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ROTATIONAL HYPERSURFACES WITH CONSTANT 2-MEAN CURVATURE IN $\mathbb{R}^{n+1}$

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Abstract. We study rotational hypersurfaces with constant 2-mean curvature in $\mathbb{R}^{n+1}$. We derive the ODE for the generating curves of such hypersurfaces, and find an integral expression for the inverse function of the solution.

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1. INTRODUCTION

In the field of submanifold geometry, principal curvatures tell us how the submanifold bends in each direction. Given a hypersurface, i.e. a submanifold of codimension one, the arithmetic mean of all principal curvatures is the mean curvature, while the product of principal curvatures is the Gauss-Kronecker curvature. It has been an active field of research to study various constraints on these notions of curvature. For example, the constant mean curvature (CMC) hypersurfaces in various ambient manifolds have been extensively studied. To explain our motivation, we mention a few results here. For rotational CMC surfaces in $\mathbb{R}^3$, Delaunay has proposed a beautiful classification theorem which indicates that the generating curves of these surfaces are formed geometrically by rolling a conic along a straight line without slippage [2]. In the 1980s, Wu-Yi Hsiang and Wenci Yu generalized Delaunay's theorem to rotational hypersurfaces in $\mathbb{R}^n$ [3, 4].

Other than the mean curvature and Gauss-Kronecker curvature, symmetric functions of principal curvatures are also interesting. Fixing an integer r between one and the dimension of the hypersurface, the r-th symmetric function of the principal curvatures is called (unnormalized) "r-mean curvature". In this paper, we consider rotational hypersurfaces in Euclidean space with constant 2-mean curvature. It can be seen as a variant of Hsiang and Yu's work mentioned above. Our main result is the following theorem:

Theorem 1. *Let $M \subset \mathbb{R}^{n+1}$ be a rotational hypersurface with constant $2-$mean curvature K such that its generating curve γ is a graph over the axis of rotation. Let $\gamma(s) = (\varphi(s), \psi(s))$ be a parametrization of the generating curve, where $\varphi(s)$ is the radius of the meridian $(n-1)$-sphere, $\psi(s)$ is the height function and s is the arclength parameter. Moreover set $u(s) = \varphi(s)^{n/2}$. Then we have the following local expression for the inverse function of $u(s)$:*

$$s = \int \pm \frac{du}{\sqrt{\frac{n}{n-1}\left(\frac{n(n-1)}{4}u^{2-4/n} - \frac{K}{2}u^2 + C\right)}}. \tag{1.1}$$

Here the sign of the integrand agrees with the sign of φ', and C is a real constant.

We note that compact embedded hypersurfaces with constant r-mean curvature have been characterized in [5], while complete ones have been studied in [1,6]. However, our examples are in general incomplete and thus are not covered by their results.

2. Rotational Hypersurfaces and its Curvatures

We set up notations and state the formulae for principal curvatures and 2-mean curvature of a rotational hypersurface in $\mathbb{R}^{n+1}$. Throughout the paper, we take $n \geq 3$ to avoid the trivial cases. The definitions of principal curvatures and higher order mean curvatures will be provided in the Appendix. Let $x_1, x_2, \ldots, x_{n+1}$ denote the standard coordinates of $\mathbb{R}^{n+1}$ and we assume that x_{n+1} is the axis of rotation. Let $f\colon \mathbb{R} \to (0, +\infty)$ be a smooth function.

Definition 1. A hypersurface M is called a *Rotational Hypersurface* if it is produced by rotating the *generating curve* $x_1 = f(x_{n+1})$ in the x_1x_{n+1}-plane around the x_{n+1} axis. It is characterized by the following equation

$$f(x_{n+1})^2 = \sum_{i=1}^{n} x_i^2.$$

Note that $f(x_{n+1})$ is the radius of the horizontal subsphere at height x_{n+1}. Throughout this paper, M will always denote a rotational hypersurface in $\mathbb{R}^{n+1}$ unless otherwise stated.

We choose an appropriate parametrization of the generating curve to facilitate the calculation. Let $\varphi(s)$ denote the radius of the $n-1$ dimensional hypersphere and $\psi(s)$ denote the corresponding height. We choose the parameter s to be the arclength parameter, that is, $\varphi'^2 + \psi'^2 = 1$. Under the above parametrization, the generating curve $x_1 = f(x_{n+1})$ is parametrized as $(x_1, x_{n+1}) = (\varphi(s), \psi(s))$. Note that $\varphi(s) \geq 0$ since it is the radius, and we require $\psi'(s) \geq 0$ so that the generating curve is a graph over the x_{n+1} axis.

We use the hypersphere coordinate $(\varphi, \theta_1, \ldots, \theta_{n-1})$ to parametrize the rotational hypersurface. The position vector field of rotational hypersurface M can be written as

$$\vec{r}(\varphi,\theta_1,\ldots,\theta_{n-1}) = (\varphi\cos\theta_1\cdots\cos\theta_{n-1}, \varphi\cos\theta_1\cdots\cos\theta_{n-2}\sin\theta_{n-1}, \ldots, \varphi\cos\theta_1\sin\theta_2, \varphi\sin\theta_1, \psi) \tag{2.1}$$

where $\theta_1 \in [-\pi/2, \pi/2]$ and $\theta_i \in [0, 2\pi]$ for $i = 2,3,\ldots,n-1$. Note that ψ can be expressed in terms of φ since $\varphi'^2 + \psi'^2 = 1$.

Under the above parametrization, the principal curvatures and the 2-mean curvature of M are:

Theorem 2. *The principal curvatures $k_1,\ldots,k_n$ of M are given below:*

(1) $k_1 = -\frac{\varphi''}{\psi'}$;

(2) $k_i = \frac{\psi'}{\varphi}$ *for* $i = 2,3,\ldots,n$.

Theorem 3. *The* 2*-mean curvature K of M is given below:*

$$K = \sum_{1\le i<j\le n} k_i k_j = -(n-1)\frac{\varphi''}{\varphi} + \binom{n-1}{2}\frac{1-\varphi'^2}{\varphi^2}. \tag{2.2}$$

Detailed calculations can be found in the Appendix.

3. Constant 2-mean Curvature

In the previous section, if we require the 2-mean curvature to be a constant K, then we obtain the following ODE:

$$(n-1)\varphi''\varphi + \binom{n-1}{2}\varphi'^2 + K\varphi^2 = \binom{n-1}{2}. \tag{3.1}$$

Set $m = \binom{n-1}{2}/(n-1) + 1 = n/2$, then

$$\frac{(\varphi^m)''}{\varphi^{m-2}} = \frac{m}{n-1}\left((n-1)\varphi''\varphi + \binom{n-1}{2}\varphi'^2\right). \tag{3.2}$$

Thus combining (3.1) and (3.2) we have

$$\frac{n-1}{m}(\varphi^m)'' + K\varphi^m = \binom{n-1}{2}\varphi^{m-2}.$$

Set $u = \varphi^m$, then:

$$\frac{n-1}{m}u'' + Ku - \binom{n-1}{2}u^{(m-2)/m} = 0.$$

Multiply both sides by u' and then integrate:

$$\frac{n-1}{2m}u'^2 = \binom{n-1}{2}\frac{m}{2m-2}u^{2-2/m} - \frac{K}{2}u^2 + C.$$

Finally solving this equation by separation of variables gives immediately (1.1) of Theorem 1.

4. Appendix

In the Appendix we provide essential definitions and notations about the curvatures of hypersurfaces in $\mathbb{R}^{n+1}$ and carry out the calculation. All notations are defined in the Euclidean Space $\mathbb{R}^{n+1}$.

A hypersurface M is a codimension one submanifold of $\mathbb{R}^{n+1}$. Let U be a domain in $\mathbb{R}^n$ and

$$\vec{r}\colon U \to M \subset \mathbb{R}^{n+1}, \quad \vec{r} = \vec{r}(x_1, x_2, \ldots, x_n)$$

be a local coordinate chart of M. We call $\vec{r}$ the *position vector field* of M in $\mathbb{R}^{n+1}$.

The *tangent vectors* of M are

$$\frac{\partial \vec{r}}{\partial x_1}, \frac{\partial \vec{r}}{\partial x_2}, \ldots, \frac{\partial \vec{r}}{\partial x_n}.$$

The vector $\vec{n}$ of length one that is perpendicular to all tangent vectors of M is the *unit normal vector* of M.

Definition 2 (First Fundamental Form)**.** Denote the first order derivatives by $\vec{r}_i = \frac{\partial \vec{r}}{\partial x_i}$. The *first fundamental form* of M is given below:

$$I = \begin{bmatrix} \vec{r}_1 \cdot \vec{r}_1 & \vec{r}_1 \cdot \vec{r}_2 & \cdots & \vec{r}_1 \cdot \vec{r}_n \\ \vec{r}_2 \cdot \vec{r}_1 & \vec{r}_2 \cdot \vec{r}_2 & \cdots & \vec{r}_2 \cdot \vec{r}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{r}_n \cdot \vec{r}_1 & \vec{r}_n \cdot \vec{r}_2 & \cdots & \vec{r}_n \cdot \vec{r}_n \end{bmatrix}$$

Definition 3 (Second Fundamental Form)**.** Denote the second order derivatives by $\vec{r}_{i,j} = \frac{\partial^2 \vec{r}}{\partial x_i \partial x_j}$. The *second fundamental form* of M is given below:

$$II = \begin{bmatrix} \vec{r}_{1,1} \cdot \vec{n} & \vec{r}_{1,2} \cdot \vec{n} & \cdots & \vec{r}_{1,n} \cdot \vec{n} \\ \vec{r}_{2,1} \cdot \vec{n} & \vec{r}_{2,2} \cdot \vec{n} & \cdots & \vec{r}_{2,n} \cdot \vec{n} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{r}_{n,1} \cdot \vec{n} & \vec{r}_{n,2} \cdot \vec{n} & \cdots & \vec{r}_{n,n} \cdot \vec{n} \end{bmatrix}.$$

Definition 4 (Principal Curvature)**.** Let matrix $A = -II \cdot I^{-1}$ where I^{-1} denotes the inverse matrix of I. The n eigenvalues of matrix A are the *principal curvatures* of M, denoted as $k_1, ..., k_n$.

Definition 5 (r-mean Curvature)**.** For an integer r with $1 \le r \le n$, the (unnormalized) r-mean Curvature H_r of M is the r-th symmetric function of the n principal curvatures. That is, $H_r = \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} \prod_{j=1}^{r} k_{i_j}$.

We use again the hypersphere coordinate (2.1) to parametrize a rotational hypersurface M. Under this parametrization, we can compute the tangent vectors, unit normal vector, first fundamental form, second fundamental form, principal curvatures, and 2-mean curvature of M as below.

Lemma 1. *Let* $\vec{r}_\varphi = \frac{\partial \vec{r}}{\partial \varphi}$ *and* $\vec{r}_i = \frac{\partial \vec{r}}{\partial \theta_i}$*. The tangent vectors of M are given below:*

$$\begin{aligned}
\vec{r}_\varphi &= (\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-1},\dots,\cos\theta_1\sin\theta_2,\sin\theta_1,\frac{\psi'}{\varphi'}),\\
\vec{r}_1 &= (-\varphi\sin\theta_1\cos\theta_2\cdots\cos\theta_{n-1},\dots,-\varphi\sin\theta_1\sin\theta_2,\varphi\cos\theta_1,0),\\
\vec{r}_2 &= (-\varphi\cos\theta_1\sin\theta_2\cos\theta_3\cdots\cos\theta_{n-1},\dots,-\varphi\cos\theta_1\cos\theta_2,0,0),\\
&\vdots\\
\vec{r}_{n-1} &= (-\varphi\cos\theta_1\cdots\cos\theta_{n-2}\sin\theta_{n-1},\varphi\cos\theta_1\cdots\cos\theta_{n-2}\cos\theta_{n-1},0,\dots,0).
\end{aligned}$$

Proof. Notice that $\frac{\partial\psi}{\partial\varphi} = \frac{\frac{\partial\psi}{\partial s}}{\frac{\partial\varphi}{\partial s}} = \frac{\psi'}{\varphi'}$, we can derive the tangent vectors by computing the first partial derivatives of the position vector field $\vec{r}(\varphi,\theta_1,\dots,\theta_{n-1})$ with respect to $\theta_1,\theta_2,\dots,\theta_{n-1}$ respectively. □

Lemma 2. *The Unit Normal Vector of M is given below:*

$$\begin{aligned}
\vec{n} = \psi'(&\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-1},\cos\theta_1\cdots\cos\theta_{n-2}\sin\theta_{n-1},\\
&\dots,\cos\theta_1\sin\theta_2,\sin\theta_1,-\frac{\varphi'}{\psi'}).
\end{aligned}$$

Proof. We only need to show that $\vec{n}\cdot\vec{r}_\varphi = 0$ and $\vec{n}\cdot\vec{r}_i = 0$ for $i = 1,\dots,n-1$. Let $x_{a,b}$ denote the value of the b^{th} coordinate of $\vec{r}_a$. First consider the value of $\vec{n}\cdot\vec{r}_\varphi$:

$$\begin{aligned}
\vec{n}\cdot\vec{r}_\varphi =&\psi'\left(\sum_{i=1}^{n}(x_{\varphi,i})^2 + \frac{\psi'}{\varphi'}\cdot(-\frac{\varphi'}{\psi'})\right)\\
=&\psi'[(\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-1})^2+\cdots+(\cos\theta_1\sin\theta_2)^2+\sin\theta_1^2-1]\\
=&\psi'[(\cos\theta_1\cdots\cos\theta_{n-2})^2+\cdots+(\cos\theta_1\sin\theta_2)^2+\sin\theta_1^2-1]\\
=&\psi'[\sin\theta_1^2+\cos\theta_1^2-1]\\
=&0.
\end{aligned}$$

From the above equation, we get

$$\sum_{i=1}^{k}(x_{\varphi,i})^2 = (\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-k})^2.$$

Notice that:

(1) $x_{i,j} = -\varphi x_{\psi,j}\tan\theta_i$ for $j = 1,2,\dots,n-i$;
(2) $x_{i,j} = \varphi x_{\psi,j}\cot\theta_i$ for $j = n-i+1$;
(3) $x_{i,j} = 0$ for $j > n-i+1$;
(4) The first n coordinates of $\vec{n}$ and $\vec{r}_\varphi$ are identical.

We get

$$\begin{aligned}\vec{n}\cdot\vec{r}_i &= \psi'\left(\sum_{j=1}^{n-i} x_{i,j}x_{\varphi,j} + x_{i,n-i+1}x_{\varphi,n-i}\right)\\ &= \psi'\left(-\varphi\tan\theta_i \sum_{j=1}^{n-i}(x_{\psi,j})^2 + \varphi\cot\theta_i(\cos\theta_1\cdots\cos\theta_{i-1}\sin\theta_i)^2\right)\\ &= \psi'\left(-\varphi\tan\theta_i(\cos\theta_1\cdots\cos\theta_i)^2 + \varphi\cot\theta_i(\cos\theta_1\cdots\cos\theta_{i-1}\sin\theta_i)^2\right) = 0.\end{aligned}$$

Moreover, it is clear that

$$|\vec{n}| = \psi'\sqrt{1+(-\frac{\varphi'}{\psi'})^2} = \sqrt{\varphi'^2+\psi'^2} = 1.$$

Therefore, $\vec{n}$ is indeed the Unit Normal Vector of M. □

Lemma 3. *The First Fundamental Form of M is a diagonal matrix in the form below:*

$$I = \begin{bmatrix} |\vec{r}_\varphi|^2 & \cdots & 0 & 0\\ \vdots & |\vec{r}_1|^2 & \cdots & 0\\ 0 & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & |\vec{r}_{n-1}|^2 \end{bmatrix}.$$

Proof. From Lemma 2, we know that

$$\vec{r}_\varphi\cdot\vec{r}_i = \psi'^{-1}\cdot\vec{n}\cdot\vec{r}_i = 0.$$

So it remains to be shown that

$$\vec{r}_i\cdot\vec{r}_j = 0 \quad (i\neq j).$$

Assume that $i > j$. From Lemma 1 we get

$$\begin{aligned}\vec{r}_i\cdot\vec{r}_j &= \sum_{k=1}^{n-i} x_{i,k}x_{j,k} + x_{i,n-i}x_{j,n-i+1}\\ &= \varphi^2\tan\theta_i\tan\theta_j\sum_{k=1}^{n-i}(x_{\psi,i})^2 - \varphi^2\cot\theta_i\tan\theta_j(\cos\theta_1\cdots\cos\theta_{i-1}\sin\theta_i)^2\\ &= \varphi^2\tan\theta_j[\tan\theta_i(\cos\theta_1\cdots\cos\theta_i)^2 - \cot\theta_i(\cos\theta_1\cdots\cos\theta_{i-1}\sin\theta_i)^2]\\ &= 0.\end{aligned}$$

The above equation indicates that I is a diagonal matrix as stated in the theorem. □

Lemma 4. *Let $\vec{r}_{x,y}$ denotes $\frac{\partial^2 \vec{r}}{\partial x \partial y}$ where θ_i is replaced by i. The Second Fundamental Form of M is another diagonal matrix in the form below:*

$$II = \begin{bmatrix} \vec{r}_{\varphi,\varphi} \cdot \vec{n} & \cdots & 0 & 0 \\ \vdots & \vec{r}_{1,1} \cdot \vec{n} & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \vec{r}_{n-1,n-1} \cdot \vec{n} \end{bmatrix}.$$

Proof. From Lemma 1 and the labels in Lemma 2, we can further derive the second derivatives as below:

(1) $\vec{r}_{\varphi,\varphi} = (0, 0, \ldots, 0, -\frac{\varphi''}{\varphi'^3 \psi'})$;
(2) $\vec{r}_{i,i} = (-\varphi x_{\varphi,1},\ -\varphi x_{\varphi,2},\ \ldots,\ -\varphi x_{\varphi,n-i+1},\ 0, \ldots, 0)$;
(3)

$$\begin{aligned} \vec{r}_{\varphi,i} = \vec{r}_{i,\varphi} = (&-x_{\varphi,1} \tan\theta_i,\ -x_{\varphi,2} \tan\theta_i, \ldots,\ -x_{\varphi,n-i} \tan\theta_i, \\ &x_{\varphi,n-i+1} \cot\theta_i,\ 0, \ldots, 0); \end{aligned}$$

(4)

$$\begin{aligned} \vec{r}_{i,j} = \vec{r}_{j,i} = (&\varphi x_{\varphi,1} \tan\theta_i \tan\theta_j, \ldots,\ \varphi x_{\varphi,n-i} \tan\theta_i \tan\theta_j, \\ &-\varphi \cot\theta_i \tan\theta_j, 0, \ldots, 0) \quad \text{for } i > j. \end{aligned}$$

So, we only need to prove the inner product of $\vec{n}$ and the derivatives in Items (3) and (4) is 0. From Lemma 2, we get

$$\begin{aligned} \vec{r}_{\varphi,i} \cdot \vec{n} &= \psi'[-\tan\theta_i \sum_{k=1}^{n-i} x_{\varphi,k}^2 + \cot\theta_i x_{\varphi,n-i+1}^2] \\ &= \psi'[-\tan\theta_i (\cos\theta_1 \cdots \cos\theta_i)^2 + \cot\theta_i (\cos\theta_1 \cdots \cos\theta_{i-1} \sin\theta_i)^2] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \vec{r}_{i,j} \cdot \vec{n} &= \psi'[\varphi \tan\theta_i \tan\theta_j \sum_{k=1}^{n-i} x_{\varphi,k}^2 + \cot\theta_i \tan\theta_j x_{\varphi,n-i+1}^2] \\ &= \psi' \varphi \tan\theta_j [\tan\theta_i (\cos\theta_1 \cdots \cos\theta_i)^2 - \cot\theta_i (\cos\theta_1 \cdots \cos\theta_{i-1} \sin\theta_i)^2] \\ &= 0. \end{aligned}$$

The above computation indicates that II is a diagonal matrix in the proposed form. □

Now we can prove Theorem 2.

Proof. From Lemma 1, we can compute the entries in I as below:

(1) $|\vec{r}_\varphi|^2 = 1 + \frac{\psi'}{\varphi'} = \frac{1}{\varphi'^2}$
(2) $|\vec{r}_i|^2 = \varphi^2 \prod_{a=1}^{i-1} \cos\theta_a^2$ (Assume that $\cos\theta_0 = 1$)

Similarly, we can compute the elements in II as below:

(1) $\vec{r}_{\varphi,\varphi}\cdot\vec{n}=\frac{\varphi''}{\varphi'^2\psi'}$

(2) $\vec{r}_{i,i}\cdot\vec{n}=-\varphi\psi'\prod_{a=1}^{i-1}\cos\theta_a^2$ (Assume that $\cos\theta_0=1$)

Then,

$$
\begin{aligned}
A &= -II\cdot I^{-1}\\
&= \begin{bmatrix} -\frac{\varphi''}{\varphi'^2\psi'} & \cdots & 0 & 0\\ \vdots & \varphi & \cdots & 0\\ 0 & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varphi\prod_{a=1}^{n-1}\cos\theta_a^2 \end{bmatrix}\cdot\begin{bmatrix} \varphi'^2 & \cdots & 0 & 0\\ \vdots & \frac{1}{\varphi^2} & \cdots & 0\\ 0 & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{1}{\varphi^2\prod_{a=1}^{n-1}\cos\theta_a^2} \end{bmatrix}\\
&= \begin{bmatrix} -\frac{\varphi''}{\psi'} & \cdots & 0 & 0\\ \vdots & \frac{\psi'}{\varphi} & \cdots & 0\\ 0 & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{\psi'}{\varphi} \end{bmatrix}
\end{aligned}
$$

Clearly, the principal curvatures are diagonal entries. □

Using the formula for principal curvatures, we have derived expression (2.2) in Theorem 3 of the 2-mean curvature of M under the φ and ψ parametrization.

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