in the unit disk $\mathbb{U}$ (see, for example, [1–5, 7–18]). In particular, in the recently-published survey-cum-expository review article by Srivastava [7], the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [7, p. 340]). In the present work, we determine certain appropriate classes of admissible functions and investigate some first-order differential subordination properties of holomorphic functions defined in unit ball $\mathbb{B}$. We also make use of the theory of biholomprphic functions in our investigation here.

2. A SET OF MAIN RESULTS

In this section, we first give the following definition.

Definition 5. Let $\Omega \subset \mathbb{C}^n$ and q be a biholomorphic function in $\mathbb{B}_\rho$ with $\rho > 1$. The class of admissible functions $\Phi_{\mathbb{B}}[\Omega,q]$ consists of those functions $\phi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{B} \longrightarrow \mathbb{C}$ that satisfy the conditions $\phi(1,0;0) \in \Omega$ and $\phi(s,t;z) \notin \Omega$ when $(s,t) \in Q(q)$ and $z \in \mathbb{B}$.

Theorem 1. *Let $\phi \in \Phi_{\mathbb{B}}[\Omega,q]$. If $f \in \mathcal{H}(\mathbb{B})$ satisfies the following inclusion relation*:

$$\left\{\phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right) \quad (z \in \mathbb{B})\right\} \subset \Omega, \tag{2.1}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

Proof. For a given $f \in \mathcal{H}(\mathbb{B})$, we define the the function p by

$$p(z) = \frac{zf'(z)}{f(z)}. \tag{2.2}$$

It is obvious that the function p is holomorphic in $\mathbb{B}$ and $p(0) = 1$. By simple calculations using (2.2), we see that

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)} + 1. \tag{2.3}$$

We now define the transformation from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}$ by

$$s(u,v) = u \quad \text{and} \quad t(u,v) = u + \frac{v}{u}.$$

We also set

$$\psi(u,v;z) = \phi(s,t;z) = \phi\left(u, u + \frac{v}{u}; z\right). \tag{2.4}$$

If we use the equations (2.2) and (2.3), it follows from (2.4) that

$$\psi\left(p(z), zp'(z); z\right) = \phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right). \tag{2.5}$$

In view of (2.1) and (2.5), we thus find that

$$\psi\left(p(z), zp'(z); z\right) \in \Omega.$$

Hence, from (2.4), we observe that the admissibility condition for $\phi \in \Phi_{\mathbb{B}}[\Omega, q]$ in Definition 5 is equivalent to the admissibility condition for ψ as given in Definition 4. Thus, clearly, $\psi \in \Psi[\Omega, q]$ and, by applying the Lemma in Section 1, we obtain $p(z) \prec q(z)$ or, equivalently,

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

This completes the proof of Theorem 1. □

We next consider the special situation when there is a biholomorphic mapping h from $\mathbb{B}$ onto Ω. In this case, $\Omega = h(\mathbb{B})$ and the class $\Phi_{\mathbb{B}}[h(\mathbb{B}), q]$ is written as $\Phi_{\mathbb{B}}[h, q]$. The following result is an immediate consequence of Theorem 1.

Theorem 2. *Let* $\phi \in \Phi_{\mathbb{B}}[h, q]$ *and* $f \in \mathcal{H}(\mathbb{B})$. *If*

$$\phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right)$$

is holomolphic in $\mathbb{B}$ *and*

$$\phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right) \prec h(z), \tag{2.6}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

The next result is an extension of Theorem 1 to the case when the behaviour of q on the boundary of $\mathbb{B}$ is not known.

Theorem 3. *Let the function q be biholomorphic on $\mathbb{B}$. Also let $\phi \in \Phi_{\mathbb{B}}[\Omega, q_\eta]$ for some $\eta \in (0,1)$, where $q_\eta(z) = q(\eta z)$. Also let $f \in \mathcal{H}(\mathbb{B})$. If*

$$\phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right)$$

is holomorphic in $\mathbb{B}$ and

$$\phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right) \in \Omega,$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

Proof. Since q is biholomorphic on $\mathbb{B}$, the function q_η is biholomorphic on $\mathbb{B}_\rho$ for some $\rho > 1$. Thus, by applying Theorem 1, we obtain

$$\frac{zf'(z)}{f(z)} \prec q_\eta(z).$$

The result asserted by Theorem 3 is now deduced from the fact that $q_\eta(z) \prec q(z)$. □

The next result gives the best dominant of the differential subordination (2.6).

Theorem 4. *Let h be a biholomorphic function on $\mathbb{B}_\rho$ with $\rho > 1$. Also let $\phi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{B} \longrightarrow \mathbb{C}$. Suppose that the following differential equation:*

$$\phi\left(q(z), q(z) + \frac{zq'(z)}{q(z)}; z\right) = h(z) \tag{2.7}$$

has a solution q with $q(0) = 1$ and satisfies one of the following conditions:

(1) $\phi \in \Phi_{\mathbb{B}}[h, q]$;
(2) $\phi \in \Phi_{\mathbb{B}}[h, q_\eta]$ *for some* $\eta \in (0,1)$.

If $f \in \mathcal{H}(\mathbb{B})$ and

$$\phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right)$$

is holomorphic in $\mathbb{B}$, then the differential subordination (2.6) implies that

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

Proof. By applying Theorem 1 and Theorem 3, we deduce that q is a dominant of the differential subordination (2.6). Since q satisfies (2.7), it is also a solution of (2.6) and hence q is the best dominant among all dominants of (2.6). □

In the particular case, we introduce a class of admissible functions, which deals with differential subordinations with dominants of the form $q:\mathbb{B}\longrightarrow\mathbb{C}^n$ given by $q(z)=Mz$, where M is a positive constant. In this case, q is a biholomorphic function on $\mathbb{B}$ and $q(\mathbb{B})$ is a ball in $\mathbb{C}^n$. The class $\Phi_{\mathbb{B}}[\Omega,Mz]$, which is denoted simply by $\Phi_{\mathbb{B}}[M]$, can be expressed in the following form.

Definition 6. Let $\Omega\subset\mathbb{C}^n$ and let M be a positive constant. The class $\Phi_{\mathbb{B}}[M]$ of admissible functions consists of those functions $\phi:\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{B}\longrightarrow\mathbb{C}^n$ that satisfy the conditions $\phi(0,0;0)\in\Omega$ and $\phi(s,t;z)\notin\Omega$ when $(s,t)\in Q(M)$ and $z\in\mathbb{B}$, where

$$Q(M)\equiv Q(M,z)=\{(s,t)\in\mathbb{C}^n\times\mathbb{C}^n:\|s\|=M \ \text{ and } \ \|t\|\geqq M\}.$$

Theorem 5. *Let $\phi\in\Phi_{\mathbb{B}}[M]$. If $f\in\mathcal{H}(\mathbb{B})$ satisfies the following condition:*

$$\phi\left(zf'(z),z^2f''(z);z\right)\in\Omega, \tag{2.8}$$

then

$$zf'(z)\prec Mz \quad (M>0).$$

Proof. For given $f\in\mathcal{H}(\mathbb{B})$, we define the the function p by

$$p(z)=zf'(z). \tag{2.9}$$

It is obvious that the function p is holomorphic in $\mathbb{B}$ and $p(0)=1$. After some computations and by using (2.9), we conclude that

$$zp'(z)-p(z)=z^2f''(z). \tag{2.10}$$

We now define the transformation from $\mathbb{C}^n\times\mathbb{C}^n$ to $\mathbb{C}^n$ by

$$s(u,v)=u \quad \text{and} \quad t(u,v)=v-u.$$

We assume that

$$\psi(u,v;z)=\phi(s,t;z)=\phi(u,v-u;z). \tag{2.11}$$

In view of (2.9) and (2.10), we find from (2.11) that

$$\psi\left(p(z),zp'(z);z\right)=\phi\left(zf'(z),z^2f''(z);z\right). \tag{2.12}$$

Hence (2.8) becomes $\psi(p(z),zp'(z);z)\in\Omega$.

We next find from (2.11) that the admissibility condition for $\phi\in\Phi_{\mathbb{B}}[M]$ in Definition 6 is equivalent to the admissibility condition for ψ as given in Definition 4 and, by applying the Lemma in Section 1, we obtain $p(z)\prec Mz$ or, equivalently,

$$zf'(z)\prec Mz \quad (M>0).$$

□

3. Concluding remarks and observations

In Geometric Function Theory of Complex Analysis, there are many interesting and fruitful usages of a wide variety of differential subordinations for holomorphic functions in the unit disk $\mathbb{U}$:

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Here, in this article, we have derived several properties of the first-order differential subordinations for holomorphic functions which are defined in the unit ball $\mathbb{B}$:

$$\mathbb{B} = \{z : z \in \mathbb{C}^n \quad \text{and} \quad \|z\| < 1\}$$

by using a certain class of admissible functions. In our investigation, we have made use of biholomorphic functions as well. We have also indicated how one can deduce a number of special cases and consequences for our main results (Theorem 1 and Theorem 5).

We conclude our investigation by remarking that, in order to motivate further researches on the subject-matter of this article, we have chosen to draw the attention of the interested readers toward a considerably large number of related recent publications on the subjects which we have discussed here.

References

[1] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Differential subordination and superordination of analytic functions defined by the multiplier transformation," *Math. Inequal. Appl.*, vol. 12, pp. 123–139, 2009.

[2] R. M. Ali, V. Ravichandran, and N. Seenivasagan, "Differential subordination and superordination of analytic functions defined by the Dziok-Srivastava operator," *J. Franklin Inst.*, vol. 347, pp. 1762–1781, 2010, doi: 10.1016/j.jfranklin.2010.08.009.

[3] M. K. Aouf and T. M. Seoudy, "Subordination and superordination of a certain integral operator on meromorphic functions," *Comput. Math. Appl.*, vol. 59, pp. 3669–3678, 2010, doi: 10.1016/j.camwa.2010.03.068.

[4] W. G. Atshan and A. K. Wanas, "Differential subordinations of multivalent analytic functions associated with Ruscheweyh derivative," *Analele Univ. Oradea Fasc. Math.*, vol. 20, pp. 27–33, 2013.

[5] N. E. Cho and H. M. Srivastava, "A class of nonlinear integral operators preserving subordination and superordination," *Integral Transforms Spec. Funct.*, vol. 18, pp. 95–107, 2007, doi: 10.1080/10652460601135342.

[6] S. G. Krantz, *Function Theory of Several Complex Variables*. Second edition, The Wadsworth and Brooks/Cole Mathematics Series, Wadsworth and Brooks (Cole Advanced Books and Software), Pacific Grove, California: Reprinted by American Mathematical Society Chelsea Publishing Complany, Providence, Rhode Island, 2001, 1992.

[7] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*. Series on Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York: Springer Verlag, 2000.

[8] H. M. Srivastava, "Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis," *Iran. J. Sci. Technol. Trans. A: Sci.*, vol. 44, pp. 327–344, 2020, doi: 10.1007/s40995-019-00815-0.

[9] H. M. Srivastava, A. Motamednezhad, and E. A. Adegan, "Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator," *Mathematics*, vol. 8, pp. 1–12, 2020, doi: 10.3390/math8020172.

[10] H. M. Srivastava, A. Prajapati, and P. Gochhayat, "Third-order differential subordination and differential superordination results for analytic functions involving the Srivastava-Attiya operator," *Appl. Math. Inform. Sci.*, vol. 12, pp. 469–481, 2018.

[11] H. M. Srivastava, S.Hussain, A. Raziq, and M. Raza, "The Fekete-Szegő functional for a subclass of analytic functions associated with quasi-subordination," *Carpathian J. Math.*, vol. 34, pp. 103–113, 2018.

[12] H. M. Srivastava and A. K. Wanas, "Strong differential sandwich results of λ-pseudo-starlike functions with respect to symmetrical points," *Math. Moravica*, vol. 23, pp. 45–58, 2019, doi: 10.5937/MatMor1902045S.

[13] H. M. Srivastava, D.-G. Yang, and N.-E. Xu, "Subordinations for multivalent analytic functions associated with the Dziok-Srivastava operator," *Integral Transforms Spec. Funct.*, vol. 20, pp. 581–606, 2009, doi: 10.1080/10652460902723655.

[14] H. Tang and E. Deniz, "Third-Order differential subordination results for analytic functions involving the generalized Bessel functions," *Acta Math. Sci. Ser. B Engl. Ed.*, vol. 34, pp. 1707–1719, 2014.

[15] H. Tang, H. M. Srivastava, and G.-T. Deng, "Some families of analytic functions in the upper half-plane and their associated differential subordination and differential superordination properties and problems," *Appl. Math. Inform. Sci.*, vol. 11, pp. 1247–1257, 2017, doi: 10.18576/amis/110502.

[16] H. Tang, H. M. Srivastava, G.-T. Deng, and S.-H.Li, "Second-order differential superordination for analytic functions in the upper half-plane," *J. Nonlinear Sci. Appl.*, vol. 10, pp. 5271–5280, 2017, doi: 10.22436/jnsa.010.10.13.

[17] A. K. Wanas and G. Murugusundaramoorthy, "Differential sandwich results for Wanas operator of analytic functions," *Math. Moravica*, vol. 24, pp. 17–28, 2020, doi: 10.5937/MatMor2001017K.

[18] Q.-H. Xu, H.-G. Xiao, and H. M. Srivastava, "Some applications of differential subordination and the Dziok-Srivastava convolution operator," *Appl. Math. Comput.*, vol. 230, pp. 496–508, 2014, doi: 10.1016/j.amc.2013.12.065.

Authors' addresses

Hari Mohan Srivastava

(**Corresponding author**) Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada and Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China and Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan and Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy

E-mail address: `harimsri@math.uvic.ca`

Abbas Kareem Wanas

Department of Mathematics, College of Science, University of Al-Qadisiyah, Al Diwaniyah, Al-Qadisiyah, Iraq

E-mail address: `abbas.kareem.w@qu.edu.iq`