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SOME APPLICATIONS OF MAIA'S FIXED POINT THEOREM IN THE CASE OF OPERATORS WITH VOLTERRA PROPERTY WITH RESPECT TO A SUBINTERVAL

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Abstract. The aim of this paper is to improve some fixed point results given by I.A. Rus in (*On a fixed point theorem of Maia*, Studia Univ. Babeş-Bolyai Math., 22(1977), 40-42), (*Basic problem for Maia's theorem*, Sem. on Fixed Point Theory, Preprint 3(1981), Babeş-Bolyai Univ. Cluj-Napoca, 112-115) and (*Data dependence of the fixed points in a set with two metrics*, Fixed Point Theory, 8(2007), no. 1, 115-123), by applying, instead of contraction principle, some variants of Maia's fixed point theorem.

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1. Introduction and Preliminaries

By following the paper of I.A. Rus [15], we give some variants of Maia's fixed point theorem and an application of these variants, concerning the existence and uniqueness of fixed points for some integral equations. Our results improve some fixed point results given in [6], [7] and [12].

First, let us recall gradually some notions, notations and results which will be used in the sequel of this paper. We will start with the notion of L-space, the most simple structure that allows us to present the Picard and weakly Picard operators. Notice that we do not need the context of metric space to do this, but we will use this context to recall the following notion and results: the G-contraction, the Fiber Contraction Principle and the Maia's fixed point theorem.

1.1. *L-space*

The notion of L-space was introduced in 1906 by M. Fréchet in [3]. It is an abstract space in which works one of the basic tools in the theory of operatorial equations, especially in the fixed point theory: the sequence of successive approximations method.

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Let X be a nonempty set. Let $s(X) := \big\{\{x_n\}_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\big\}$. Let $c(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

(i) if $x_n = x$, for all $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$.
(ii) if $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$, then for all subsequences $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$, we have that $\{x_{n_i}\}_{i\in\mathbb{N}} \in c(X)$ and $Lim\{x_{n_i}\}_{i\in\mathbb{N}} = x$.

A simple example of an L-space is the pair $(X, \stackrel{d}{\to})$, where X is a nonempty set and $\stackrel{d}{\to}$ is the convergence structure induced by a metric d on X.

In general, an L-space is any nonempty set endowed with a structure implying a notion of convergence for sequences. Other examples of L-spaces are: Hausdorff topological spaces, generalized metric spaces ($d(x,y) \in \mathbb{R}^m_+$ or $d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$), K-metric spaces ($d(x,y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces.

1.2. *Picard operators and weakly Picard operators*

Let $(X, \to)$ be an L-space. An operator $f : X \to X$ is called weakly Picard operator (WPO) if the sequence of successive approximations, $\{f^n(x)\}_{n\in\mathbb{N}}$, converges for all $x \in X$ and its limit (which generally depend on x) is a fixed point of f.

If an operator f is WPO and the fixed point set of f is a singleton, $F_f = \{x^*\}$, then by definition, f is called Picard operator (PO).

For a WPO, $f : X \to X$, we define the operator $f^\infty : X \to X$, by $f^\infty(x) := \lim_{n\to\infty} f^n(x)$. Notice that, $f^\infty(X) = F_f$, i.e., f^∞ is a set retraction of X on F_f.

If X is a nonempty set, then the triple $(X, \to, \leq)$ is an ordered L-space if $(X, \to)$ is an L-space and $\leq$ is a partial order relation on X which is closed with respect to the convergence structure of the L-space.

In the setting of ordered L-spaces, we have some properties concerning $WPOs$ and POs.

Lemma 1 (Abstract Gronwall Lemma.). *Let $(X, \to, \leq)$ be an ordered L-space and $f : X \to X$ be an increasing WPO. Then:*

(i) $\forall\, x \in X,\ x \leq f(x) \Rightarrow x \leq f^\infty(x)$;
(ii) $\forall\, x \in X,\ x \geq f(x) \Rightarrow x \geq f^\infty(x)$.

Lemma 2 (Abstract Comparison Lemma.). *Let $(X, \to, \leq)$ be an ordered L-space and $f, g, h : X \to X$ be such that:*

(1) $f \leq g \leq h$;
(2) *the operators f, g, h are $WPOs$;*
(3) *the operator g is increasing.*

Then $\forall\, x, y, z \in X,\ x \leq y \leq z \Rightarrow f^\infty(x) \leq g^\infty(y) \leq h^\infty(z)$.

Regarding the theory of $WPOs$ and POs see [9], [10], [13], [14], [17], [8], [16] and [1].

1.3. *G-contractions*

Let (X,d) be a metric space and $G \subset X \times X$ be a nonempty subset. An operator $f : X \to X$ is a G-contraction if there exists $l \in]0,1[$ such that,

$$d(f(x), f(y)) \leq l d(x,y), \forall\ (x,y) \in G.$$

Notice that if $G = X \times X$ then the notion of G-contraction is identical with the notion of contraction (or l-contraction, if we want explicitly to mention the contraction constant l), i.e., there exists $l \in]0,1[$ such that, $d(f(x), f(y)) \leq l d(x,y)$, $\forall\ x,y \in X$.

Example 1. Let $a,b,c \in \mathbb{R}$, $a < c < b$. We consider the set of all continuous real-valued functions defined on the interval $[a,b]$, $X := C[a,b]$, endowed with the metric

$$\rho(x,y) := \left(\int_a^b |x(s) - y(s)|^2 ds\right)^{\frac{1}{2}}.$$

For $K, H \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$, let $f : C[a,b] \to C[a,b]$ be defined by

$$f(x)(t) := \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds,\ t \in [a,b].$$

We suppose that there exists $L_H \in C([a,b] \times [a,b])$ such that

$$|H(t,s,u) - H(t,s,v)| \leq L_H(t,s)|u - v|,\ \forall\ t,s \in [a,b],\ \forall\ u,v \in \mathbb{R}.$$

If $G := \{(x,y) \in C[a,b] \times C[a,b] \mid x\big|_{[a,c]} = y\big|_{[a,c]}\}$ and $\int_c^b \int_c^b |L_H(t,s)|^2 dtds < 1$, then f is a G-contraction with respect to ρ.

Indeed, let $(x,y) \in G$. Then, $\rho(f(x), f(y)) = \left(\int_c^b |f(x)(t) - f(y)(t)|^2 dt\right)^{\frac{1}{2}}$ and

$$\begin{aligned}
|f(x)(t) - f(y)(t)| &\leq \int_c^t \big|H(t,s,x(s)) - H(t,s,y(s))\big| ds \\
&\leq \int_c^b L_H(t,s)|x(s) - y(s)| ds \\
&\overset{\substack{\text{Hölder's}\\ \text{inequality}}}{\leq} \left(\int_c^b |L_H(t,s)|^2 ds\right)^{\frac{1}{2}} \left(\int_c^b |x(s) - y(s)|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\rho(f(x), f(y)) &\leq \left(\int_c^b \left(\int_c^b |L_H(t,s)|^2 ds\right) \rho(x,y)^2 dt\right)^{\frac{1}{2}} \\
&= \left(\int_c^b \int_c^b |L_H(t,s)|^2 dtds\right)^{\frac{1}{2}} \rho(x,y),\ \text{for all } (x,y) \in G.
\end{aligned}$$

For other examples of G-contractions see [15], [11] and [16], pp. 282-284.

1.4. *Fiber Contraction Principle*

Another important result (see [13], [17]) concerning $WPOs$ and POs, is the following one.

Lemma 3 (Fiber Contraction Theorem.). *Let $(X,\to)$ be an L-space, (Y,ρ) be a metric space, $g : X \to X$, $h : X \times Y \to Y$ and $f : X \times Y \to X \times Y$, $f(x,y) := (g(x),h(x,y))$. We suppose that:*

(1) *(Y,ρ) is a complete metric space;*
(2) *g is WPO;*
(3) *$h(x,\cdot) : Y \to Y$ is a contraction, $\forall\, x \in X$;*
(4) *$h : X \times Y \to Y$ is continuous.*

Then, f is WPO. Moreover, if g is a PO, then f is a PO.

Comment 1 (Generalized Fiber Contraction Theorem.). Let $(X,\to)$ be an L-space, (X_i,d_i), $i=\overline{1,m}$, $m \geq 1$ be metric spaces. Let, $f_i : X_0 \times \ldots \times X_i \to X_i$, $i=\overline{0,m}$, be some operators. We suppose that:

(1) (X_i,d_i), $i=\overline{1,m}$, are complete metric spaces;
(2) f_0 is a WPO;
(3) $f_i(x_0,\ldots,x_{i-1},\cdot) : X_i \to X_i$, $i=\overline{1,m}$, are l_i-contractions;
(4) f_i, $i=\overline{1,m}$, are continuous.

Then, the operator $f : X_0 \times \ldots \times X_m \to X_0 \times \ldots \times X_m$, defined by, $f(x_0,\ldots,x_m) := (f_0(x_0), f_1(x_0,x_1),\ldots, f_m(x_0,\ldots,x_m))$ is a WPO.

If f_0 is a PO, then f is a PO.

1.5. *Maia's fixed point theorem*

The following result was introduced by M.G. Maia in [4].

Theorem 1. *Let X be a nonempty set, d and ρ be two metrics on X and $V : X \to X$ be an operator. We suppose that:*

(1) *there exists $c>0$ such that, $d(x,y) \leq c\rho(x,y)$, $\forall\, x,y \in X$;*
(2) *(X,d) is a complete metric space;*
(3) *$V : (X,d) \to (X,d)$ is continuous;*
(4) *$V : (X,\rho) \to (X,\rho)$ is a contraction.*

Then:

(i) *$F_V = \{x^*\}$;*
(ii) *$V : (X,d) \to (X,d)$ is PO.*

Maia's Theorem 1 remains true if we replace the condition (1) with the following one:

$(1')$ there exists $c>0$ such that, $d(V(x),V(y)) \leq c\rho(x,y)$, $\forall\, x,y \in X$.

Hence, we obtain the so called Rus' variant of Maia's fixed point theorem. More considerations can be found in [8], [6], [7], [12], [5] and [2].

2. Fixed point theorems for operators with Volterra property with respect to a subinterval

By following [15], we present first some notions and notations.

Let $a,b,c \in \mathbb{R}$, with $a < c < b$.

Let $C[a,b] := \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b]\}$.

On $C[a,b]$ and $C[a,c]$ we consider norms of uniform convergence such as the max-norms, $\|\cdot\|$, or the Bielecki norm, $\|\cdot\|_\tau$.

In $C[a,b] \times C[a,b]$ we consider the subset

$$G := \{(x,y) \in C[a,b] \times C[a,b] \mid x\big|_{[a,c]} = y\big|_{[a,c]}\}$$

and for each $x \in C[a,b]$ we consider the subset

$$X_x := \{y \in C[a,b] \mid y\big|_{[a,c]} = x\}.$$

Definition 1. An operator, $V : C[a,b] \to C[a,b]$, has the Volterra property on the subinterval, $[a,c]$, if the following implication holds,

$$x,y \in C[a,b],\ x\big|_{[a,c]} = y\big|_{[a,c]} \Rightarrow V(x)\big|_{[a,c]} = V(y)\big|_{[a,c]}.$$

Definition 2. An operator, $V : C[a,b] \to C[a,b]$, has the Volterra property if it has the Volterra property on each subinterval, $[a,t]$, for $a < t < b$.

For example, in the terms of the above considerations, let $K,H \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})$ and $V : C[a,b] \to C[a,b]$ be defined by,

$$V(x)(t) := \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds,\ t \in [a,b].$$

This operator V has the Volterra property on the subinterval $[a,c]$, but V has not the Volterra property.

Remark 1. If $V : C[a,b] \to C[a,b]$ is an operator with Volterra property on $[a,c]$, then the operator V induces an operator, V_1, on $C[a,c]$, defined by

$$V_1(x) := V(\tilde{x})\big|_{[a,c]},\ \text{where } \tilde{x} \in C[a,b] \text{ with, } \tilde{x}\big|_{[a,c]} = x.$$

Remark 2. If $V : C[a,b] \to C[a,b]$ has the Volterra property on $[a,c]$ and V is a G-contraction, then the operator

$$V\big|_{X_x} : X_x \to X_{V_1(x)},$$

is a contraction on X_x, for all $x \in C[a,c]$.

If $x^* \in F_{V_1}$, then $V(X_{x^*}) \subset X_{x^*}$.

The first abstract main result of our paper is the following.

Theorem 2. *Let $a,b,c \in \mathbb{R}$, with $a<c<b$. On $C[a,b]$ we consider the max-norm, $\|\cdot\|$ (or the Bielecki norm, $\|\cdot\|_\tau$). On $C[a,c]$ we consider the max-norm, $\|\cdot\|$ and a metric $\rho : C[a,c]\times C[a,c] \to \mathbb{R}_+$.*

Let $V : C[a,b] \to C[a,b]$ be an operator. We suppose that:

(1) *V has the Volterra property on $[a,c]$;*
(2) *V_1 is a contraction with respect to $\|\cdot\|$;*
(3) *$\exists\, M>0$ such that $\|x-y\| \le M\rho(x,y)$, $\forall\, x,y \in C[a,c]$;*
(4) *V is a G-contraction with respect to ρ.*

Then

(i) *$F_V = \{x^*\}$;*
(ii) *V_1 is PO with respect to the uniform convergence and $x^*\big|_{[a,c]} = V_1^\infty(x)$, $\forall\, x \in C[a,c]$;*
(iii) *$x^* = V^\infty(x)$, $\forall\, x \in X_{x^*|_{[a,c]}}$.*

Proof. Let $x_0 \in C[a,c]$. By (1) and (4), V_1 is a contraction with respect to ρ on $C[a,c]$. By following the proof of Maia's Theorem 1, it can be shown that the sequence of successive approximations $\{V_1^n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C[a,c],\rho)$. By (3), we get that $\{V_1^n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C[a,c],\|\cdot\|)$, so it converges uniformly in the Banach space $(C[a,c],\|\cdot\|)$, to an element $x_1^* \in C[a,c]$, i.e., $V_1^\infty(x_0) = x_1^*$. By (2), $x_1^* \in F_{V_1}$ and by (4) we get $F_{V_1} = \{x_1^*\}$. Hence, V_1 is *PO* with respect to the uniform convergence on $(C[a,c],\|\cdot\|)$.

By (1), (4) and the Remark 2, the operator $W := V\big|_{X_{x_1^*}} : X_{x_1^*} \to X_{x_1^*}$, is a contraction with respect to ρ, where $X_{x_1^*} = \{y \in C[a,b] \mid y\big|_{[a,c]} = x_1^*\}$.

Let $y_0 \in X_{x_1^*}$. Since W is a contraction with respect to ρ, it can be shown that $\{W^n(y_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X_{x_1^*},\rho)$, and by (3), we get that $\{W^n(y_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(X_{x_1^*},\|\cdot\|)$, i.e., it converges uniformly to an element $x^* \in X_{x_1^*}$. Since W is a contraction, W is also continuous, so $F_W = \{x^*\}$. Hence, $x^*\big|_{[a,c]} = x_1^*$.

From these we have (i), (ii) and (iii). □

From the above theorem, the following conjecture arises.

Conjecture 1. *In the conditions of Theorem 2, the operator V is PO with respect to the uniform convergence on $(C[a,b],\|\cdot\|)$, i.e., $x^* = V^\infty(x)$, $\forall\, x \in C[a,b]$.*

Also, we have the Rus' variant for Theorem 2.

Theorem 3. *Let $a,b,c \in \mathbb{R}$, with $a<c<b$. On $C[a,b]$ we consider the max-norm, $\|\cdot\|$ (or the Bielecki norm, $\|\cdot\|_\tau$). On $C[a,c]$ we consider the max-norm, $\|\cdot\|$ and a metric $\rho : C[a,c]\times C[a,c] \to \mathbb{R}_+$.*

Let $V : C[a,b] \to C[a,b]$ be an operator. We suppose that:

(1) *V has the Volterra property on $[a,c]$;*

(2) *V_1 is a contraction with respect to $\|\cdot\|$;*
(3) *$\exists\, M > 0$ such that $\|V_1(x) - V_1(y)\| \leq M\rho(x,y)$, $\forall\, x,y \in C[a,c]$;*
(4) *V is a G-contraction with respect to ρ.*

Then

(i) *$F_V = \{x^*\}$;*
(ii) *V_1 is PO with respect to the uniform convergence and $x^*\big|_{[a,c]} = V_1^\infty(x)$, $\forall\, x \in C[a,c]$;*
(iii) *$x^* = V^\infty(x)$, $\forall\, x \in X_{x^*\big|_{[a,c]}}$.*

Proof. Let $x_0 \in C[a,c]$. By (4), $\{V_1^n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to ρ. On the other hand, for $n,p \in \mathbb{N}$, we have:

$$\begin{aligned}
&\rho(V_1^{n+p}(x_0), V_1^n(x_0)) \leq \\
&\qquad \leq \rho(V_1^{n+p}(x_0), V_1^{n+p-1}(x_0)) + \ldots + \rho(V_1^{n+1}(x_0), V_1^n(x_0)) \\
&\qquad \leq l^{n+p-1}\rho(V_1(x_0), x_0) + \ldots + l^n\rho(V_1(x_0), x_0) \\
&\qquad \leq \frac{l^n}{1-l}\rho(V_1(x_0), x_0) \to 0 \text{ as } n,p \to \infty.
\end{aligned}$$

By (3), it follows that $\|V_1^{n+1}(x_0), V_1^{n+p+1}(x_0)\| \to 0$ as $n,p \to \infty$. Hence, $\{V_1^n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|$. We follow the proof of Theorem 2. □

3. Applications

In this section, we present an application for Theorem 2, Fiber Contraction Theorem and Abstract Gronwall Lemma.

Let $a,b,c \in \mathbb{R}$, with $a < c < b$.

For $K,H \in C([a,b]\times[a,b]\times\mathbb{R},\mathbb{R})$, we consider the following functional integral equation,

$$x(t) = \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds,\ t \in [a,b]. \tag{3.1}$$

We are looking for solutions for this equation in $C[a,b]$.

We suppose that there exists $L_K \in C([a,b]\times[a,b])$ such that

$$|K(t,s,u) - K(t,s,v)| \leq L_K(t,s)|u-v|,\ \forall\, t \in [a,b],\ \forall\, s \in [a,c],\ \forall\, u,v \in \mathbb{R}$$

and there exists $L_H \in C([a,b]\times[a,b])$ such that

$$|H(t,s,u) - H(t,s,v)| \leq L_H(t,s)|u-v|,\ \forall\, t,s \in [a,b],\ \forall\, u,v \in \mathbb{R}.$$

In addition, we suppose that $\displaystyle\int_c^b\int_c^b |L_H(t,s)|^2 dtds < 1$.

Let $V : C[a,b] \to C[a,b]$ be the operator defined by,

$$V(x)(t) := \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds,\ \forall\, t \in [a,b].$$

The operator V_1, induced by V on the interval $[a,c]$ is

$$V_1(x)(t) := \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds,\ \forall\, t \in [a,c].$$

On $C[a,b]$ we consider the max-norm, $\|x-y\| := \max_{t\in[a,b]} |x(t)-y(t)|$ and the metric $\rho(x,y) := \left(\int_a^b |x(t)-y(t)|^2 ds\right)^{\frac{1}{2}}$.

It is clear that V has the Volterra property on $[a,c]$ and V_1 is continuous with respect to $\|\cdot\|$.

Let $x,y \in C[a,c]$. We have

$$\begin{aligned}
&|V_1(x)(t) - V_1(y)(t)| \leq \\
&\leq \int_a^c |K(t,s,x(s)) - K(t,s,y(s))|ds + \int_a^t |H(t,s,x(s)) - H(t,s,y(s))|ds \\
&\leq \int_a^c L_K(t,s)|x(s)-y(s)|ds \ + \int_a^t L_H(t,s)|x(s)-y(s)|ds \\
&\overset{\substack{\text{Hölder's}\\ \text{inequality}}}{\leq} \left(\int_a^c |L_K(t,s)|^2 ds\right)^{\frac{1}{2}} \left(\int_a^c |x(s)-y(s)|^2 ds\right)^{\frac{1}{2}} + \\
&\qquad + \left(\int_a^t |L_H(t,s)|^2 ds\right)^{\frac{1}{2}} \left(\int_a^t |x(s)-y(s)|^2 ds\right)^{\frac{1}{2}} \\
&\leq \left(\left(\int_a^c |L_K(t,s)|^2 ds\right)^{\frac{1}{2}} + \left(\int_a^t |L_H(t,s)|^2 ds\right)^{\frac{1}{2}}\right)\rho(x,y).
\end{aligned}$$

By taking the $\max_{t\in[a,c]}$ in the above inequalities, there exists

$$M := \max_{t\in[a,c]} \left(\left(\int_a^c |L_K(t,s)|^2 ds\right)^{\frac{1}{2}} + \left(\int_a^t |L_H(t,s)|^2 ds\right)^{\frac{1}{2}}\right) > 0$$

such that $\|V_1(x) - V_1(y)\| \leq M\rho(x,y)$, for all $x,y \in C[a,c]$.

On the other hand, by Example 1, the operator V is a G-contraction with respect to ρ, where $G := \{(x,y) \in C[a,b]\times C[a,b] \mid x\big|_{[a,c]} = y\big|_{[a,c]}\}$.

By applying Theorem 3, it follows that the equation (3.1) has a unique solution x^* in $C[a,b]$. Moreover, for $t \in [a,c]$, $x^*(t) = \lim_{n\to\infty} x_n(t)$, for each $x_0 \in C[a,c]$, where $\{x_n\}_{n\in\mathbb{N}}$ is defined by,

$$x_{n+1}(t) = \int_a^c K(t,s,x_n(s))ds + \int_a^t H(t,s,x_n(s))ds,$$

and for $t \in [a,b]$, $x^*(t) = \lim_{n\to\infty} y_n(t)$, where $\{y_n\}_{n\in\mathbb{N}}$, is defined by

$y_0 \in C[a,b]$, with $y_0\big|_{[a,c]} = x^*\big|_{[a,c]}$, and

$$y_{n+1}(t) = \int_a^c K(t,s,x^*(s))ds + \int_a^t H(t,s,\ y_n(s))ds.$$

Remark 3. In the case of the operator V, considered above, Conjecture 1 is a theorem. Indeed, let $X := C[a,c]$, $Y := C[c,b]$ and $C[a,b]$ be endowed with max-norm. We take, $g : C[a,c] \to C[a,c]$, defined by $g := V_1$ and $h : C[a,c] \times C[c,b] \to C[c,b]$ be defined by

$$h(x,y)(t) := \int_a^c K(t,s,x(s))ds + \int_a^c H(t,s,x(s))ds + \int_c^t H(t,s,y(s))ds.$$

We remark that g is PO when

$$\max_{t\in[a,c]} \left(\int_a^c L_K(t,s)ds + \int_a^t L_H(t,s)ds \right) < 1 \tag{3.2}$$

and the operator $h(x,\cdot) : C[c,b] \to C[c,b]$ is a contraction, when

$$l := \max_{t\in[c,b]} \int_c^t L_H(t,s)ds < 1 \tag{3.3}$$

By the Fiber Contraction Theorem, in the conditions (3.2) and (3.3), the operator $f : C[a,c] \times C[c,b] \to C[a,c] \times C[c,b]$, defined by $f(x,y) = (g(x),h(x,y))$, for all $(x,y) \in C[a,c] \times C[c,b]$ is PO.

Now, let

$$x_0 \in C[a,c],\ x_{n+1} = g(x_n),\ n \in \mathbb{N},$$

and

$$y_0 \in C[c,b],\ y_{n+1} = h(x_n,y_n),\ n \in \mathbb{N}.$$

Then, $x_n \to x^*\big|_{[a,c]}$ as $n \to \infty$ and $y_n \to x^*\big|_{[c,b]}$ as $n \to \infty$.

We denote,

$$u_n(t) = \begin{cases} x_n(t),\ t \in [a,c], \\ y_n(t),\ t \in [c,b]. \end{cases}$$

Then, $u_n \in C[a,b]$, for $n \in \mathbb{N}^*$, and, $u_{n+1} = V(u_n)$ with $u_n \to x^*$ as $n \to \infty$, i.e., V is a PO.

This result is very important because we can apply for V, the Abstract Gronwall Lemma. So we have:

Theorem 4. *Let us consider the equation* (3.1) *in the following conditions:* (3.2), (3.3) *and $K(t,s,\cdot)$, $H(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ be increasing functions, for all $t,s \in [a,b]$. Let us denote by x^* the unique solution of* (3.1)*. Then the following implications hold:*

(i) $x \in C[a,b]$, $x(t) \le \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds$, $t \in [a,b]$, $\Rightarrow x \le x^*$;

(ii) $x \in C[a,b]$, $x(t) \ge \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds$, $t \in [a,b]$, $\Rightarrow x \ge x^*$.

Also, from the Abstract Comparison Lemma it can be established a comparison result for equation (3.1).

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