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YAKUBOVICH'S THEOREM FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT OF GENERALIZED TYPE

KUO-SHOU CHIU

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Abstract. The main goal of our paper is to obtain sufficient conditions for the asymptotic equivalence of a linear differential system and a quasilinear system of impulsive differential equations with piecewise constant argument of generalized type, in short IDEPCAG. A deviating argument is of the advanced and delayed type.

As an auxiliary result, the structure of the set of solutions of the quasilinear system is specified. Several examples are also given to show the feasibility of results.

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1. INTRODUCTION

The problem of asymptotic equivalence between two differential (or difference) systems is one of the most important part in the study of asymptotic property for solutions of differential (or difference) systems, and it shows an asymptotic relationship between solutions of differential (or difference) systems.

If we know that two systems are asymptotically equivalent, and if we can understand the asymptotic behavior of the solutions of one of them, then we will get information about the asymptotic behavior of the solutions of the other system. Apparently, the first results regarding the asymptotic behaviour of systems on the basis of one-to-one correspondence between sets of solutions were obtained in [13, 17, 21], see also [12, 25].

The purpose of this paper is to generalize well-known theorems of Yakubovich, valid for asymptotically equivalent of solutions of linear ODE system and quasilinear ODE system, to a linear ODE system and a perturbed system of impulsive differential equations with piecewise alternately advanced and retarded argument of generalized type.

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Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers, respectively. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{C}^n$, $n \in \mathbb{N}$ and $\mathbb{R}^+ = [0,\infty)$. Fix two real valued non-negative sequences $\{t_i\}$, $\{\gamma_i\}$, $i \in \mathbb{N}$ such that $t_i < t_{i+1}$, $t_i \le \gamma_i \le t_{i+1}$ and $t_i \to \infty$ as $i \to \infty$. Let γ: $\mathbb{R}^+ \to \mathbb{R}^+$, be a given general step function $\gamma|_{I_i} = \gamma_i$, $I_i = [t_i, t_{i+1})$, $I_{-1} = [0, t_0]$, $\gamma(s) = s$ for s in I_{-1} and $\mathbb{R}^+ = \bigcup_{i=-1}^{\infty} I_i$. In this case we speak of differential equations with piecewise constant arguments of generalized type, in short DEPCAG. Theory and practice of the DEPCAGs have been discussed extensively in [1,4,8–11,22].

The theory of impulsive differential equations developed rapidly, in recent years. This development, in particular, is due to the fact that many phenomena and processes arise in a variety of real world applications such as in the study of theoretical physics, population dynamics, biological systems, mechanical systems, and control theory. Furthermore, moment problem approaches appear also as a natural instrument in control theory of neutral type systems. For a good account on these theories, which have seen a significant development over the past decades we refer the interested reader to the monographs [16,26], the papers [14,15,18–20] and the references cited therein.

To the best of our knowledge, there are only a few papers involving the DEPCAG with impulsive effects (IDEPCAG) [3,5–7] to investigate the problem of asymptotic equivalence between the solutions of two differential systems.

In 2008, M. U. Akhmet [1] demonstrated the asymptotic equivalence of solutions of the linear ODE system and the DEPCAG system

$$\begin{aligned} x'(t) &= Cx(t), \\ z'(t) &= Cz(t) + f(t, z(t), z(\gamma(t))), \end{aligned}$$

where $x, z \in \mathbb{R}^n$, $t \in \mathbb{R}$, C is a constant $n \times n$ real valued matrix and $f \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ is a real valued $n \times 1$ function, $\gamma(t) = \gamma_i$, $t \in [t_i, t_{i+1})$, $i \in \mathbb{N}$.

In 2009, M. Pinto [22] proved the asymptotic equivalence of solutions of the linear ODE system and the following DEPCAG system

$$\begin{aligned} x'(t) &= A(t)x(t), \\ z'(t) &= A(t)z(t) + f(t, z(t), z(\gamma(t))), \end{aligned}$$

where $x, z \in \mathbb{C}^n$, $A = A(t)$ is a $n \times n$ complex matrix on $\mathbb{R}^+$, $f : \mathbb{R}^+ \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function and $\gamma(t) = \gamma_i$, $t \in [t_i, t_{i+1})$, if $i \in \mathbb{N}$.

In the present paper, we consider the following linear differential system

$$x'(t) = Ax(t) \tag{1.1}$$

and the linear perturbed IDEPCAG system

$$y'(t) = Ay(t) + f(t, y(t), y(\gamma(t))), \quad t \ne t_k, \tag{1.2a}$$

$$\Delta y|_{t=t_k} = \mathfrak{J}_k(y(t_k^-)), \quad k \in \mathbb{N}, \tag{1.2b}$$

where $x,y \in \mathbb{C}^n$, $t \in \mathbb{R}^+$, A is a constant $n \times n$ complex valued matrix and $f\colon \mathbb{R}^+ \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function and

$$\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-), \quad y(t_k^+) = \lim_{t\to t_k^+} y(t) \quad \text{and} \quad y(t_k^-) = \lim_{t\to t_k^-} y(t).$$

We assume the following hypotheses.

($\mathfrak{L}_1$) There exist two positive locally integrable functions η_1 and η_2 on $\mathbb{R}^+$ such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le \eta_1(t)|x_1 - x_2| + \eta_2(t)|y_1 - y_2|$$

for all $t \in \mathbb{R}^+$, $x_1, x_2, y_1, y_2 \in \mathbb{C}^n$ and $f(t,0,0) = 0$.

($\mathfrak{L}_2$) *The impulsive operator* $\mathfrak{J}_k$, $k \in \mathbb{N}$, satisfies

$$|\mathfrak{J}_k(u) - \mathfrak{J}_k(v)| \le \mathfrak{L}_k|u - v|$$

for the positive constant $\mathfrak{L}_k$, for all u, $v \in \mathbb{C}^n$ and $\mathfrak{J}_k(0) = 0$.

($\mathfrak{D}$) Let $\rho_i^+(A) = \exp(A \cdot (\gamma(t_i) - t_i))$, $\rho_i^-(A) = \exp(A \cdot (t_{i+1} - \gamma(t_i)))$, $\rho_i(A) = \rho_i^+(A)\rho_i^-(A)$, $i \in \mathbb{N}$. Suppose

$$\rho(A) = \sup_{i\in\mathbb{N}} \rho_i(A) < \infty$$

and

$$\rho = \rho(A) \sup_{i\in\mathbb{N}} \left(\int_{t_i}^{\gamma_i} [\eta_1(s) + \eta_2(s)]ds \right) < 1.$$

To investigate the asymptotic properties of the solutions, the following definitions can be efficiently applied.

Definition 1. A function y is a solution of the IDEPCAG system (1.2a)–(1.2b) in $[\tau,\infty)$ if

(i) $y(t)$ is continuous for $t \in [\tau,\infty)$ with the possible exception of the points $t = t_k$, $k \in \mathbb{N}$.

(ii) $y(t)$ is right continuous and has left-hand limits at the points $t = t_k$, $k \in \mathbb{N}$.

(iii) $y(t)$ is differentiable and satisfies (1.2a) for any $t \in [\tau,\infty)$, with the possible exception of the points $t = t_k$, $k \in \mathbb{N}$, where one-sided derivatives exist.

(iv) $y(t_k)$ satisfies (1.2b), $k \in \mathbb{N}$.

Definition 2. ([21, Page 288]) Systems (1.1) and (1.2a)–(1.2b) will be called equivalent if there exists a homeomorphism between the sets of solutions x and y, and systems (1.1) and (1.2a)–(1.2b) will be called asymptotically equivalent if, in addition, $x(t) - y(t) \to 0$ as $t \to \infty$.

The main purpose of this paper is to obtain a theorem of asymptotic equivalence of the stable solutions of the ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) by using the IDEPCAG's integral inequalities of Gronwall type and the method of investigation introduced by K.-S. Chiu [2] (2013). We also remark that our results extend their asymptotic formulae given by M. U. Akhmet [1] (2008). See Remark 4.

The paper is organized as follows. In the next section, we focus on some preliminary results which will be used to obtain the criteria for the existence and uniqueness of the solutions of the IDEPCAG system (1.2a)–(1.2b). Here, the IDEPCAG Gronwall-type inequality is very useful. The third section is devoted to the problem of the asymptotic equivalence of the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b). The fourth section, two examples are given to illustrate the validity of our results.

2. Integral Equations and Gronwall Integral Inequality

To study the nonlinear IDEPCAG system, we will use the approach based on the construction of an equivalent integral equation. Let us give the following proposition. We omit the proof of this assertion, since it can be proved in the same way as Proposition 2 in [2].

For the sake of convenience, we adopt the following notation. For every $t \in \mathbb{R}$, let $i = i(t) \in \mathbb{N}$ be the unique integer such that $t \in I_i = [t_i, t_{i+1})$.

Proposition 2.1. *For any* $(\tau, y_0) \in \mathbb{R}^+ \times \mathbb{C}^n$ *the solution* $y(t) = y(t, \tau, y_0)$ *of the IDEPCAG system* (1.2a)–(1.2b) *is defined on* $[\tau, \infty)$ *and given by*

$$y(t) = \begin{cases} e^{A(t-\tau)}y_0 + \int_\tau^t e^{A(t-s)} f(s, y(s), y(\gamma(s)))ds \\ \qquad + \sum_{k=i(\tau)+1}^{i(t)} e^{A(t-t_k)} \mathfrak{J}_k(y(t_k^-)), & i(t) > i(\tau), \\ e^{A(t-\tau)}y_0 + \int_\tau^t e^{A(t-s)} f(s, y(s), y(\gamma_{i(\tau)}))ds, & i(t) = i(\tau). \end{cases}$$

In the next, we give the following lemma about IDEPCAG integral inequality of Gronwall type, which is one of the most important auxiliary results of the present paper. The proof of the assertion is similar to that of Lemma 3 in K.-S. Chiu [2] (2013).

Lemma 2.1 (IDEPCAG's Gronwall Inequality)**.** *Let* $v, \alpha_j \colon \mathbb{R}^+ \to \mathbb{R}^+$, $j = 1, 2$, *be three functions such that* v *is continuous with possible points of discontinuity of the first kind at* $t = t_k$, $k \in \mathbb{N}$ *and* α_j *are locally integrable, for which the inequality satisfying*

$$v(t) \le v(\tau) + \left| \int_\tau^t [\alpha_1(s)v(s) + \alpha_2(s)v(\gamma(s))]\, ds \right| + \sum_{k=i(\tau)+1}^{i(t)} \rho_k v(t_k^-),$$

where $i(t) > i(\tau)$ *and* ρ_k *are non-negative constants. Then for* $t \ge \tau$,

$$v(t) \le v(\tau) \left\{ \prod_{k=i(\tau)+1}^{i(t)} (1 + \rho_k) \right\} \exp\left(\int_\tau^t \left[\alpha_1(s) + \frac{\alpha_2(s)}{1 - \upsilon} \right] ds \right) \tag{2.1}$$

and

$$v(\gamma(t)) \leq \frac{v(\tau)}{1-\upsilon}\left\{\prod_{k=i(\tau)+1}^{i(t)}(1+\rho_k)\right\}\exp\left(\int_\tau^t\left[\alpha_1(s)+\frac{\alpha_2(s)}{1-\upsilon}\right]ds\right)$$

where

$$\sup_{i(\tau)\leq k}\int_{t_k}^{\gamma_k}[\alpha_1(s)+\alpha_2(s)]\,ds \leq \upsilon < 1.$$

Remark 1. Consider $\rho_k \equiv 0$ in the IDEPCAG's Gronwall Inequality (2.1), we can obtain the following DEPCAG's inequality of Gronwall type:

$$u(t) \leq u(\tau)\exp\left(\int_\tau^t\left(\alpha_1(s)+\frac{\alpha_2(s)}{1-\upsilon}\right)ds\right).$$

The following theorem provides the existence of a unique solution when the initial moment is an arbitrary positive real number τ. Proof of this affirmation is omitted, as it can be demonstrated in the same way as Theorem 3.1 in K.-S. Chiu [7] (2021).

Theorem 2.1. *Let us assume that the conditions* $(\mathfrak{L}_1)$, $(\mathfrak{L}_2)$ *and* $(\mathfrak{D})$ *are satisfied. Then, given an initial condition* $(\tau, y_0) \in \mathbb{R}\times\mathbb{C}^n$, *there exists a unique solution* $y(\cdot) = y(\cdot,\tau,y_0)$ *of the IDEPCAG system* (1.2a)–(1.2b) *in the sense of Definition 1 such that* $y(\tau) = y_0$.

3. Asymptotic Equivalence

In this section we establish sufficient conditions under which the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are asymptotically equivalent. Following V. Yakubovich's theorem [21, 27] and Ráb [23, 24], as in [1, 22, 28], we do the use of analogical change of variable $y(t) = \exp(At)u(t)$. See Lemma 3.1. Along the lines we also derive a formula for asymptotic representation of solutions of the IDEPCAG system (1.2a)–(1.2b).

Consider the following nonlinear impulsive differential system with piecewise constant argument of generalized type

$$u'(t) = \hat{g}(t,u(t),u(\gamma((t)))) := e^{-At}f(t,e^{At}u(t),e^{A\gamma(t)}u(\gamma(t))), \quad t \neq t_k, \tag{3.1a}$$

$$\Delta u|_{t=t_k} = \hat{\mathfrak{J}}_k(u(t_k^-)) := e^{-At_k}\mathfrak{J}_k(e^{At_k^-}u(t_k^-)), \quad k\in\mathbb{N}, \tag{3.1b}$$

where $u\in\mathbb{C}^n$, e^{At} is the fundamental matrix of solutions of the linear ODE system (1.1) and $f\colon \mathbb{R}^+\times\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ satisfies the Lipschitz condition $(\mathfrak{L}_1)$ and the impulsive operator $\mathfrak{J}_k$, $k\in\mathbb{N}$, satisfies the Lipschitz condition $(\mathfrak{L}_2)$.

Assume the following hypotheses.

$(\mathfrak{H}_1)$ The two integrable functions $\hat{\eta}_1, \hat{\eta}_2$ are defined on $\mathbb{R}^+$, where

$$\hat{\eta}_1(t) = |e^{-At}||e^{At}|\eta_1(t), \qquad \hat{\eta}_2(t) = |e^{-At}||e^{A\gamma(t)}|\eta_2(t).$$

($\mathfrak{H}_2$) There exists a summable sequence of non-negative real numbers $(\mathfrak{L}_k)_{k=1}^{\infty}$ such that

$$\left|e^{-At_k}\right|\left|e^{At_k^-}\right|\mathfrak{L}_k \in l^1(\mathbb{R}^+).$$

($\mathfrak{H}_3$) There exist positive constants κ_1, κ_2 such that

$$\left|e^{At}\right| \le \kappa_1 t^{m_+-1}e^{\mu_+ t} \quad \text{and} \quad \left|e^{-At}\right| \le \kappa_2 t^{m_--1}e^{-\mu_- t} \quad \text{for all } t \in \mathbb{R}^+,$$

where $\mu_- = \min_{1\le k\le n} \Re\lambda_k$, $\mu_+ = \max_{1\le k\le n} \Re\lambda_k$ ($\Re\lambda_k$ is the real part of the eigenvalue λ_k of the matrix A) and $m_\pm$ are the maxima of the degrees of elementary divisors of the matrix A corresponding to eigenvalues λ with $\Re\lambda$ equal to $\mu_\pm$, respectively.

Let

$$\alpha(t) = \exp\left(\int_\tau^t \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{\nu}}\right] ds\right)\left\{\exp\left(\int_t^\infty \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{\nu}}\right] ds\right) - 1\right\},$$

where

$$\bar{\nu} = \sup_{i\in\mathbb{N}} \int_{t_i}^{\gamma_i} [\hat{\eta}_1(s) + \hat{\eta}_2(s)]ds < 1 \tag{3.2}$$

is the smallness condition for the IDEPCAG system (3.1a)–(3.1b). From the Lipschitz condition ($\mathfrak{L}_1$), the function $\hat{g}$ satisfies

$$|\hat{g}(t,u_1,v_1) - \hat{g}(t,u_2,v_2)| \le \hat{\eta}_1(t)|u_1-u_2| + \hat{\eta}_2(t)|v_1-v_2| \tag{3.3}$$

where $t \in \mathbb{R}^+$, $u_1,u_2,v_1,v_2 \in \mathbb{C}^n$ and $\hat{g}(t,0,0) = f(t,0,0) = 0$.
From the Lipschitz condition ($\mathfrak{L}_2$), the function $\hat{\mathfrak{J}}_k$ satisfies

$$|\hat{\mathfrak{J}}_k(u) - \hat{\mathfrak{J}}_k(v)| \le \hat{\mathfrak{L}}_k|u-v| \tag{3.4}$$

where $k \in \mathbb{N}$, $u,v \in \mathbb{C}^n$, $\hat{\mathfrak{L}}_k := |e^{-At_k}||e^{At_k^-}|\mathfrak{L}_k$ and $\hat{\mathfrak{J}}_k(0) = \mathfrak{J}_k(0) = 0$.

The following lemma can be easily proved by direct substitution.

Lemma 3.1. *If $y(t)$ is a solution of the IDEPCAG system (1.2a)–(1.2b), then there is a solution $u(t)$ of the IDEPCAG system (3.1a)–(3.1b) such that*

$$y(t) = e^{At}u(t). \tag{3.5}$$

Conversely, if $u(t)$ is a solution of the IDEPCAG system (3.1a)–(3.1b), then $y(t)$ in (3.5) is a solution of the IDEPCAG system (1.2a)–(1.2b).

Theorem 3.1. *Under the conditions ($\mathfrak{H}_1$), ($\mathfrak{H}_2$), (3.2), (3.3) and (3.4), all the solutions u of the IDEPCAG system (3.1a)–(3.1b) are defined on $I_\tau = [\tau,\infty)$ and convergent to some $u_\infty \in \mathbb{C}^n$ as $t \to \infty$*

$$u(t) = u_\infty + |u(\tau)|0\left(\left\{\exp\left(\int_t^\infty \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{\nu}}\right] ds\right) - 1\right\} + \sum_{k=i(t)+1}^{\infty} \hat{\mathfrak{L}}_k\right). \tag{3.6}$$

Conversely, for any $u_\infty \in \mathbb{C}^n$, there exists a unique solution u of the IDEPCAG system (3.1a)–(3.1b) *defined on $I_\tau = [\tau,\infty)$ for τ big enough such that $u(t) \to u_\infty$ as $t \to \infty$ and* (3.6) *holds. Furthermore, the correspondence $u \to u_\infty$ is an asymptotic equivalence.*

Proof. Let $u(t) = u(t,\tau,u_0)$ denote a solution of the nonlinear IDEPCAG system (3.1a)–(3.1b) satisfying $u(\tau) = u_0$. By the smallness condition (3.2) and using the same technique of Theorem 2.1, the solution

$$
\begin{aligned}
u(t) &= u_0 + \int_\tau^t e^{-As} f(s, e^{As}u(s), e^{A\gamma(s)}u(\gamma(s)))ds + \sum_{k=i(\tau)+1}^{i(t)} e^{-At_k}\mathfrak{J}_k(e^{At_k^-}u(t_k^-)) \\
&= u_0 + \int_\tau^t \hat{g}(s,u(s),u(\gamma(s)))ds + \sum_{k=i(\tau)+1}^{i(t)} \hat{\mathfrak{J}}_k(u(t_k^-)), \quad i(t) > i(\tau),
\end{aligned}
$$

exists on $I_\tau = [\tau,\infty)$ and is unique. By using (3.3) and (3.4), $\hat{g}(t,0,0) = 0$ and $\hat{\mathfrak{J}}_k(0) = 0$, we have

$$
|u(t)| \leq |u_0| + \int_\tau^t [\hat{\eta}_1(s)\,|u(s)| + \hat{\eta}_2(s)|u(\gamma(s))|]ds + \sum_{k=i(\tau)+1}^{i(t)} \hat{\mathfrak{L}}_k \left|u(t_k^-)\right|, \quad i(t) > i(\tau).
$$

By virtue of the IDEPCAG integral inequality of Gronwall type (Lemma 2.1), we obtain

$$
|u(t)| \leq |u_0| \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} \left(1+\hat{\mathfrak{L}}_\kappa\right) \right\} \exp\left(\int_\tau^t \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{\nu}}\right] ds \right)
$$

and

$$
|u(\gamma(t))| \leq \frac{1}{1-\bar{\nu}}|u_0| \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} \left(1+\hat{\mathfrak{L}}_\kappa\right) \right\} \exp\left(\int_\tau^t \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{\nu}}\right] ds \right)
$$

where $\bar{\nu}$ is the smallness condition (3.2).

By $(\mathfrak{H}_2)$, we have $\hat{\mathfrak{L}}_k \in l^1([\tau,\infty))$, then $\prod_{k\geq i(\tau)+1} \left(1+\hat{\mathfrak{L}}_k\right) < \infty$. As $\hat{\eta}_1, \hat{\eta}_2 \in l^1([\tau,\infty))$ and $\prod_{k\geq i(\tau)+1} \left(1+\hat{\mathfrak{L}}_k\right) < \infty$, we conclude that u is bounded on $\mathbb{R}^+$, i.e., $|u(t)| \leq M$, for all $t \in [\tau,\infty)$. Therefore

$$
u_\infty = u_0 + \int_\tau^\infty \hat{g}(s,u(s),u(\gamma(s)))ds + \sum_{k=i(\tau)+1}^{\infty} \hat{\mathfrak{J}}_k(u(t_k^-)) \tag{3.7}
$$

exists and

$$
u(t) = u_\infty - \int_t^\infty \hat{g}(s,u(s),u(\gamma(s)))ds - \sum_{k=i(t)+1}^{\infty} \hat{\mathfrak{J}}_k(u(t_k^-)). \tag{3.8}
$$

Moreover

$$
\begin{aligned}
&\int_t^{\infty} |\hat{g}(s,u(s),u(\gamma(s)))|ds \\
&\leq |u_0| \int_t^{\infty} \left[\left(\hat{\eta}_1(s) + \frac{1}{1-\bar{v}} \hat{\eta}_2(s) \right) \ \exp\left(\int_{\tau}^{s} \left[\hat{\eta}_1(r) + \frac{1}{1-\bar{v}} \hat{\eta}_2(r) \right] dr \right) \right] ds \\
&= |u_0| \exp\left(\int_{\tau}^{t} \left[\hat{\eta}_1(s) + \frac{1}{1-\bar{v}} \hat{\eta}_2(s) \right] ds \right) \\
&\quad \times \left\{ \exp\left(\int_t^{\infty} \left[\hat{\eta}_1(s) + \frac{1}{1-\bar{v}} \hat{\eta}_2(s) \right] ds \right) - 1 \right\} \\
&= |u_0| 0 \left\{ \exp\left(\int_t^{\infty} \left[\hat{\eta}_1(s) + \frac{1}{1-\bar{v}} \hat{\eta}_2(s) \right] ds \right) - 1 \right\}, \quad \text{as} \quad t \to \infty.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=i(t)+1}^{\infty} \left| \hat{\mathfrak{J}}_k(u(t_k^-)) \right| &\leq \sum_{k=i(t)+1}^{\infty} \hat{\mathfrak{L}}_k \left| u(t_k^-) \right| \leq M \sum_{k=i(t)+1}^{\infty} \hat{\mathfrak{L}}_k \\
&= |u_0| \cdot \left(\rho \sum_{k=i(t)+1}^{\infty} \hat{\mathfrak{L}}_k \right) = |u_0| 0 \left\{ \sum_{k=i(t)+1}^{\infty} \hat{\mathfrak{L}}_k \right\}, \quad \text{as} \quad t \to \infty,
\end{aligned}
$$

where $\rho = M/|u_0|$. Then the first conclusion and (3.6) follow.

For $u_0^i \in \mathbb{C}^n$, $i = 1,2$ and $u_i(t) = u_i(t,\tau,u_0^i)$, we have

$$
\begin{aligned}
|u_1(t) - u_2(t)| \leq |u_0^1 - u_0^2| &+ \int_{\tau}^{t} |\hat{g}(s,u_1(s),u_1(\gamma(s))) - \hat{g}(s,u_2(s),u_2(\gamma(s)))| ds \\
&+ \sum_{k=i(\tau)+1}^{i(t)} \left| \hat{\mathfrak{J}}_k(u_1(t_k^-)) - \hat{\mathfrak{J}}_k(u_2(t_k^-)) \right| \\
&\leq |u_0^1 - u_0^2| \\
&+ \int_{\tau}^{t} [\hat{\eta}_1(s)|u_1(s) - u_2(s)| + \hat{\eta}_2(s)|u_1(\gamma(s)) - u_2(\gamma(s))|)] ds \\
&+ \sum_{k=i(\tau)+1}^{i(t)} \hat{\mathfrak{L}}_k \left| u_1(t_k^-) - u_2(t_k^-) \right|.
\end{aligned}
$$

From the IDEPCAG integral inequality of Gronwall type (Lemma 2.1), for $i(t) > i(\tau)$

$$
|u_1(t) - u_2(t)| \leq |u_0^1 - u_0^2| \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} (1 + \hat{\mathfrak{L}}_\kappa) \right\} \exp\left(\int_{\tau}^{t} \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{v}} \right] ds \right)
$$

and

$$
|u_1(\gamma(t)) - u_2(\gamma(t))| \leq \frac{1}{1-\bar{v}} |u_0^1 - u_0^2| \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} (1 + \hat{\mathfrak{L}}_\kappa) \right\}
$$

$$\times \exp\left(\int_\tau^t \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{\nu}}\right] ds\right).$$

Furthermore, as in (3.6), by (3.7), it is not difficult to obtain

$$\left(1-\alpha(\tau) - M \sum_{k=i(\tau)+1}^{\infty} \hat{\mathfrak{L}}_k\right) |u_0^1 - u_0^2| \leq |u_\infty^1 - u_\infty^2|$$

$$\leq |u_0^1 - u_0^2| \left(1+\alpha(\tau) + M \sum_{k=i(\tau)+1}^{\infty} \hat{\mathfrak{L}}_k\right),$$

if $\alpha(\tau) + M \sum_{k=i(\tau)+1}^{\infty} \hat{\mathfrak{L}}_k < 1$. This establishes the equivalence of $u_0 \to u_\infty$. Finally, for any $u_\infty \in \mathbb{C}^n$, Eq. (3.8) has a unique solution u defined on $I_\tau = [\tau, \infty)$, if, for example, τ satisfies

$$\int_\tau^\infty \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{\nu}}\right] ds + M \sum_{k=i(\tau)+1}^{\infty} \hat{\mathfrak{L}}_k < 1.$$

Then the smallness condition (3.2) holds and hence the same technique of Theorem 2.1 is applicable to guarantee the uniqueness of solution u. This proves the asymptotic equivalence of the correspondence $u(t)$ and u_∞. □

Corollary 3.1. *If conditions ($\mathfrak{H}_1$), ($\mathfrak{H}_2$), (3.2), (3.3) and (3.4) are valid, then every solution of the IDEPCAG system (3.1a)–(3.1b) is bounded on $I_\tau = [\tau, \infty)$ and for each solution u of the IDEPCAG system (3.1a)–(3.1b), there exists a constant vector u_∞ such that $u(t) \to u_\infty$ as $t \to \infty$.*

Theorem 3.2. *If conditions ($\mathfrak{H}_1$), ($\mathfrak{H}_2$), (3.2), (3.3) and (3.4) are fulfilled, then each solution y of the IDEPCAG system (1.2a)–(1.2b) is defined on $I_\tau = [\tau, \infty)$. Furthermore, the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are equivalent and we have the following asymptotic formula*

$$y(t) = e^{At}[v + \varepsilon(t)], \quad as \quad t \to \infty, \tag{3.9}$$

where $v \in \mathbb{C}^n$ is a constant vector and the error function ε verifies

$$\varepsilon(t) = 0\left\{\left(\exp\left(\int_t^\infty \left[|e^{-As}||e^{As}|\eta_1(s) + \frac{|e^{-As}||e^{A\cdot\gamma(s)}|\eta_2(s)}{1-\bar{\nu}}\right] ds\right) - 1\right)\right.$$
$$\left. + \sum_{k=i(t)+1}^{\infty} |e^{-At_k}||e^{At_k^-}|\mathfrak{L}_k\right\}. \tag{3.10}$$

Moreover, if $\varepsilon_0(t) \to 0$ as $t \to \infty$, where

$$\varepsilon_0(t) = \int_t^\infty \left|e^{A\cdot(t-s)}\right| \left[\left|e^{As}\right|\eta_1(s) + \left|e^{A\cdot\gamma(s)}\right|\eta_2(s)\right] ds$$

$$+\sum_{k=i(t)+1}^{\infty}\left|e^{A\cdot(t-t_k)}\right|\left|e^{A\cdot t_k^-}\right|\mathfrak{L}_k, \tag{3.11}$$

then the linear ODE system (1.1) *and the IDEPCAG system* (1.2a)–(1.2b) *are asymptotically equivalent and we have the asymptotic formula*

$$y(t) = x(t) + 0(\varepsilon_0(t)), \quad as \quad t\to\infty, \tag{3.12}$$

for some constant vector $v\in\mathbb{C}^n$.

Proof. By (3.1) and $y = e^{At}u$, we get

$$y(t) = e^{At}\left[u_\infty - \int_t^\infty \hat{g}(s,u(s),u(\gamma(s)))ds - \sum_{k=i(t)+1}^{\infty}\hat{\mathfrak{J}}_k(u(t_k^-))\right], \tag{3.13}$$

where $u = u(t,\tau,u_0)$ is a solution of (3.1). Remark that in (3.13) $x(t) = e^{At}u_\infty$ is a solution of the linear ODE system (1.1) and $y = e^{At}u$ and so $y(\tau) = u_0$. By Theorem 3.1, the correspondence $u_0 \leftrightarrow u_\infty$ is a homeomorphism for some τ sufficiently large. Then, the correspondence $x\leftrightarrow y$ is also a homeomorphism and formulae (3.9) and (3.12) follow. □

Remark 2. Suppose now that

$$|e^{At}| = 0(\varphi_+(t)), \quad |e^{-At}| = 0(\varphi_-(t)) \quad \text{as} \quad t\to\infty \tag{3.14}$$

and

$$0(\gamma(t)) = 0(t) \quad \text{as} \quad t\to\infty. \tag{3.15}$$

Thus, by (3.14) and (3.15), the functions ε and ε_0 in (3.10) and (3.11) of Theorem 3.2 may be considered as

$$\varepsilon(t) = 0\left\{\left(\exp\left[\int_t^\infty \varphi_-(s)\varphi_+(s)\left(\eta_1(s)+\frac{\eta_2(s)}{1-\bar{\nu}}\right)ds\right]-1\right) + \sum_{k=i(t)+1}^{\infty}\varphi_-(t_k)\varphi_+(t_k)\mathfrak{L}_k\right\}$$

and

$$\varepsilon_0(t) = \int_t^\infty \varphi_-(s-t)\varphi_+(s)\left(\eta_1(s)+\eta_2(s)\right)ds + \sum_{k=i(t)+1}^{\infty}\varphi_-(t_k-t)\varphi_+(t_k)\mathfrak{L}_k.$$

This gives a practical way to apply Theorem 3.2.

Corollary 3.2. *If conditions* $(\mathfrak{H}_1)$, $(\mathfrak{H}_2)$, (3.2), (3.3) *and* (3.4) *are satisfied, then every solution y of the IDEPCAG system* (1.2a)–(1.2b) *possesses an asymptotic representation of the form*

$$y(t) = e^{At}[v+\varepsilon(t)], \quad as \quad t\to\infty,$$

where $v \in \mathbb{C}^n$ is a constant vector and for a solution u of the IDEPCAG system (3.1a)–(3.1b).

$$\varepsilon(t) = -\int_t^\infty e^{-As} f(s, e^{As}u(s), e^{A\gamma(s)}u(\gamma(s)))ds - \sum_{k=i(t)+1}^{\infty} e^{-At_k}\mathfrak{J}_k(e^{At_k^-}u(t_k^-)).$$

Remark 3. Note that we can easily deduce the condition (3.2) for the following form

$$\nu = \sup_{i\in\mathbb{N}} \int_{t_i}^{\gamma_i} \left|e^{-As}\right| \left(\eta_1(s)\left|e^{As}\right| + \eta_2(s)\left|e^{A\gamma(s)}\right|\right)ds < 1. \tag{3.16}$$

By $(\mathfrak{H}_3)$, the condition (3.16) can be deduced by

$$\nu = \kappa_1\kappa_2 \sup_{i\in\mathbb{N}} \int_{t_i}^{\gamma_i} s^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)s}\left(\eta_1(s) + \eta_2(s)\frac{\gamma(s)^{m_+-1}e^{\mu_+\gamma(s)}}{s^{m_+-1}e^{\mu_+ s}}\right)ds < 1 \tag{3.17}$$

and we can find a positive number κ such that

$$\frac{\gamma(s)^{m_+-1}e^{\mu_+\gamma(s)}}{s^{m_+-1}e^{\mu_+ s}} = \left(\frac{\gamma(s)}{s}\right)^{m_+-1} e^{\mu_+(\gamma(s)-s)} \le \sup_{i\in\mathbb{N}}\left(\frac{\gamma_i}{t_i}\right)^{m_+-1} e^{\mu_+ \sup_{j\in\mathbb{N}}(\gamma_j - t_j)} = \kappa.$$

Then the conditions $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$ can be deduced by the following forms

$(\mathfrak{H}_1')$ The two integrable functions η_1, η_2 are defined on $\mathbb{R}^+$ such that

$$t^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)\cdot t}\left(\eta_1(t) + \eta_2(t)\right) \in L_1(\mathbb{R}^+).$$

$(\mathfrak{H}_2')$ There exists a summable sequence of non-negative real numbers $(\mathfrak{L}_k)_{k=1}^\infty$ such that

$$t_k^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)\cdot t_k}\mathfrak{L}_k \in l^1(\mathbb{R}^+).$$

Theorem 3.3. *Under the conditions $(\mathfrak{H}_1')$, $(\mathfrak{H}_2')$,* (3.2)*,* (3.3) *and* (3.4)*, all the solutions u of the IDEPCAG system* (3.1a)–(3.1b) *are defined on $I_\tau = [\tau, \infty)$ and convergent to some $u_\infty \in \mathbb{C}^n$ as $t \to \infty$*

$$u(t) = u_\infty + |u(\tau)| 0\left(\left\{\exp\left(\int_t^\infty \left[s^{m_+ + m_- - 2}e^{(\mu_+ - \mu_-)\cdot s}\left(\eta_1(s) + \frac{\kappa}{1-\nu}\eta_2(s)\right)\right]ds\right) - 1\right\} + \sum_{k=i(t)+1}^{\infty} t_k^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)\cdot t_k}\mathfrak{L}_k\right). \tag{3.18}$$

Conversely, for any $u_\infty \in \mathbb{C}^n$, there exists a unique solution u of the IDEPCAG system (3.1a)–(3.1b) *defined on $I_\tau = [\tau, \infty)$ for τ big enough such that $u(t) \to u_\infty$ as $t \to \infty$ and* (3.18) *holds. Furthermore, the correspondence $u \to u_\infty$ is an asymptotic equivalence.*

The proof follows from Lemma 3.1, Theorem 3.1 and Remark 3.

Theorem 3.4. *If the conditions ($\mathfrak{H}_1'$), ($\mathfrak{H}_2'$), (3.2), (3.3) and (3.4) are fulfilled, then each solution y of the IDEPCAG system (1.2a)–(1.2b) is defined on $I_\tau = [\tau, \infty)$. Furthermore, the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are equivalent and we have the following asymptotic formula*

$$y(t) = e^{At}[c + \varepsilon(t)], \quad as \quad t \to \infty,$$

where $c \in \mathbb{C}^n$ is a constant vector and the error function ε verifies

$$\varepsilon(t) = 0\left\{\left\langle \exp\left(\int_t^\infty \left[s^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)\cdot s}\left(\eta_1(s) + \frac{\kappa}{1-\nu}\eta_2(s)\right)\right] ds\right) - 1\right\rangle + \sum_{k=i(t)+1}^{\infty} t_k^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)\cdot t_k} \mathfrak{L}_k \right\}.$$

Moreover, if $\varepsilon_0(t) \to 0$ as $t \to \infty$, where

$$\varepsilon_0(t) = \int_t^\infty (s-t)^{m_- - 1} e^{-\mu_-\cdot(s-t)} s^{m_+ - 1} e^{\mu_+ s} (\eta_1(s) + \eta_2(s)) ds + \sum_{k=i(t)+1}^{\infty} (t_k - t)^{m_- - 1} e^{-\mu_-\cdot(t_k - t)} t_k^{m_+ - 1} e^{\mu_+ t_k} \mathfrak{L}_k, \tag{3.19}$$

then the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are asymptotically equivalent and we have the asymptotic formula

$$y(t) = e^{At}c + 0(\varepsilon_0(t)), \quad as \quad t \to \infty, \tag{3.20}$$

for some constant vector $c \in \mathbb{C}^n$.

The proof follows from Lemma 3.1, Theorem 3.2 and Remark 3.

Remark 4. Let $\eta(s) = \max_{i=1,2} \eta_i(s)$ and assume that conditions ($\mathfrak{H}_1'$), ($\mathfrak{H}_2'$), (3.3), (3.4) and (3.17) are fulfilled. Moreover

$$\int_t^\infty (s-t)^{m_- - 1} e^{-\mu_-(s-t)} s^{m_+ - 1} e^{\mu_+ s} \eta(s) ds + \sum_{k=i(t)+1}^{\infty} (t_k - t)^{m_- - 1} e^{-\mu_-(t_k - t)} t_k^{m_+ - 1} e^{\mu_+ t_k} \mathfrak{L}_k \to 0 \quad \text{as} \quad t \to \infty,$$

then the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are asymptotically equivalent.

Let $\eta(s) \equiv \eta_i(s)$, $i = 1, 2$, and assume that conditions ($\mathfrak{H}_1'$), (3.3) and (3.17) are fulfilled and

$$\int_t^\infty (s-t)^{m_- - 1} e^{-\mu_-(s-t)} s^{m_+ - 1} e^{\mu_+ s} \eta(s) ds \to 0 \quad \text{as} \quad t \to \infty. \tag{3.21}$$

Then the linear ODE system (1.1) and the DEPCAG system (1.2a) are asymptotically equivalent.

The previous result has been obtained in [1, Theorems 2.1 and 2.2], our condition (3.17) to guarantee the existence and uniqueness of the quasilinear DEPCAG system (1.2a) is less restrictive than **(C3)** mentioned in [1].

Remark 5. In [27], we can find Yakubovich's theorem on the asymptotic equivalence of the linear ODE system and quasilinear ODE system. The sufficient condition for asymptotic equivalence was

$$\int_t^\infty s^{m_+ + m_- - 2} e^{\mu_+ s} \eta(s) ds \to 0 \quad \text{as} \quad t \to \infty, \tag{3.22}$$

where $\eta(s) = \max_{i=1,2} \eta_i(s)$.

We can easily to prove that the conditions $(\mathfrak{H}_1')$ or (3.21) with $\mu_- > 0$ are less restrictive than (3.22) in Ref. [27].

Then Theorem 3.4 extends and improves some existing Yakubovich's results on asymptotic equivalence of different classes of linear and quasilinear differential systems studied in [21, 27]. Also, with this research we extend the results for the DEPCAG system in [1] and the DEPCAG system to the impulsive case (the IDEPCAG system).

4. Examples

As a direct application of Theorem 3.2 and Theorem 3.4, here we will present two particular cases.

Example 4.1. Consider the following scalar IDEPCAG

$$\begin{aligned} y'(t) &= ay + b(t)y(\gamma(t)), \quad t \neq t_k \\ \Delta y|_{t=t_k} &= \mathfrak{L}_k \cdot y(t_k^-), \quad k \in \mathbb{N}, \end{aligned} \tag{4.1}$$

with $a > 0$ is a constant, $b(t) \in L_1(\mathbb{R}^+)$, $\mathfrak{L}_k \in l^1(\mathbb{R}^+)$ and $\hat{b}(t) = 0(e^{-at})$ where

$$\hat{b}(t) = \int_t^\infty |b(s)| ds + \sum_{k=i(t)+1}^\infty \mathfrak{L}_k.$$

Then the IDEPCAG (4.1) is equivalent to the linear scalar ODE

$$x'(t) = ax(t) \tag{4.2}$$

and all solutions $y(t)$ of the IDEPCAG (4.1) have the asymptotic formula

$$y(t) = e^{at}\left(v + \varepsilon(t)\right), \quad \text{as} \quad t \to \infty,$$

where $v \in \mathbb{R}$ and the error function ε verifies

$$\varepsilon(t) = 0\left\{\left\langle \exp\left(\int_t^\infty e^{a\cdot(\gamma(s)-s)} |b(s)| ds\right) - 1\right\rangle + \sum_{k=i(t)+1}^\infty \mathfrak{L}_k\right\}.$$

Moreover, if $\varepsilon_0(t) \to 0$ as $t \to \infty$, where

$$\varepsilon_0(t) = 0\left(\int_t^{\infty}\left[e^{a\cdot(t-s)}|b(s)|e^{a\cdot\gamma(s)}\right]ds + \sum_{k=i(t)+1}^{\infty} e^{at}\mathfrak{L}_k\right),$$

then, by Theorem 3.2, the linear scalar ODE (4.2) and the scalar IDEPCAG (4.1) are asymptotically equivalent and we have the asymptotic formula

$$y(t) = x(t) + 0(\varepsilon_0(t)), \quad \text{as} \quad t \to \infty.$$

Example 4.2. Consider the following impulsive second-order differential equations with piecewise alternately advanced and retarded argument:

$$\begin{aligned} y''(t) - 5y' + 6y &= \eta(t)\cos^2\left(y\left(3\left[\frac{t+1}{3}\right]\right)\right), \quad t \neq 3k-1, \\ \Delta y|_{t=t_k} &= \left(\frac{1}{4}\right)^{3k} \cdot y(3k-1^-), \\ \Delta y'|_{t=t_k} &= \left(\frac{1}{5}\right)^{3k} \cdot y'(3k-1^-), \quad k \in \mathbb{N}, \end{aligned} \tag{4.3}$$

where $[\cdot]$ is the greatest integer function, $\eta(t)$ is a continuous function defined on $\mathbb{R}^+$ and

$$|\eta(t)| < \frac{\kappa}{1+t^2}e^{-3t}, \quad \text{for } t \in \mathbb{R}^+,$$

where $\kappa \in \mathbb{R}^+$. According to the IDEPCA (4.3), we have $\gamma(t) = 3\left[\frac{t+1}{3}\right]$, then $t_j = 3j-1$, $\gamma_j = 3j$ for all $j \in \mathbb{N}$.

Letting $z(\cdot) = (z_1(\cdot), z_2(\cdot))^T = (y(\cdot), y'(\cdot))^T$, we write the IDEPCA (4.3) in the system form

$$\begin{aligned} z'(t) &= \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ \eta(t)\cos^2\left(z_1\left(3\left[\frac{t+1}{3}\right]\right)\right) \end{pmatrix}, \\ \Delta z|_{3k-1} &= \begin{pmatrix} \left(\frac{1}{4}\right)^{3k} & 0 \\ 0 & \left(\frac{1}{5}\right)^{3k} \end{pmatrix} z(3k-1^-), \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}, \quad f\left(t, z(t), z\left(3\left[\frac{t+1}{3}\right]\right)\right) = \begin{pmatrix} 0 \\ \eta(t)\cos^2\left(z_1\left(3\left[\frac{t+1}{3}\right]\right)\right) \end{pmatrix}.$$

We can easily verify that $\mu_- = 2$, $\mu_+ = 3$, $m_\pm = 1$ and the conditions $(\mathfrak{H}_1')$, $(\mathfrak{H}_2')$, (3.2), (3.3) and (3.4) are fulfilled. Moreover we can check that the condition (3.19) holds.

$$\begin{aligned} \int_t^{\infty} (s-t)^{m_- -1} e^{-\mu_-(s-t)} s^{m_+ -1} e^{\mu_+ s}(\eta(s))ds &\leq \int_t^{\infty} e^{-2(s-t)} e^{3s} \frac{\kappa}{1+s^2} e^{-3s} ds \\ &\leq \frac{\kappa}{1+t^2} \underset{t\to\infty}{\to} 0, \end{aligned}$$

and

$$\sum_{k=i(t)+1}^{\infty} (t_k - t)^{m_- - 1} e^{-\mu_-(t_k - t)} t_k^{m_+ - 1} e^{\mu_+ t_k} \mathfrak{L}_k \leq \sum_{k=i(t)+1}^{\infty} e^{2t+(3k-1)} \left(\frac{1}{4}\right)^{3k} \underset{t\to\infty}{\to} 0.$$

Thus, by Theorem 3.4, the IDEPCA (4.3) is asymptotically equivalent to the following ODE equation

$$x''(t) - 5x' + 6x = 0$$

which has solutions

$$x(t) = c_1 e^{2t} + c_2 e^{3t}.$$

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Author’s address

Kuo-Shou Chiu
Departamento de Matemática, Facultad de Ciencias Básicas, Universidad Metropolitana de Ciencias de la Educación, José Pedro Alessandri 774, Santiago, Chile
E-mail address: kschiu@umce.cl