Miskolc Mathematical Notes
Vol. 23 (2022), No. 2, pp. 913–927

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3699

EVENTUAL PERIODICITY OF A MAX-TYPE SYSTEM OF DIFFERENCE EQUATIONS OF HIGHER ORDER WITH FOUR VARIABLES

GUANGWANG SU, TAIXIANG SUN, CAIHONG HAN, BIN QIN, AND WEIZHEN QUAN

Received 04 February, 2021

Abstract. The aim of this paper is to investigate eventual periodicity of the following max-type system of difference equations of higher order with four variables

$$\begin{cases} u_n = \max\left\{A, \frac{s_{n-k}}{v_{n-1}}\right\}, \\ v_n = \max\left\{B, \frac{t_{n-k}}{u_{n-1}}\right\}, \\ s_n = \max\left\{C, \frac{u_{n-k}}{t_{n-1}}\right\}, \\ t_n = \max\left\{D, \frac{v_{n-k}}{s_{n-1}}\right\}, \end{cases} \qquad n \in \{0,1,2,\cdots\},$$

where k is a positive integer, $A,B,C,D \in (0,+\infty)$ with $A \le B$ and $C \le D$, and the initial values $u_{-i}, v_{-i}, s_{-i}, t_{-i} \in (0,+\infty)$ for $i \in \{1,2,\cdots,k\}$. We show that:

(1) If $AC < 1$ or $A = B = C = D = 1$, then there exists a solution $\{(u_n, v_n, s_n, t_n)\}_{n=-k}^{+\infty}$ of this system which is not eventually periodic.

(2) If $BD = AC = 1$ with $A \neq C$ or $BD > AC = 1$ or $AC > 1$, then every solution of this system is eventually periodic.

2010 *Mathematics Subject Classification:* 39A10; 39A11.

Keywords: Max-type system; Difference equation; Positive solution; Higher order; Eventual periodicity

1. INTRODUCTION

The aim of this paper is to investigate eventual periodicity of the following max-type system of difference equations of higher order with four variables

$$\begin{cases} u_n = \max\left\{A, \frac{s_{n-k}}{v_{n-1}}\right\}, \\ v_n = \max\left\{B, \frac{t_{n-k}}{u_{n-1}}\right\}, \\ s_n = \max\left\{C, \frac{u_{n-k}}{t_{n-1}}\right\}, \\ t_n = \max\left\{D, \frac{v_{n-k}}{s_{n-1}}\right\}, \end{cases} \qquad n \in \mathbb{N}_0 \equiv \{0,1,2,\cdots\}, \tag{1.1}$$

where $k \in \mathbb{N} \equiv \{1, 2 \cdots\}$, $A, B, C, D \in \mathbb{R}^+ \equiv (0, +\infty)$ satisfying

$$A \leq B \quad \text{and} \quad C \leq D, \tag{1.2}$$

and the initial values $u_{-i}, v_{-i}, s_{-i}, t_{-i} \in (0, +\infty)$ $(i \in \mathbb{Z}(1,k))$, where $\mathbb{Z}(a,b) \equiv \{a, a+1, \cdots, b\}$ for any integer $a < b$.

If $x_n = u_n = v_n$ and $y_n = s_n = t_n$ and $A = B$ and $C = D$, then (1.1) reduces to the following max-type system

$$\begin{cases} x_n = \max\left\{A, \frac{y_{n-k}}{x_{n-1}}\right\}, \\ y_n = \max\left\{C, \frac{x_{n-k}}{y_{n-1}}\right\}, \end{cases} \qquad n \in \mathbb{N}_0. \tag{1.3}$$

In [24], Su et al. investigated the periodicity of (1.3) and showed that if $AC > 1$ or $AC = 1$ and $A \neq C$, then every positive solution of (1.3) is eventually periodic.

Recently, the study of the properties of the max-type difference equations and systems, such as global behavior, eventual periodicity and boundedness, has aroused a great deal of interest (see [1–5, 7–17, 19–21, 23, 26, 28, 30]). For example, Fotiades and Papaschinopoulos [6] investigated the following max-type system of difference equations

$$\begin{cases} x_n = \max\left\{A, \frac{y_{n-1}}{x_{n-2}}\right\}, \\ y_n = \max\left\{B, \frac{x_{n-1}}{y_{n-2}}\right\}, \end{cases} \qquad n \in \mathbb{N}_0, \tag{1.4}$$

and showed that every positive solution of (1.4) is eventually periodic.

In [25], Su et al., inspired by above results of (1.4), investigated the periodicity of positive solutions of the following max-type systems of difference equations

$$\begin{cases} x_n = \max\left\{A_n, \frac{y_{n-1}}{x_{n-2}}\right\}, \\ y_n = \max\left\{B_n, \frac{x_{n-1}}{y_{n-2}}\right\}, \end{cases} \qquad n \in \mathbb{N}_0, \tag{1.5}$$

where $A_n, B_n \in \mathbb{R}^+$ are periodic with period 2 and showed that every positive solution of (1.5) is eventually periodic.

In 2015, Yazlik et al. [31] investigated the periodicity of positive solutions of the following system

$$\begin{cases} x_n &= \max\left\{\frac{1}{x_{n-1}}, \min\left\{1, \frac{A}{y_{n-1}}\right\}\right\}, \\ y_n &= \max\left\{\frac{1}{y_{n-1}}, \min\left\{1, \frac{A}{x_{n-1}}\right\}\right\} \end{cases} \qquad n \in \mathbb{N}_0 \tag{1.6}$$

and the general solution of (1.6) is obtained in an elegant manner.

The above results of (1.6) motivated Sun and Xi [27] in 2016 to investigate the following more general system

$$\begin{cases} x_n &= \max\left\{\frac{1}{x_{n-m}}, \min\left\{1, \frac{A}{y_{n-r}}\right\}\right\}, \\ y_n &= \max\left\{\frac{1}{y_{n-m}}, \min\left\{1, \frac{B}{x_{n-t}}\right\}\right\}, \end{cases} \qquad n \in \mathbb{N}_0, \tag{1.7}$$

where $A,B \in \mathbb{R}^+$, $m,r,t \in \mathbb{N}$ and the initial values $x_{-i}, y_{-i} \in \mathbb{R}^+$ $(i \in \mathbb{Z}(1,d))$ with $d = \max\{m,r,t\}$ and it is shown that every positive solution of (1.7) is eventually periodic with period $2m$.

In 2013, Stević [18] investigated the boundedness character and global attractivity of the following symmetric system

$$\begin{cases} x_n &= \max\left\{B, \frac{y^p_{n-1}}{x^p_{n-2}}\right\}, \\ y_n &= \max\left\{B, \frac{x^p_{n-1}}{y^p_{n-2}}\right\}, \end{cases} \qquad n \in \mathbb{N}_0, \tag{1.8}$$

where $B,p \in \mathbb{R}^+$ and the initial values $x_{-i}, y_{-i} \in \mathbb{R}^+ (i \in \mathbb{Z}(1,2))$.

Also above results of (1.8) motivated Stević [22] to continue studying the behavior of the following system

$$\begin{cases} x_n &= \max\left\{B, \frac{y^p_{n-1}}{z^p_{n-2}}\right\}, \\ y_n &= \max\left\{B, \frac{z^p_{n-1}}{x^p_{n-2}}\right\}, \\ z_n &= \max\left\{B, \frac{x^p_{n-1}}{y^p_{n-2}}\right\}. \end{cases} \qquad n \in \mathbb{N}_0, \tag{1.9}$$

where $B,p \in \mathbb{R}^+$ and the initial values $x_{-i}, y_{-i}, z_{-i} \in \mathbb{R}^+ (i \in \mathbb{Z}(1,2))$, and showed that system (1.9) is permanent when $p \in (0,4)$.

In this paper, we investigate eventual periodicity of (1.1) and obtain the following theorem.

Theorem 1.

(1) *If $AC < 1$, then there exists a solution $(u_n, v_n, s_n, t_n)_{n=-k}^{+\infty}$ of (1.1) such that $u_n = A$ and $s_n = C$ for any $n \geq -k$ and $\lim_{n\longrightarrow\infty} v_n = \lim_{n\longrightarrow\infty} t_n = \infty$*

(2) *If $A = B = C = D = 1$, then there exists a solution $\{(u_n, v_n, s_n, t_n)\}_{n=-k}^{\infty}$ of (1.1) such that $u_n = v_n = s_n = t_n$ and $1 < u_{n+1} < u_n$ for any $n \geq -k$ and $\lim_{n\longrightarrow\infty} u_n = 1$.*

(3) *If $BD = AC = 1$ and $A \neq C$, then every solution of (1.1) is eventually periodic with period $2k$.*

(4) *If $BD > AC = 1$ and $\{(u_n, v_n, s_n, t_n)\}_{n=-k}^{+\infty}$ is a solution of (1.1), then u_n and s_n are eventually periodic with period 2 and v_n and t_n are eventually periodic with period $2k$.*

(5) *If $AC > 1$, then every solution of (1.1) is eventually periodic with period 1.*

2. Proof of Theorem 1

In this section, we study eventual periodicity of positive solutions of system (1.1). Let $u_n = Ax_n, v_n = By_n, s_n = Cp_n, t_n = Dq_n$ for any $n \geq -k$. Then (1.1) reduces to the following system

$$\begin{cases} x_n &= \max\left\{1, \frac{Cp_{n-k}}{ABy_{n-1}}\right\}, \\ y_n &= \max\left\{1, \frac{Dq_{n-k}}{ABx_{n-1}}\right\}, \\ p_n &= \max\left\{1, \frac{Ax_{n-k}}{CDq_{n-1}}\right\}, \\ q_n &= \max\left\{1, \frac{By_{n-k}}{CDp_{n-1}}\right\}, \end{cases} \qquad n \in \mathbb{N}_0, \tag{2.1}$$

where the initial values $x_{-i}, y_{-i}, p_{-i}, q_{-i} \in \mathbb{R}^+$ $(i \in \mathbb{Z}(1,k))$. Let $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ be a positive solution of (2.1). To show Theorem 1, we need the following lemmas and propositions.

Lemma 1.

(1) *For any* $n \in \mathbb{N}_0$,

$$x_n \geq 1, \quad y_n \geq 1, \quad p_n \geq 1, \quad q_n \geq 1. \tag{2.2}$$

(2) *If* $AC \geq 1$, *then for any* $r \in \mathbb{N}$ *and* $n \geq 2rk$,

$$\begin{cases} x_n &= \max\left\{1, \frac{C}{ABy_{n-1}}, \left(\frac{1}{BD}\right)^r \frac{x_{n-2rk}}{y_{n-1}q_{n-k-1}y_{n-2k-1}q_{n-3k-1}\cdots y_{n-2(r-1)k-1}q_{n-(2r-1)k-1}}\right\}, \\ y_n &= \max\left\{1, \frac{D}{ABx_{n-1}}, \left(\frac{1}{AC}\right)^r \frac{y_{n-2rk}}{x_{n-1}p_{n-k-1}x_{n-2k-1}p_{n-3k-1}\cdots x_{n-2(r-1)k-1}p_{n-(2r-1)k-1}}\right\}, \\ p_n &= \max\left\{1, \frac{A}{CDq_{n-1}}, \left(\frac{1}{BD}\right)^r \frac{p_{n-2rk}}{q_{n-1}y_{n-k-1}q_{n-2k-1}y_{n-3k-1}\cdots q_{n-2(r-1)k-1}y_{n-(2r-1)k-1}}\right\}, \\ q_n &= \max\left\{1, \frac{B}{CDp_{n-1}}, \left(\frac{1}{AC}\right)^r \frac{q_{n-2rk}}{p_{n-1}x_{n-k-1}p_{n-2k-1}px_{n-3k-1}\cdots p_{n-2(r-1)k-1}x_{n-(2r-1)k-1}}\right\}. \end{cases} \tag{2.3}$$

Proof.

(1) It follows from (2.1).

(2) Note $BD \geq AC \geq 1$. From this, (2.1) and (2.2) it follows that for any $r \in \mathbb{N}$ and $n \geq 2rk$,

$$\begin{aligned} x_n &= \max\left\{1, \frac{Cp_{n-k}}{ABy_{n-1}}\right\} \\ &= \max\left\{1, \frac{C}{ABy_{n-1}} \max\left\{1, \frac{Ax_{n-2k}}{CDq_{n-k-1}}\right\}\right\} \\ &= \max\left\{1, \frac{C}{ABy_{n-1}}, \frac{x_{n-2k}}{BDy_{n-1}q_{n-k-1}}\right\} \\ &= \max\left\{1, \frac{C}{ABy_{n-1}}, \frac{1}{BDy_{n-1}q_{n-k-1}} \max\left\{1, \frac{C}{ABy_{n-2k-1}}, \frac{x_{n-4k}}{BDy_{n-2k-1}q_{n-3k-1}}\right\}\right\} \\ &= \max\left\{1, \frac{C}{ABy_{n-1}}, \left(\frac{1}{BD}\right)^2 \frac{x_{n-4k}}{y_{n-1}q_{n-k-1}y_{n-2k-1}q_{n-3k-1}}\right\} \\ &\cdots \\ &= \max\left\{1, \frac{C}{ABy_{n-1}}, \left(\frac{1}{BD}\right)^r \frac{x_{n-2rk}}{y_{n-1}q_{n-k-1}y_{n-2k-1}q_{n-3k-1}\cdots y_{n-(2r-2)k-1}q_{n-(2r-1)r-1}}\right\}. \end{aligned}$$

In a similar way, also we can obtain the other three formulas. The proof is complete. □

Proposition 1. *If $AC < 1$, then there exists a solution $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ of (2.1) such that $x_n = p_n = 1$ for any $n \geq -k$ and $\lim_{n\longrightarrow\infty} y_n = \lim_{n\longrightarrow\infty} q_n = \infty$.*

Proof. Let $x_{-i} = p_{-i} = 1$ and $y_{-i} = q_{-i} = \max\{\frac{AB}{D}, \frac{A}{CD}, \frac{C}{AB}, \frac{CD}{B}\} + 1$ for any $i \in \mathbb{Z}(1,k)$. Then by a simple calculation it follows from (1.2) and (2.1) that

$$\begin{cases} x_0 = \max\left\{1, \frac{Cp_{-k}}{ABy_{-1}}\right\} = 1, \\ y_0 = \max\left\{1, \frac{Dq_{-k}}{ABx_{-1}}\right\} = \frac{Dq_{-k}}{AB} > 1, \\ p_0 = \max\left\{1, \frac{Ax_{-k}}{CDq_{-1}}\right\} = 1, \\ q_0 = \max\left\{1, \frac{By_{-k}}{CDp_{-1}}\right\} = \frac{By_{-k}}{CD} > 1. \end{cases}$$

$$\begin{cases} x_1 = \max\left\{1, \frac{Cp_{1-k}}{ABy_0}\right\} = 1, \\ y_1 = \max\left\{1, \frac{Dq_{1-k}}{ABx_0}\right\} = \frac{Dq_{1-k}}{AB} > 1, \\ p_1 = \max\left\{1, \frac{Ax_{1-k}}{CDq_0}\right\} = 1, \\ q_1 = \max\left\{1, \frac{By_{1-k}}{CDp_0}\right\} = \frac{By_{1-k}}{CD} > 1. \end{cases}$$

$$\cdots\cdots$$

$$\begin{cases} x_{k-1} = \max\left\{1, \frac{Cp_{-1}}{ABy_{k-2}}\right\} = 1, \\ y_{k-1} = \max\left\{1, \frac{Dq_{-1}}{ABx_{k-2}}\right\} = \frac{Dq_{-1}}{AB} > 1, \\ p_{k-1} = \max\left\{1, \frac{Ax_{-1}}{CDq_{k-2}}\right\} = 1, \\ q_{k-1} = \max\left\{1, \frac{By_{-1}}{CDp_{k-2}}\right\} = \frac{By_{-1}}{CD} > 1. \end{cases}$$

$$\begin{cases} x_k = \max\left\{1, \frac{Cp_0}{ABy_{k-1}}\right\} = 1, \\ y_k = \max\left\{1, \frac{Dq_0}{ABx_{k-1}}\right\} = \frac{y_{-k}}{AC} > 1, \\ p_k = \max\left\{1, \frac{Ax_0}{CDq_{k-1}}\right\} = 1, \\ q_k = \max\left\{1, \frac{By_0}{CDp_{k-1}}\right\} = \frac{q_{-k}}{AC} > 1. \end{cases}$$

$$\begin{cases} x_{k+1} = \max\left\{1, \frac{Cp_1}{ABy_k}\right\} = 1, \\ y_{k+1} = \max\left\{1, \frac{Dq_1}{ABx_k}\right\} = \frac{By_{1-k}}{CD} > 1, \\ p_{k+1} = \max\left\{1, \frac{Ax_1}{CDq_k}\right\} = 1, \\ q_{k+1} = \max\left\{1, \frac{By_1}{CDp_k}\right\} = \frac{q_{1-k}}{AC} > 1. \end{cases}$$

$$\cdots\cdots$$

$$\begin{cases} x_{2k-1} = \max\left\{1, \frac{Cp_{k-1}}{ABy_{2k-2}}\right\} = 1, \\ y_{2k-1} = \max\left\{1, \frac{Dq_{k-1}}{ABx_{2k-2}}\right\} = \frac{y_{-1}}{AC} > 1, \\ p_{2k-1} = \max\left\{1, \frac{Ax_{k-1}}{CDq_{2k-2}}\right\} = 1, \\ q_{2k-1} = \max\left\{1, \frac{By_{k-1}}{CDp_{2k-2}}\right\} = \frac{q_{-1}}{AC} > 1. \end{cases}$$

By mathematical induction, we can obtain that for any $\lambda \in \mathbb{N}_0$ and any $r \in \mathbb{Z}(0, k-1)$,

$$\begin{cases} x_{2\lambda k+r} = \max\left\{1, \frac{Cp_{2\lambda k-k+r}}{ABy_{2\lambda k+r-1}}\right\} = 1, \\ y_{2\lambda k+r} = \max\left\{1, \frac{Dq_{2\lambda k-k+r}}{ABx_{2\lambda k+r-1}}\right\} = \frac{D}{AB}(\frac{1}{AC})^{\lambda} q_{r-k} > 1, \\ p_{2\lambda k+r} = \max\left\{1, \frac{Ax_{2\lambda k-k+r}}{CDq_{2\lambda k+r-1}}\right\} = 1, \\ q_{2\lambda k+r} = \max\left\{1, \frac{By_{2\lambda k-k+r}}{CDp_{2\lambda k+r-1}}\right\} = \frac{B}{CD}(\frac{1}{AC})^{\lambda} y_{r-k} > 1 \end{cases}$$

and

$$\begin{cases} x_{(2\lambda+1)k+r} = \max\left\{1, \frac{Cp_{2\lambda k+r}}{ABy_{(2\lambda+1)k+r-1}}\right\} = 1, \\ y_{(2\lambda+1)k+r} = \max\left\{1, \frac{Dq_{2\lambda k+r}}{ABx_{(2\lambda+1)k+r-1}}\right\} = (\frac{1}{AC})^{\lambda+1} y_{r-k} > 1, \\ p_{(2\lambda+1)k+r} = \max\left\{1, \frac{Ax_{2\lambda k+r}}{CDq_{(2\lambda+1)k+r-1}}\right\} = 1, \\ q_{(2\lambda+1)k+r} = \max\left\{1, \frac{By_{2\lambda k+r}}{CDq_{(2\lambda+1)k+r-1}}\right\} = (\frac{1}{AC})^{\lambda+1} q_{r-k} > 1. \end{cases}$$

From the above we have $x_n = p_n = 1$ for any $n \geq -k$ and $\lim_{n\longrightarrow\infty} y_n = \lim_{n\longrightarrow\infty} q_n = \infty$. The proof is complete. □

In Example 3.1 of [29], we showed that the following equation

$$x_n = \frac{x_{n-k}}{x_{n-1}} \tag{2.4}$$

has a positive solution z_n $(n \geq -k)$ with $1 < z_{n+1} < z_n$ for any $n \geq -k$ and $\lim_{n\longrightarrow\infty} z_n = 1$. From Example 3.1 of [29], we obtain the following proposition.

Proposition 2. *If $A = B = C = D = 1$ and z_n $(n \geq -k)$ is a positive solution of (2.4) with $1 < z_{n+1} < z_n$ for any $n \geq -k$ and $\lim_{n\longrightarrow\infty} z_n = 1$, then there exists a solution $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{\infty}$ of (2.1) such that $x_n = y_n = p_n = q_n = z_n$ and $1 < x_{n+1} < x_n$ for any $n \geq -k$ and $\lim_{n\longrightarrow\infty} x_n = 1$.*

Now we assume $A = B > AC = 1 > C = D$. From Lemma 1, we see that for any $i \in \mathbb{Z}(0, 2k-1)$ and $n \in \mathbb{N}$,

$$\begin{cases} x_{2nk+i} = \max\left\{1, \frac{x_{2(n-1)k+i}}{y_{2nk+i-1}q_{2nk+i-k-1}}\right\}, \\ y_{2nk+i} = \max\left\{1, \frac{y_{2(n-1)k+i}}{x_{2nk+i-1}p_{2nk+i-k-1}}\right\}, \\ p_{2nk+i} = \max\left\{1, \frac{A}{C^2 q_{2nk+i-1}}, \frac{p_{2(n-1)k+i}}{q_{2nk+i-1}y_{2nk+i-k-1}}\right\}, \\ q_{2nk+i} = \max\left\{1, \frac{A}{C^2 p_{2nk+i-1}}, \frac{q_{2(n-1)k+i}}{p_{2nk+i-1}x_{2nk+i-k-1}}\right\} \end{cases} \tag{2.5}$$

since $\frac{C}{A^2 y_{2nk+i-1}} < 1$ and $\frac{C}{A^2 x_{2nk+i-1}} < 1$. From (2.5) and (2.2) it follows that for any $i \in \mathbb{Z}(0, 2k-1)$ and $n \in \mathbb{N}$,

$$\begin{cases} 1 & \leq x_{2nk+i} & \leq \max\left\{1, x_{2(n-1)k+i}\right\} \leq x_{2(n-1)k+i}, \\ 1 & \leq y_{2nk+i} & \leq \max\left\{1, y_{2(n-1)k+i}\right\} \leq y_{2(n-1)k+i}. \end{cases}$$

Write

$$\begin{cases} \lim_{n\longrightarrow\infty} x_{2nk+i} = A_i \geq 1, \\ \lim_{n\longrightarrow\infty} y_{2nk+i} = B_i \geq 1. \end{cases}$$

Lemma 2. *Let $A = B > AC = 1 > C = D$.*

(1) *If $A_i > 1$ (resp. $B_i > 1$) for some $i \in \mathbb{Z}(0, 2k-1)$, then $x_{2nk+i+2r}$ and $p_{2nk-k+i+2r}$ (resp. $y_{2nk+i+2r}$ and $q_{2nk-k+i+2r}$) are constant sequences eventually for any $r \in \mathbb{N}$, and $q_{2nk-k+i+2r+1} = y_{2nk+i+2r+1} = 1$ (resp. $p_{2nk-k+i+2r+1} = x_{2nk+i+2r+1} = 1$) eventually for any $r \in \mathbb{N}_0$.*

(2) *If $A_i = 1$ (resp. $B_i = 1$) for some $i \in \mathbb{Z}(0, 2k-1)$, then $x_{2nk+i+2r} = 1$ (resp. $y_{2nk+i+2r} = 1$) eventually for any $r \in \mathbb{N}_0$.*

Proof.

(1) If $A_i > 1$ for some $i \in \mathbb{Z}(0, 2k-1)$, then by (2.5) one has

$$x_{2nk+i} = \frac{x_{2(n-1)k+i}}{y_{2nk+i-1} q_{2nk+i-k-1}} \tag{2.6}$$

eventually and

$$y_{2nk+i+1} = \max\left\{1, \frac{y_{2(n-1)k+i+1}}{x_{2nk+i} p_{2nk+i-k}}\right\} = 1 \tag{2.7}$$

eventually since $p_{2nk+i-k} \geq 1$ and

$$\frac{y_{2(n-1)k+i+1}}{x_{2nk+i} p_{2nk+i-k}} \leq \frac{y_{2(n-1)k+i+1}}{x_{2nk+i}} \longrightarrow \frac{B_{i+1}}{A_i} < B_{i+1} = \lim_{n\longrightarrow\infty} y_{2nk+i+1}.$$

It follows from (2.6) that

$$\lim_{n\longrightarrow\infty} y_{2nk+i-1} = \lim_{n\longrightarrow\infty} q_{2nk+i-k-1} = 1. \tag{2.8}$$

On the other hand, by (2.1) we see

$$x_{2nk+i} = \frac{C p_{2nk+i-k}}{A^2 y_{2nk+i-1}} \tag{2.9}$$

eventually. Furthermore by (1.2) and (2.1) and (2.8) we have

$$y_{2nk+i-1} = \max\left\{1, \frac{C q_{2nk+i-k-1}}{A^2 x_{2nk+i-1}}\right\} = 1 \tag{2.10}$$

eventually since $\frac{C q_{2nk+i-k-1}}{A^2 x_{2nk+i-1}} \longrightarrow \frac{C}{A^2 A_{i-1}} < 1$, which with (2.9) implies

$$x_{2nk+i} = \frac{C p_{2nk+i-k}}{A^2}. \tag{2.11}$$

It follows from (2.1) and (2.7) and (2.11) that

$$q_{2nk+i-k+1} = \max\left\{1, \frac{Ay_{2nk+i-2k+1}}{C^2 p_{2nk+i-k}}\right\} = 1 \tag{2.12}$$

eventually since

$$\lim_{n\longrightarrow\infty} \frac{Ay_{2nk+i-2k+1}}{C^2 p_{2nk+i-k}} = \frac{1}{A_i} < 1.$$

Thus by (2.2) and (2.5) and (2.7) and (2.12) we have

$$x_{2nk+i+2} = \max\left\{1, \frac{x_{2(n-1)k+i+2}}{y_{2nk+i+1}q_{2nk+i-k+1}}\right\} = x_{2(n-1)k+i+2}$$

eventually.

We claim that $x_{2nk+i+2} > 1$ eventually. Indeed, if $x_{2nk+i+2} = 1$ eventually, then by (2.1) and (2.2) and (2.7) and (2.11) one has

$$q_{2nk+k+i+1} = \max\left\{1, \frac{Ay_{2nk+i+1}}{C^2 p_{2nk+k+i}}\right\} = \max\left\{1, \frac{1}{x_{2(n+1)k+i}}\right\} = 1$$

eventually and

$$p_{2nk+k+i+2} = \max\left\{1, \frac{A}{C^2 q_{2nk+k+i+1}}\right\} = \frac{A}{C^2}$$

eventually and

$$q_{2nk+k+i+3} = \max\left\{1, \frac{Ay_{2nk+i+3}}{C^2 p_{2nk+k+i+2}}\right\} = y_{2nk+i+3}$$

eventually and

$$y_{2nk+2k+i+3} = \max\left\{1, \frac{Aq_{2nk+k+i+3}}{C^2 x_{2nk+2k+i+2}}\right\} = \max\left\{1, \frac{Ay_{2nk+i+3}}{C^2}\right\} = \frac{Ay_{2nk+i+3}}{C^2}$$

eventually since $\frac{Ay_{2nk+i+3}}{C^2} \geq \frac{AB_{i+3}}{C^2} > 1$, which leads to a contradiction that $B_{i+3} = \frac{AB_{i+3}}{C^2} > B_{i+3}$.

By $x_{2nk+i+2} > 1$ eventually, in a similar way as the above also we have

$$y_{2nk+2k+i+3} = q_{2nk-k+i+3} = 1, \quad x_{2nk+i+4} = x_{2nk-2k+i+4}, \quad p_{2nk-k+i+2} = \frac{A^2}{C} x_{2nk+i+2}$$

eventually.

Continuing in a similar way, we can obtain that $x_{2nk+i+2r}$ and $p_{2nk+i+2r}$ are constant sequences eventually for any $r \in \mathbb{N}$, and $q_{2nk-k+i+2r+1} = y_{2nk+i+2r+1} = 1$ eventually for any $r \in \mathbb{N}$. The other case is treated similarly, so we omit the detail.

(2) Indeed, if $A_i = 1$ and $x_{2nk+i} > 1$ for some $i \in \mathbb{Z}(0, 2k-1)$ and any $k \in \mathbb{N}$, then by (2.1) we have

$$x_{2nk+i} = \max\left\{1, \frac{Cp_{2nk-k+i}}{A^2 y_{2nk+i-1}}\right\} = \frac{Cp_{2nk-k+i}}{A^2 y_{2nk+i-1}} > 1$$

eventually and by (2.5) we have

$$y_{2nk+i+1} = \max\left\{1, \frac{Cy_{2nk-2k+i+1}}{A^2 x_{2nk+i} p_{2nk+i-k}}\right\} = 1 \tag{2.13}$$

eventually since $x_{2nk+i}p_{2nk+i-k} = x^2_{2nk+i} y_{2nk+i-1}\frac{A^2}{C} \geq \frac{A^2}{C}$ and

$$\frac{Cy_{2nk-2k+i+1}}{A^2 x_{2nk+i} p_{2nk+i-k}} \leq \frac{C^2 y_{2nk-2k+i+1}}{A^4} \longrightarrow \frac{C^2 B_{i+1}}{A^4} < B_{i+1} = \lim_{n\longrightarrow\infty} y_{2nk+i+1}.$$

By (2.1) and (2.13) we have

$$q_{2nk+k+i+1} = \max\left\{1, \frac{Ay_{2nk+i+1}}{C^2 p_{2nk+i+k}}\right\} = \max\left\{1, \frac{AC}{C^2 A^2 x_{2nk+2k+i} y_{2nk+2k+i-1}}\right\} = 1$$

eventually and

$$p_{2nk+k+i+2} = \max\left\{1, \frac{Ax_{2nk+i+2}}{C^2 q_{2nk+i+k+1}}\right\} = \frac{Ax_{2nk+i+2}}{C^2} \tag{2.14}$$

eventually. Thus it follows from (2.13) and (2.14) that

$$x_{2nk+2k+i+2} = \max\left\{1, \frac{Cp_{2nk+k+i+2}}{A^2 y_{2nk+i+2k+1}}\right\} = x_{2nk+2k+i+2}$$

eventually.

Using arguments similar to ones developed in (1), also we can show that $x_{2(n+1)k+i+2r}$ is constant sequence eventually for any $r \in \mathbb{N}$. Thus one has

$$x_{2(n+1)k+i+2k} = x_{2(n+2)k+i} = 1$$

eventually, which leads to a contradiction. The other case is treated similarly, so we omit the detail. The proof is complete. □

Proposition 3. *If $BD = AC = 1$ and $A \neq C$, then $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ is eventually periodic with period $2k$.*

Proof. Without loss of generality we assume $A > C$. There are the following three cases to consider.

Case 1. $A_{2i} > 1$ for some $i \in \mathbb{Z}(0,k-1)$ and $A_{2j+1} > 1$ for some $j \in \mathbb{Z}(0,k-1)$ or $B_{2i} > 1$ for some $i \in \mathbb{Z}(0,k-1)$ and $B_{2j+1} > 1$ for some $j \in \mathbb{Z}(0,k-1)$ or $A_{2i} > 1$ for some $i \in \mathbb{Z}(0,k-1)$ and $B_{2j} > 1$ for some $j \in \mathbb{Z}(0,k-1)$ or $A_{2i+1} > 1$ for some $i \in \mathbb{Z}(0,k-1)$ and $B_{2j+1} > 1$ for some $j \in \mathbb{Z}(0,k-1)$. By Lemma 2 we see easily that $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ is eventually periodic with period $2k$.

Case 2. $A_{2i} > 1$ for some $i \in \mathbb{Z}(0,k-1)$ and $A_{2j+1} = B_{2j} = B_{2j+1} = 1$ for any $j \in \mathbb{Z}(0,k-1)$ or $B_{2i} > 1$ for some $i \in \mathbb{Z}(0,k-1)$ and $A_{2j+1} = A_{2j} = B_{2j+1} = 1$ for any $j \in \mathbb{Z}(0,k-1)$ or $A_{2i+1} > 1$ for some $i \in \mathbb{Z}(0,k-1)$ and $A_{2j} = B_{2j} = B_{2j+1} = 1$ for any $j \in \mathbb{Z}(0,k-1)$ or $B_{2i+1} > 1$ for some $i \in \mathbb{Z}(0,k-1)$ and $A_{2j} = B_{2j} = A_{2j+1} = 1$ for any $j \in \mathbb{Z}(0,k-1)$.

Without loss of generality we assume $A_{2i} > 1$ for some $i \in \mathbb{Z}(0,k-1)$ and $A_{2j+1} = B_{2j} = B_{2j+1} = 1$ for any $j \in \mathbb{Z}(0,k-1)$. Then by Lemma 2 we see that x_{2nk+2r}

and $p_{2nk-k+2r}$ are eventually constant sequences for any $r \in \mathbb{N}$, and $q_{2nk-k+2r+1} = x_{2nk+2r+1} = y_{2nk+r} = 1$ eventually for any $r \in \mathbb{N}_0$. Thus it follows from (2.1) that

$$\begin{cases} p_{2nk+k+2r+1} = \max\left\{1, \frac{Ax_{2nk+2r+1}}{C^2 q_{2nk+k+2r}}\right\} = \max\left\{1, \frac{A}{C^2 q_{2nk+k+2r}}\right\}, \\ q_{2nk+k+2r+2} = \max\left\{1, \frac{Ay_{2nk+2r+2}}{C^2 p_{2nk+k+2r+1}}\right\} = \max\left\{1, \frac{A}{C^2 p_{2nk+k+2r+1}}\right\} \end{cases} \tag{2.15}$$

eventually, from which it follows that

$$\begin{cases} p_{2nk+k+2r+1} q_{2nk+k+2r} \geq \frac{A}{C^2}, \\ q_{2nk+k+2r+2} p_{2nk+k+2r+1} \geq \frac{A}{C^2} \end{cases} \tag{2.16}$$

eventually. By (2.15) and (2.16) one has

$$\begin{cases} 1 & \leq p_{2nk+k+2r+1} = \max\left\{1, \frac{Ap_{2nk+k+2r-1}}{C^2 q_{2nk+k+2r} p_{2nk+k+2r-1}}\right\} \\ & \leq \max\left\{1, p_{2nk+k+2r-1}\right\} = p_{2nk+k+2r-1}, \\ 1 & \leq q_{2nk+k+2r+2} = \max\left\{1, \frac{Aq_{2nk+k+2r}}{C^2 p_{2nk+k+2r+1} q_{2nk+k+2r}}\right\} \\ & \leq \max\left\{1, q_{2nk+k+2r}\right\} = q_{2nk+k+2r} \end{cases} \tag{2.17}$$

eventually. On the other hand, it follows from (2.15) and (2.17) that

$$\begin{cases} p_{2nk+k+2r+1} = \max\left\{1, \frac{A}{C^2 q_{2nk+k+2r}}\right\} \geq \max\left\{1, \frac{A}{C^2 q_{2nk+k+2r-2}}\right\} = p_{2nk+k+2r-1}, \\ q_{2nk+k+2r+2} = \max\left\{1, \frac{A}{C^2 p_{2nk+k+2r+1}}\right\} \geq \max\left\{1, \frac{A}{C^2 p_{2nk+k+2r-1}}\right\} = q_{2nk+k+2r} \end{cases} \tag{2.18}$$

eventually. $p_{2nk-k+2r}$ is eventually constant sequence for any $r \in \mathbb{N}$ and $q_{2nk-k+2r+1} = 1$ eventually for any $r \in \mathbb{N}_0$ and (2.17) and (2.18) imply that p_n and q_n are eventually periodic with period $2k$.

Case 3. $A_{2i} = A_{2i+1} = B_{2i} = B_{2i+1} = 1$ for any $i \in \mathbb{Z}(0, k-1)$. Then by Lemma 2 we see that $x_n = y_n = 1$ eventually. Thus it follows from (2.1) that

$$\begin{cases} p_n = \max\left\{1, \frac{A}{C^2 q_{n-1}}\right\}, \\ q_n = \max\left\{1, \frac{A}{C^2 p_{n-1}}\right\} \end{cases} \tag{2.19}$$

and $p_n q_{n-1} \geq \frac{A}{C^2}$ and $q_n p_{n-1} \geq \frac{A}{C^2}$ eventually, from which we have

$$\begin{cases} 1 \leq p_n = \max\left\{1, \frac{Ap_{n-2}}{C^2 q_{n-1} p_{n-2}}\right\} \leq \max\left\{1, p_{n-2}\right\} = p_{n-2}, \\ 1 \leq q_n = \max\left\{1, \frac{Aq_{n-2}}{C^2 p_{n-1} q_{n-2}}\right\} \leq \max\left\{1, q_{n-2}\right\} = q_{n-2} \end{cases} \tag{2.20}$$

eventually. On the other hand, it follows from (2.19) and (2.20) that

$$\begin{cases} p_n = \max\left\{1, \frac{A}{C^2 q_{n-1}}\right\} \geq \max\left\{1, \frac{A}{C^2 q_{n-3}}\right\} = p_{n-2}, \\ q_n = \max\left\{1, \frac{A}{C^2 p_{n-1}}\right\} \geq \max\left\{1, \frac{A}{C^2 p_{n-3}}\right\} = q_{n-2} \end{cases} \tag{2.21}$$

eventually. By (2.20) and (2.21) we see that p_n and q_n are eventually periodic with period 2. The proof is complete. □

Proposition 4. *If $BD > AC = 1$ and $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ is a solution of (2.1), then x_n and p_n are eventually periodic with period 2 and y_n and q_n are eventually periodic with period $2k$.*

Proof. If $BD > AC = 1$, then from Lemma (2.3) we see that there exists $N \in \mathbb{N}$ such that for any $n \geq N+2$, we have

$$\begin{cases} x_n = \max\left\{1, \frac{C}{ABy_{n-1}}\right\}, \\ y_n = \max\left\{1, \frac{D}{ABx_{n-1}}, \frac{y_{n-2rk}}{x_{n-1}p_{n-k-1}x_{n-2k-1}p_{n-3k-1}\cdots x_{n-2(r-1)k-1}p_{n-(2r-1)k-1}}\right\}, \\ p_n = \max\left\{1, \frac{A}{CDq_{n-1}}\right\}, \\ q_n = \max\left\{1, \frac{B}{CDp_{n-1}}, \frac{q_{n-2rk}}{p_{n-1}x_{n-k-1}p_{n-2k-1}px_{n-3k-1}\cdots p_{n-2(r-1)k-1}x_{n-(2r-1)k-1}}\right\}. \end{cases} \tag{2.22}$$

Now we show that p_n and x_n are eventually periodic with period 2.

If $p_M = 1$ for some $M \geq N+2$, then from (2.22) it follows that

$$q_{M+1} \geq \frac{B}{CD}.$$

and

$$1 \leq p_{M+2} = \max\left\{1, \frac{A}{CDq_{M+1}}\right\} \leq \max\left\{1, \frac{A}{B}\right\} = 1.$$

By mathematical induction, we can obtain $s_{M+2r} = 1$ for any $r \geq 0$,

If $p_{M+2r} = \frac{A}{CDq_{M+2r-1}} > 1$ for some $M \geq N+2$ and any $r \geq 0$, then from (2.1) and (2.22) and Lemma 1 we see

$$\begin{aligned} \max\left\{\frac{B}{CD}, q_1, q_2, \cdots, q_k\right\} \geq q_{M+2r+1} &= \max\left\{1, \frac{By_{M+1+2r-k}}{CDp_{M+2r}}\right\} \\ &= \max\left\{1, \frac{By_{M+1+2r-k}}{A}q_{M+2r-1}\right\} = \frac{B}{A}y_{M+1+2r-k}q_{M+2r-1} \\ &\geq \frac{B}{A}q_{M+2r-1}, \end{aligned}$$

which implies $A = B$ (since $B > A$ leads to $\lim_{r\longrightarrow\infty} q_{M+2r+1} = \infty$) and

$$q_{M+2r+1} \geq q_{M+2r-1}. \tag{2.23}$$

By (2.22) we see that for any $i \in \mathbb{Z}(1,k)$,

$$\begin{aligned} q_{M+2rk+2i+1} = \max\Bigg\{1, &\frac{B}{CDp_{M+2rk+2i}}, \\ &\frac{q_{M+2i+1}}{p_{M+2rk+2i}x_{M+(2r-1)k+2i}p_{M+(2r-2)k+2i}px_{M+(2r-3)k+2i}\cdots p_{M+2k+2i}x_{M+k+2i}}\Bigg\} \\ = \max\Bigg\{&q_{M+2rk+2i-1}, \end{aligned}$$

$$\left.\frac{q_{M+2i+1}}{p_{M+2rk+2i}x_{M+(2r-1)k+2i}p_{M+(2r-2)k+2i}px_{M+(2r-3)k+2i}\cdots p_{M+2k+2i}x_{M+k+2i}}\right\}$$

By (2.23) and (2.2) one has that $\mu(i,r) = q_{M+2rk+2i-1}$ is increasing and

$$\lambda(i,r) = \frac{q_{M+2i+1}}{p_{M+2rk+2i}x_{M+(2r-1)k+2i}p_{M+(2r-2)k+2i}px_{M+(2r-3)k+2i}\cdots p_{M+2k+2i}x_{M+k+2i}}$$

is decreasing for any $i \in \mathbb{Z}(1,k)$.

If for some $i \in \mathbb{Z}(1,k)$, $q_{M+2rk+2i+1} = \lambda(i,r) > \mu(i,r)$ for any $r \in \mathbb{N}$, then by (2.23) we see that q_{M+2r+1} is an eventually constant sequence.

If for any $i \in \mathbb{Z}(1,k)$, $q_{M+2rk+2i+1} = \mu(i,r) = q_{M+2rk+2i-1} \geq \lambda(i,r)$ eventually, then we have $q_{M+2rk+2i+1} = q_{M+2rk+2i-1}$ eventually for any $i \in \mathbb{Z}(1,k)$.

From the above we see that p_n is eventually periodic with period 2. In a similar way we can show that x_n is eventually periodic with period 2. Let $L > N+1$ such that $p_n = p_{n+2}$ and $x_n = x_{n+2}$ for any $n \geq L$. By (2.3) we see that for any $i \in \mathbb{Z}(1,k)$,

$$q_{L+2rk+2i+1} = \max\left\{1, \frac{B}{CDp_{L+2rk+2i}}, \frac{q_{L+2i+1}}{p_{L+2rk+2i}x_{L+(2r-1)k+2i}p_{L+(2r-2)k+2i}px_{L+(2r-3)k+2i}\cdots p_{L+2k+2i}x_{L+k+2i}}\right\}.$$

If $p_{L+2k}x_{L+k} = 1$, then for any $i \in \mathbb{Z}(1,k)$,

$$q_{L+2rk+2i+1} = \max\left\{1, \frac{B}{CDp_{L+2rk+2i}}, q_{L+2i+1}\right\}.$$

If $p_{L+2k}x_{L+k} > 1$, then

$$\lim_{r\longrightarrow\infty} \frac{q_{L+2i+1}}{p_{L+2rk+2i}x_{L+(2r-1)k+2i}p_{L+(2r-2)k+2i}px_{L+(2r-3)k+2i}\cdots p_{L+2k+2i}x_{L+k+2i}} = 0.$$

Thus $q_{L+2rk+2i+1} = \max\left\{1, \frac{B}{CDp_{L+2rk+2i}}\right\}$ eventually.

From the above we see that q_n is eventually periodic with period $2k$. In a similar way we can show that y_n is eventually periodic with period $2k$. The proof is complete. □

Proposition 5. *If $AC > 1$, then $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ is eventually periodic with period 1. Furthermore, the following statements hold:*

(1) *$x_n = 1$ eventually and $y_n = \max\left\{1, \frac{D}{AB}\right\}$ eventually or $y_n = 1$ eventually and $x_n = \max\left\{1, \frac{C}{AB}\right\}$ eventually.*

(2) *$p_n = 1$ eventually and $q_n = \max\left\{1, \frac{B}{CD}\right\}$ eventually or $q_n = 1$ eventually and $p_n = \max\left\{1, \frac{A}{CD}\right\}$ eventually.*

Proof. If $AC > 1$, then from (2.3) it follows that there exists $N \in \mathbb{N}$ such that for any $n \geq N+2$, we have

$$\begin{cases} x_n = \max\left\{1, \frac{C}{ABy_{n-1}}\right\}, \\ y_n = \max\left\{1, \frac{D}{ABx_{n-1}}\right\}, \\ p_n = \max\left\{1, \frac{A}{CDq_{n-1}}\right\}, \\ q_n = \max\left\{1, \frac{B}{CDp_{n-1}}\right\}. \end{cases} \tag{2.24}$$

We claim that $x_n = 1$ for any $n \geq N+2$ or $p_n = 1$ for any $n \geq N+2$. Indeed, if $x_n = \frac{C}{ABy_{n-1}} > 1$ for some $n \geq N+2$ and $p_m = \frac{A}{CDq_{m-1}} > 1$ for some $m \geq N+2$, then $\frac{1}{BD} = \frac{C}{AB}\frac{A}{CD} > 1$ since $y_{n-1} \geq 1$ and $q_{m-1} \geq 1$, which leads to a contradiction. In a similar way, also we can obtain that $y_n = 1$ for any $n \geq N+2$ or $q_n = 1$ for any $n \geq N+2$.

If $x_n = 1$ eventually, then $y_n = \max\left\{1, \frac{D}{AB}\right\}$ eventually. If $y_n = 1$ eventually, then $x_n = \max\left\{1, \frac{C}{AB}\right\}$ eventually. If $p_n = 1$ eventually, then $q_n = \max\left\{1, \frac{B}{CD}\right\}$ eventually. If $q_n = 1$ eventually, then $p_n = \max\left\{1, \frac{A}{CD}\right\}$ eventually. The proof is complete. □

From Proposition 1, Proposition 2, Proposition 3, Proposition 4 and Proposition 5, we get Theorem 1 immediately.

Acknowledgements

Project supported by NNSF of China (11761011) and NSF of Guangxi (2022GXNSFAA035552; 2020GXNSFAA297010) and PYMRBAP for Guangxi CU (2022KY0652; 2021KY0651).

References

[1] K. S. Berenhaut, J. D. Foley, and S. Stević, "Boundedness character of positive solutions of a max difference equation," *J. Difference Equ. Appl.*, vol. 12, no. 12, pp. 1193–1199, 2006, doi: 10.1080/10236190600949766.

[2] D. W. Cranston and C. M. Kent, "On the boundedness of positive solutions of the reciprocal MAX-type difference equation $x_n = \max\left\{\frac{A^1_{n-1}}{x_{n-1}}, \frac{A^2_{n-1}}{x_{n-2}}, \cdots, \frac{A^t_{n-1}}{x_{n-t}}\right\}$ with periodic parameters," *Appl. Math. Comput.*, vol. 221, pp. 144–151, 2013, doi: 10.1016/j.amc.2013.06.040.

[3] M. M. El-Dessoky, "On the periodicity of solutions of max-type difference equation," *Math. Methods Appl. Sci.*, vol. 38, no. 15, pp. 3295–3307, 2015, doi: 10.1002/mma.3296.

[4] E. M. Elsayed, B. Iričanin, and S. Stević, "On the max-type equation $x_{n+1} = \max\{A_n/x_n, x_{n-1}\}$." *Ars Comb.*, vol. 95, pp. 187–192, 2010.

[5] E. M. Elsayed and B. D. Iričanin, "On a max-type and a min-type difference equation," *Appl. Math. Comput.*, vol. 215, no. 2, pp. 608–614, 2009, doi: 10.1016/j.amc.2009.05.045.

[6] N. Fotiades and G. Papaschinopoulos, "On a system of difference equations with maximum," *Appl. Math. Comput.*, vol. 221, pp. 684–690, 2013, doi: 10.1016/j.amc.2013.07.014.

[7] W. Liu and S. Stević, "Global attractivity of a family of nonautonomous max-type difference equations," *Appl. Math. Comput.*, vol. 218, no. 11, pp. 6297–6303, 2012, doi: 10.1016/j.amc.2011.11.108.

[8] B. Qin, T. Sun, and H. Xi, "Dynamics of the max-type difference equation $x_{n+1} = \max\left\{\frac{A}{x_n}, x_{n-k}\right\}$," *J. Comput. Anal. Appl.*, vol. 14, no. 5, pp. 856–861, 2012.

[9] T. Sauer, "Global convergence of max-type equations," *J. Difference Equ. Appl.*, vol. 17, no. 1-2, pp. 1–8, 2011, doi: 10.1080/10236190903002149.

[10] Q. Shi, X. Su, G. Yuan, and Q. Xiao, "Characters of the solutions to a generalized nonlinear max-type difference equation," *Chin. Q. J. Math.*, vol. 28, no. 2, pp. 284–289, 2013.

[11] S. Stević, "Global stability of a difference equation with maximum," *Appl. Math. Comput.*, vol. 210, no. 2, pp. 525–529, 2009, doi: 10.1016/j.amc.2009.01.050.

[12] S. Stević, "Global stability of a max-type difference equation," *Appl. Math. Comput.*, vol. 216, no. 1, pp. 354–356, 2010, doi: 10.1016/j.amc.2010.01.020.

[13] S. Stević, "On a generalized max-type difference equation from automatic control theory," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 72, no. 3-4, pp. 1841–1849, 2010, doi: 10.1016/j.na.2009.09.025.

[14] S. Stević, "Periodicity of max difference equations," *Util. Math.*, vol. 83, pp. 69–71, 2010.

[15] S. Stević, "Periodicity of a class of nonautonomous max-type difference equations," *Appl. Math. Comput.*, vol. 217, no. 23, pp. 9562–9566, 2011, doi: 10.1016/j.amc.2011.04.022.

[16] S. Stević, "On some periodic systems of max-type difference equations," *Appl. Math. Comput.*, vol. 218, no. 23, pp. 11 483–11 487, 2012, doi: 10.1016/j.amc.2012.04.077.

[17] S. Stević, "Solutions of a max-type system of difference equations," *Appl. Math. Comput.*, vol. 218, no. 19, pp. 9825–9830, 2012, doi: 10.1016/j.amc.2012.03.057.

[18] S. Stević, "On a symmetric system of max-type difference equations," *Appl. Math. Comput.*, vol. 219, no. 15, pp. 8407–8412, 2013, doi: 10.1016/j.amc.2013.02.008.

[19] S. Stevic, A. M. Alghamdi, A. Alotaibi, and N. Shahzad, "Boundedness character of a max-type system of difference equations of second order," *Electron. J. Qual. Theory Differ. Equ.*, vol. 2014, p. 12, 2014, id/No 45. [Online]. Available: www.math.u-szeged.hu/ejqtde/p3150.pdf

[20] S. Stević, M. A. Alghamdi, A. Alotaibi, and N. Shahzad, "Eventual periodicity of some systems of MAX-type difference equations," *Appl. Math. Comput.*, vol. 236, pp. 635–641, 2014, doi: 10.1016/j.amc.2013.12.149.

[21] S. Stević, M. A. Alghamdi, A. Alotaibi, and N. Shahzad, "Long-term behavior of positive solutions of a system of max-type difference equations," *Appl. Math. Comput.*, vol. 235, pp. 567–574, 2014, doi: 10.1016/j.amc.2013.11.045.

[22] S. Stević, A. Alotaibi, N. Shahzad, and M. A. Alghamdi, "On positive solutions of a system of max-type difference equations," *J. Comput. Anal. Appl.*, vol. 16, no. 5, pp. 906–915, 2014.

[23] G. Su, C. Han, T. Sun, and L. Li, "On the solutions of a max-type system of difference equations of higher order," *Adv. Difference Equ.*, vol. 2020, p. 13, 2020, id/No 213, doi: 10.1186/s13662-020-02673-2.

[24] G. Su, T. Sun, and B. Qin, "Eventually periodic solutions of a max-type system of difference equations of higher order," *Discrete Dyn. Nat. Soc.*, vol. 2018, p. 6, 2018, id/No 8467682, doi: 10.1155/2018/8467682.

[25] G. Su, T. Sun, and B. Qin, "On the solutions of a max-type system of difference equations with period-two parameters," *Adv. Difference Equ.*, vol. 2018, p. 13, 2018, id/No 358, doi: 10.1186/s13662-018-1826-1.

[26] T. Sun, Q. He, X. Wu, and H. Xi, "Global behavior of the max-type difference equation $x_n = \max\{\frac{1}{x_{n-m}}, \frac{A_n}{x_{n-r}}\}$," *Appl. Math. Comput.*, vol. 248, pp. 687–692, 2014, doi: 10.1016/j.amc.2014.10.018.

[27] T. Sun and H. Xi, "On the solutions of a system of difference equations with maximum," *Appl. Math. Comput.*, vol. 290, pp. 292–297, 2016, doi: 10.1016/j.amc.2016.06.020.

[28] T. Sun, H. Xi, C. Han, and B. Qin, "Dynamics of the max-type difference equation $x_n = \max\{\frac{1}{x_{n-m}}, \frac{A_n}{x_{n-r}}\}$," *J. Appl. Math. Comput.*, vol. 38, no. 1-2, pp. 173–180, 2012, doi: 10.1007/s12190-010-0471-y.

[29] T. Sun, H. Xi, and W. Quan, "Existence of monotone solutions of a difference equation," *Discrete Dyn. Nat. Soc.*, vol. 2008, p. 8, 2008, id/No 917560, doi: 10.1155/2008/917560.

[30] Q. Xiao and Q.-H. Shi, "Eventually periodic solutions of a max-type equation," *Math. Comput. Modelling*, vol. 57, no. 3-4, pp. 992–996, 2013, doi: 10.1016/j.mcm.2012.10.010.

[31] Y. Yazlik, D. T. Tollu, and N. Taskara, "On the solutions of a max-type difference equation system," *Math. Methods Appl. Sci.*, vol. 38, no. 17, pp. 4388–4410, 2015, doi: 10.1002/mma.3377.

Authors' addresses

Guangwang Su
College of Information and Statistics, Guangxi Univresity, Finance and Economics, Nanning, 530003, China, and Guangxi Key Laboratory Cultivation Base of Cross-border E-commerce, Intelligent Information Processing, Nanning, 530003, China

Taixiang Sun
Guangxi (ASEAN) Research Center of Finance and Economics, Nanning, 530003, China, and Guangxi Key Laboratory Cultivation Base of Cross-border E-commerce Intelligent Information Processing, Nanning, 530003, China

Caihong Han
(corresponding) College of Information and Statistics, Guangxi Univresity, Finance and Economics, Nanning, 530003, China, and Guangxi Key Laboratory Cultivation Base of Cross-border E-commerce, Intelligent Information Processing, Nanning, 530003, China
E-mail address: h198204c@163.com

Bin Qin
(corresponding) Guangxi (ASEAN) Research Center of Finance and Economics, Nanning, 530003, China, and Guangxi Key Laboratory Cultivation Base of Cross-border E-commerce, Intelligent Information Processing, Nanning, 530003, China
E-mail address: q3009b@163.com

Weizhen Quan
Zhanjiang Preschool Education College, Zhanjiang, 524300, China