# Sub-lattice entanglement in an exactly solvable anyon-like spin ladder 

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#### Abstract

We introduce an integrable spin ladder model and study its exact solution, correlation functions, and entanglement properties. The model supports two particle types (corresponding to the even and odd sub-lattices), such that the scattering phases are constants: particles of the same type scatter as free fermions, whereas the inter-particle phase shift is a constant tuned by an interaction parameter. Therefore, the spin ladder bears similarities with anyonic models. We present exact results for the spectrum and correlation functions, and we study the sub-lattice entanglement by numerical means.


## INTRODUCTION

Classical simulations of quantum many body systems are limited by the growth of entanglement, both in equilibrium and out-of-equilibrium situations. This motivates the study of models where exact solutions can be found, at least for certain physical quantities. Important classes of solvable many body systems are the free theories, the one dimensional integrable models [1, 2], and also the recently discovered dual unitary quantum circuits [3].

Integrable models have been studied extensively over many decades, and in the last 10 years their nonequilibrium dynamics also received considerable attention. Now it is understood that isolated integrable models equilibrate to the Generalized Gibbs Ensemble [4, 5], and their transport properties are described by Generalized Hydrodynamics [6]. However, these results describe only the large time limit for large system sizes, and generally they lack a completely rigorous proof. Furthermore, they don't provide access to certain exotic features of the dynamics, such as anomalous current fluctuations [7-11].

This motivates the study of selected integrable models with even simpler dynamics, where there is some interaction in the system, nevertheless closed form results can be derived for the real time evolution of certain physical quantities. Such models include the Rule54 cellular automaton [12-17], the box-ball systems [10, 18, 19], classical cellular automata of the XXC type [9, 20-24], nontrivial strong coupling limits of known models [25-27] including the folded XXZ model [28-31], or quantum circuits that are both integrable and dual-unitary [32]. A common property of these models is that the scattering of the particles (either classical or quantum) is rather simple compared to a generic integrable model.

What are the simplest possible interacting $S$-matrices for integrable (Hermitian) quantum spin chains? An especially simple case is when the $S$-matrix is constant (momentum-independent). For a single particle species the only possibilities are phase factors $\pm 1$ corresponding
to free bosons/fermions. However, for multiple particle species we can explore a wider range of options.

In this paper we introduce a new spin ladder model, which supports two particle types which propagate on the two legs of the spin ladder. In this model the scattering phases are constant statistical factors. This bears strong similarity with anyonic models [33-37], or parafermionic chains [38-40]. A crucial difference is that in our model the anyon-like phases arise from a local interaction defined in the standard spin basis. This provides a unique opportunity to study the entangling effects of constant scattering phases.

## THE MODEL AND ITS INTEGRABILITY

We consider a spin chain made of qubits, using the notation $X_{j}, Y_{j}$ and $Z_{j}$ for operators given by the Pauli matrices, acting on site $j=1 \ldots L$, where $L$ (even) is the length of the chain.

Our model can be seen as a spin chain or as a spin ladder in a zig-zag geometry, see Fig. 1. We consider a hopping model on the ladder, such that the two legs have a minimal coupling between them. Particles can propagate on the two legs separately, but the local hopping phases on one leg also depend on the local occupation numbers on the other leg, leading to a model Hamiltonian with three site interaction:

$$
\begin{equation*}
H(\gamma)=\sum_{j=1}^{L / 2} h_{2 j, 2 j+1,2 j+2}(\gamma)+h_{2 j+1,2 j+2,2 j+3}(-\gamma) \tag{1}
\end{equation*}
$$

Here $h_{a, b, c}(\gamma)$ is the Hamiltonian density with a real coupling constant $\gamma \in \mathbb{R}$, given by

$$
\begin{equation*}
h_{1,2,3}(\gamma)=-\left[\sigma_{1}^{-} e^{i \gamma Z_{2}} \sigma_{3}^{+}+\sigma_{1}^{+} e^{-i \gamma Z_{2}} \sigma_{3}^{-}\right] \tag{2}
\end{equation*}
$$

where $\sigma_{j}^{ \pm}=\left(X_{j} \pm i Y_{j}\right) / 2$ are the standard raising/lowering operators. The Hamiltonian is invariant
with respect to a global spin-flip in the $Z$-basis, and it is also space reflection invariant for reflections that also exchange the two sub-lattices.

In the following interpret the up spins as a vacuum and the down spins as particles. The Hamiltonian generates hopping on the two sub-lattices, so that the sub-lattice magnetizations

$$
\begin{equation*}
S^{A}=\sum_{j=1}^{L / 2} Z_{2 j}, \quad S^{B}=\sum_{j=1}^{L / 2} Z_{2 j+1} \tag{3}
\end{equation*}
$$

are separately conserved. Here and in the following $A$ and $B$ stand for the even and odd sub-lattices, respectively. Furthermore the hopping phase is $e^{ \pm i \gamma}$ depending on the position and the occupation of the sites involved.


Figure 1. The geometry of the spin zigzag ladder. $A$ and $B$ denote the two sub-lattices on which the particles can hop.

To our best knowledge this model has not yet appeared in the literature. Similar models include the so-called extended XX model [41, 42], a super-symmetric hopping model treated in [35], or the Bariev model [43]. However, our model is different, as the exact solution below shows.

The model has two free fermion points, with different physical interpretation. For $\gamma=0$ the interaction between the two legs disappears, leading to two uncoupled XX spin chains. For $\gamma=\pi / 2$ the model becomes a special case of the extended XX models [42], which can be solved by a single Jordan-Wigner transformation, see below.

The model is integrable for any coupling $\gamma$. It has an infinite set of conserved charges, which are given by a diagonal dressing of the known conserved charges of the XX spin chains [44]. The charges are organized into four infinite families, corresponding to the two sub-lattices and two "chiralities". The charges can be expressed in terms of densities as

$$
\begin{equation*}
Q_{\alpha}^{A+}=\sum_{j=1}^{L / 2} q_{\alpha}^{A+}(2 j), \quad Q_{\alpha}^{B+}=\sum_{j=1}^{L / 2} q_{\alpha}^{B+}(2 j+1) \tag{4}
\end{equation*}
$$

together with $Q_{\alpha}^{A-}=\left(Q_{\alpha}^{A+}\right)^{\dagger}$ and $Q_{\alpha}^{B-}=\left(Q_{\alpha}^{B+}\right)^{\dagger}$. The index $\alpha$ denotes the range of the given operator density.

The shortest charge densities appear for $\alpha=3$ and they are simply just terms from the Hamiltonian:

$$
\begin{align*}
q_{3}^{A+}(2 k) & =\sigma_{2 k}^{+} D_{2 k+1}^{\dagger} \sigma_{2 k+2}^{-} \\
q_{3}^{B+}(2 k+1) & =\sigma_{2 k+1}^{+} D_{2 k+2} \sigma_{2 k+3}^{-} \tag{5}
\end{align*}
$$

where we defined

$$
\begin{equation*}
D_{j}=e^{i \gamma Z_{j}}=\cos (\gamma)+i \sin (\gamma) Z_{j} \tag{6}
\end{equation*}
$$

Higher charges are constructed by a mixed diagonal dressing of the hopping terms of the type $\sigma_{k}^{+} \sigma_{k+\alpha-1}^{-}$. The dressing is such that for each site between $k$ and $k+\alpha-1$ we include a $Z$ operator if the site is from the same sub-lattice, and a $D$ operator otherwise. For example, for range $\alpha=5$ we get

$$
\begin{align*}
q_{5}^{A+}(2 k) & =\sigma_{2 k}^{+} D_{2 k+1}^{\dagger} Z_{2 k+2} D_{2 k+3}^{\dagger} \sigma_{2 k+4}^{-} \\
q_{5}^{B+}(2 k+1) & =\sigma_{2 k+1}^{+} D_{2 k+2} Z_{2 k+3} D_{2 k+4} \sigma_{2 k+5}^{-} \tag{7}
\end{align*}
$$

together with $q_{5}^{A-}(2 k)=\left(q_{5}^{A+}(2 k)\right)^{\dagger}$ and $q_{5}^{B-}(2 k+1)=$ $\left(q_{5}^{B+}(2 k+1)\right)^{\dagger}$. Higher charges can be constructed in an analogous way. The commutativity of all of these charges is proven as follows. First of all, all charges belonging to either sub-lattice $A$ or $B$ necessarily commute with each other, because they are just diagonal dressings of the charges of an XX model localized on one of the sublattices [44]. The commutativity of the charges corresponding to different sub-lattices is less obvious. However, in this case the charge densities actually commute, for example direct computation gives

$$
\begin{equation*}
\left[q_{3}^{A+}(2 k), q_{3}^{B+}(2 l+1)\right]=0 \tag{8}
\end{equation*}
$$

for all $k, l$. A more complete proof can be given using a similarity transformation discussed below.

The existence of an infinite family of commuting charges implies that the model is integrable and it has a completely elastic and factorized $S$-matrix $[45,46]$. We derive this $S$-matrix below. The charges above can be embedded into a transfer matrix constructed from local Lax operators, using the framework of [47], but we do not treat this approach here.

## THE SOLUTION OF THE MODEL

For $\gamma=0$ the model can be solved by two independent Jordan-Wigner (JW) transformations performed on the two sub-lattices. It is then a natural idea to construct a generalized JW transformation also for finite $\gamma$. To this order let us consider the model with free boundary conditions (or alternatively, a half-infinite chain). We introduce creation and annihilation operators for the two sub-lattices as

$$
\begin{align*}
c^{A}(2 j) & =D_{1} Z_{2} D_{3} Z_{4} \ldots Z_{2 j-2} D_{2 j-1} \sigma_{2 j}^{+} \\
c^{B}(2 j+1) & =Z_{1} D_{2}^{\dagger} Z_{3} D_{4}^{\dagger} \ldots Z_{2 j-1} D_{2 j}^{\dagger} \sigma_{2 j+1}^{+} \tag{9}
\end{align*}
$$

together with their adjoints.
Direct computation shows that

$$
\begin{equation*}
\left\{c^{A}(2 j), c^{A}(2 k)\right\}=0, \quad\left\{c^{A}(2 j), c^{A \dagger}(2 k)\right\}=\delta_{j k} \tag{10}
\end{equation*}
$$

and similarly for the $B$ sub-lattice. However, for the cross-commutation terms we get for example

$$
\begin{equation*}
c^{A}(2 j) c^{B}(2 k+1)=e^{2 i \gamma} c^{B}(2 k+1) c^{A}(2 j), \quad j<k \tag{11}
\end{equation*}
$$

This means that the model is partially anyonic. The Hamiltonian can be rewritten for all $\gamma$ as

$$
\begin{equation*}
H(\gamma)=-\sum_{j=1}^{L / 2-1}\left[\left(c_{2 j+2}^{A}\right)^{\dagger} c_{2 j}^{A}+\left(c_{2 j+3}^{B}\right)^{\dagger} c_{2 j+1}^{B}\right]+c . c . \tag{12}
\end{equation*}
$$

This means that all the interaction is included now in the definition of the creation/annihilation operators. This is similar to parafermionic models [38-40], but in our case the non-trivial commutation relations arise from the local interaction in the model and not from pre-defined operator algebras.

We recover the usual JW transformation in two special cases. If $\gamma=0$ then we get two independent JW transformations on the two sub-lattices. In contrast, for $\gamma=\pi / 2$ we get a single JW transformation on the whole chain. Thus the model interpolates between two uncoupled XX chains and a single XX chain, although in the latter case the Hamiltonian is actually a higher charge of the usual XX model [42].

While for the free cases the model can be solved by Fourier transform, this does not work for generic $\gamma$ due to the mixed anyon-like interactions. In these cases we can use the fact that the phase factors do not depend on the momenta of the particles, only on the relative position of the particles on the two sub-lattices. This leads to a simple explicit construction of the wave functions.

Let us consider a state with $N_{A}$ and $N_{B}$ particles on the two sub-lattices. Lattice momenta of the particles will be denoted as $p_{j}^{A}$ and $p_{k}^{B}$. For writing down the wave function we will use coordinates $\mathbf{a}=\left\{a_{j}\right\}_{j=1, \ldots, N_{A}}$ and $\mathbf{b}=\left\{b_{k}\right\}_{j=1, \ldots, N_{B}}$ which run over the even and odd numbers, respectively. First we consider periodic boundary conditions. The wave function is then given by

$$
\begin{equation*}
\Psi(\mathbf{a}, \mathbf{b})=\operatorname{det}(\mathcal{A}) \operatorname{det}(\mathcal{B}) \prod_{a<b} e^{-i \gamma} \prod_{b<a} e^{i \gamma} \tag{13}
\end{equation*}
$$

Here $\mathcal{A}$ and $\mathcal{B}$ are matrices of sizes $N_{A} \times N_{A}$ and $N_{B} \times N_{B}$, respectively. They describe free fermionic wave functions localized on the two sub-lattices, with components given by

$$
\begin{equation*}
\mathcal{A}_{j k}=e^{i\left(p_{j}^{A}-\gamma\right) a_{k} / 2}, \quad \mathcal{B}_{j k}=e^{i\left(p_{j}^{B}+\gamma\right) b_{k} / 2} \tag{14}
\end{equation*}
$$

The interpretation of this wave function is the following: The model supports two particle species moving on the two sub-lattices, with momenta $p_{j}^{A}$ and $p_{j}^{B}$. The scattering in the model is factorized and diagonal, with momentum independent phase shifts given by

$$
\begin{equation*}
\mathcal{S}_{A A}=\mathcal{S}_{B B}=-1, \quad \mathcal{S}_{A B}=\left(\mathcal{S}_{B A}\right)^{-1}=e^{2 i \gamma} \tag{15}
\end{equation*}
$$

The phase shifts reflect the commutation relations (10)(11).

Periodicity implies that the momenta have to satisfy the Bethe equations

$$
\begin{align*}
e^{i p_{j}^{A} L / 2} & =(-1)^{N_{A}-1} e^{i \gamma\left(L / 2-2 N_{B}\right)} \\
e^{i p_{j}^{B} L / 2} & =(-1)^{N_{B}-1} e^{-i \gamma\left(L / 2-2 N_{A}\right)} \tag{16}
\end{align*}
$$

These equations are almost free: the only coupling between the two sets of momenta is simply just a twist, which depends on the overall particle numbers. The energy eigenvalues are then

$$
\begin{equation*}
E=\sum_{j=1}^{N_{A}} e\left(p_{j}^{A}\right)+\sum_{j=1}^{N_{B}} e\left(p_{j}^{B}\right) \tag{17}
\end{equation*}
$$

with $e(p)=-2 \cos (p)$.
Now we consider the model with free boundary conditions, and show that in this case the wave functions are found simply using a global similarity transformation. The diagonal operator

$$
\begin{equation*}
\mathcal{D}=\prod_{2 j<2 k+1} e^{i \gamma Z_{2 j} Z_{2 k+1} / 4} \prod_{2 j>2 k+1} e^{-i \gamma Z_{2 k+1} Z_{2 j} / 4} \tag{18}
\end{equation*}
$$

completely decouples the two legs of the ladder:

$$
\begin{equation*}
\mathcal{D} H(\gamma) \mathcal{D}^{-1}=H(0) \tag{19}
\end{equation*}
$$

This implies that the spectrum of the open chain is the same as that of two uncoupled XX chains for all $\alpha$. In this sense the model is free, and it belongs to the class of models investigated in [48, 49]. Nevertheless, the operator $\mathcal{D}$ is highly non-local, and it makes the two legs of the ladder highly entangled, both in equilibrium and out-ofequilibrium situations.

We also study the thermodynamic limit (TDL). We introduce root densities $\rho^{A}(p)$ and $\rho^{B}(p)$, and the limiting value of the ground state energy density becomes

$$
\begin{equation*}
\lim _{T D L} \frac{E}{L}=\int_{-\pi}^{\pi} \frac{d p}{4 \pi} e(p)\left(\rho^{A}(p)+\rho^{B}(p)\right) \tag{20}
\end{equation*}
$$

The ground state is given by the half filled state

$$
\rho^{A, B}(p)= \begin{cases}1 & \text { for }|p|<\pi / 2  \tag{21}\\ 0 & \text { for }|p|>\pi / 2\end{cases}
$$

The ground state energy density is $-2 / \pi$, which is identical to that of the XX model.

## CORRELATION FUNCTIONS

The coupling between the two sub-lattices makes them strongly entangled. We demonstrate this by computing a selected short range correlation function. Due to the similarity transformation (19) the correlation functions of $Z$ operators will be the same as in two uncoupled XX chains. Therefore, non-trivial information is seen in correlation functions with hopping terms. We choose the following connected correlation function:

$$
\begin{equation*}
C_{\Psi} \equiv\langle\Psi| \sigma_{0}^{-} Z_{1} \sigma_{2}^{+}|\Psi\rangle-\langle\Psi| \sigma_{0}^{-} \sigma_{2}^{+}|\Psi\rangle\langle\Psi| Z_{1}|\Psi\rangle \tag{22}
\end{equation*}
$$

Here $|\Psi\rangle$ is an arbitrary eigenstate with $N_{A}$ and $N_{B}$ particles on the two sub-lattices. A non-zero value of
$C_{\Psi}$ demonstrates the entanglement between the two sublattices.

We introduce the magnetization on the odd sub-lattice:

$$
\begin{equation*}
m_{B}=\langle\Psi| Z_{1}|\Psi\rangle=\frac{L-4 N_{B}}{L} \tag{23}
\end{equation*}
$$

A certain combination of the operators above is simply the density of a conserved charge (one term in the Hamiltonian), therefore we get

$$
\begin{equation*}
\langle\Psi| \cos (\gamma) \sigma_{0}^{-} \sigma_{2}^{+}+i \sin (\gamma) \sigma_{0}^{-} Z_{1} \sigma_{2}^{+}|\Psi\rangle=2 W \tag{24}
\end{equation*}
$$

with

$$
\begin{equation*}
W=\frac{1}{L} \sum_{j=1}^{N_{A}} e^{i p_{j}^{A}} \tag{25}
\end{equation*}
$$

Now we apply the Hellmann-Feynman theorem for the corresponding charge, from which we can obtain the mean values of the $\gamma$-derivative of the operators on the l.h.s. above. Combining this with the mean value above and with the Bethe equations (16) we get the result

$$
\begin{equation*}
C_{\Psi}=2 i \sin (\gamma) W\left(m_{B}^{2}-1\right) \tag{26}
\end{equation*}
$$

In the thermodynamic limit we get

$$
\begin{equation*}
W \rightarrow \int_{-\pi}^{\pi} \frac{d p}{4 \pi} \rho^{A}(p) e^{i p} \tag{27}
\end{equation*}
$$

This retains a finite value unless the root distribution is constant, therefore we obtain a finite correlation between the sub-lattices for almost all states.

The correlation function vanishes if either lattice is fully polarized, having magnetization equal to $\pm 1$. If the odd sub-lattice is polarized, then the vanishing is guaranteed by the factor $\left(m_{B}^{2}-1\right)$, whereas if the even sublattice is completely polarized, then $W=0$ (because for a fully polarized state $\rho$ is constant).

Analogous results can be obtained for combinations similar to (22).

## SUB-LATTICE ENTANGLEMENT

We also study the entanglement properties of the model, in both equilibrium and out-of-equilibrium situations. In the literature the most often studied entanglement is that of connected sub-systems [50-55]. However, it is expected that in this model the usual bipartite entanglement behaves very similar to that of the XX model. Therefore we focus on the sub-lattice entanglement, which is a highly non-trivial quantity that can be tuned by the coupling constant $\gamma$. Sub-lattice entanglement was studied earlier in a number of situations [56$61]$.

As before, let $A$ and $B$ denote the sites of the even and odd sub-lattices, and we define the sub-lattice (von Neumann) entanglement entropy as

$$
\begin{equation*}
S=-\operatorname{Tr}\left(\rho_{A} \log \rho_{A}\right) \tag{28}
\end{equation*}
$$

where $\rho_{A}=\operatorname{Tr}_{B} \rho$, with $\rho$ being the density matrix of the system, either in equilibrium or in an out-of-equilibrium process. The entanglement entropy is expected to be extensive, and we introduce the entropy density

$$
\begin{equation*}
s=\frac{S}{L} \tag{29}
\end{equation*}
$$

It is expected that $s$ should not depend on $L$ apart from minor finite size effects.

We numerically study $s$ in the ground state for different values of $\gamma$ and $L$, results are plotted in Fig. 2. We find that finite size effects are indeed small, and the entropy density obtains its maximum value at the free fermion point $\gamma=\pi / 2$.

We also consider non-equilibrium time evolution started from a selected initial state, namely the ferromagnetic state with polarization in the $x$-direction:

$$
\begin{equation*}
|X\rangle=\bigotimes_{j=1}^{L} \frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle) \tag{30}
\end{equation*}
$$

We study the real time evolution of $s$, results are plotted in Fig 2. It can be seen that the entanglement is indeed extensive, but now there are bigger finite size effects. For the largest system size $L=24$ we see that entanglement reaches a plateau relatively soon for all values of $\gamma$, with the height of the plateau behaving in a similar way as before: the maximum entanglement is seen for $\gamma=\pi / 2$. Qualitatively similar behaviour can also be found for other initial states.

## DISCUSSION

We introduced a new exactly solvable spin ladder, which is one of the simplest quantum integrable models with a tunable coupling between particles. The model interpolates between two free fermion points, in which the two sub-lattices are either uncoupled or maximally coupled. The entanglement between the sub-lattices was demonstrated analytically by an exact result for a correlation function, and numerically by the sub-lattice entanglement (28) defined in the real space basis, which was examined both in and out of equilibrium.

In the special case $\gamma=\pi / 2$ the entanglement can also be studied in terms of the fermionic degrees of freedom. It was pointed out in [62] that for free fermionic chains these two definitions of entanglement give generally different results. They only agree for connected sub-systems, because for disconnected sub-systems the Jordan-Wigner


Figure 2. Numerical results for the sub-lattice entanglement density $s=S / L$. Left: Ground state values as a function of the coupling $\gamma$ for different values of $L$. Middle: Time evolution of the entanglement in a quench problem (see main text), for different volumes and $\gamma=\pi / 2$. Right: Time evolution in a quench problem for $L=24$ and different values of $\gamma$.
transformation causes differences between the two types of entanglement. Our model can be regarded as an extreme example for this phenomenon: for $\gamma=\pi / 2$ the two sub-lattices are completely decoupled if one considers the fermions, see eq. (12). Therefore, the sub-lattice entanglement in terms of the fermions is exactly zero. In contrast, we find that the real space entanglement is non-zero, and in fact it is maximal for the free fermion point $\gamma=\pi / 2$ !

Finally we note that the model is partially anyonic for a generic $\gamma$, and it seems to be one of the simplest nontrivial scattering theories. This could lead to interesting applications, for example in the realization of interacting Bethe states in quantum computers [63-67].

Note added: After this work was completed we became aware of the recent work [68] which treats a closely related model. Our results about the integrability and exact solvability of the model (together with the exact result for a correlation function) appear to be new, whereas the results for sub-lattice entanglement are partly overlapping.

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[1] B. Sutherland, Beautiful Models (World Scientific Publishing Company, 2004).
[2] V. Korepin, N. Bogoliubov, and A. Izergin, Quantum inverse scattering method and correlation functions (Cam-
bridge University Press, 1993).
[3] L. Piroli, B. Bertini, J. I. Cirac, and T. Prosen, Exact dynamics in dual-unitary quantum circuits, Phys. Rev. B 101, 094304 (2020), arXiv:1911.11175 [cond-mat.statmech].
[4] L. Vidmar and M. Rigol, Generalized Gibbs ensemble in integrable lattice models, J. Stat. Mech. 6, 064007 (2016), arXiv:1604.03990 [cond-mat.stat-mech].
[5] F. H. L. Essler and M. Fagotti, Quench dynamics and relaxation in isolated integrable quantum spin chains, J. Stat. Mech. 6, 064002 (2016), arXiv:1603.06452 [cond-mat.quant-gas].
[6] A. Bastianello, B. Bertini, B. Doyon, and R. Vasseur, Introduction to the special issue on emergent hydrodynamics in integrable many-body systems, JSTAT 2022 (1), 014001.
[7] M. Žnidarič, Anomalous nonequilibrium current fluctuations in the Heisenberg model, Phys. Rev. B 90, 115156 (2014), arXiv:1405.5541 [cond-mat.str-el].
[8] Z̈. Krajnik, E. Ilievski, and T. Prosen, Absence of Normal Fluctuations in an Integrable Magnet, Phys. Rev. Lett. 128, 090604 (2022), arXiv:2109.13088 [cond-mat.statmech].
[9] Ž. Krajnik, J. Schmidt, V. Pasquier, E. Ilievski, and T. Prosen, Exact anomalous current fluctuations in a deterministic interacting model, Phys. Rev. Lett. 128, 160601 (2022), arXiv:2201.05126 [cond-mat.stat-mech].
[10] A. Kuniba, G. Misguich, and V. Pasquier, Current correlations, Drude weights and large deviations in a box-ball system, arXiv e-prints (2022), arXiv:2201.13126 [mathph .
[11] S. Gopalakrishnan, A. Morningstar, R. Vasseur, and V. Khemani, Theory of anomalous full counting statistics in anisotropic spin chains, arXiv e-prints (2022), arXiv:2203.09526 [cond-mat.stat-mech].
[12] A. Bobenko, M. Bordemann, C. Gunn, and U. Pinkall, On two integrable cellular automata, Comm. Math. Phys. 158, 127 (1993).
[13] B. Buča, K. Klobas, and T. Prosen, Rule 54: Exactly solvable model of nonequilibrium statistical mechanics, J. Stat. Mech. 2021, 074001 (2021), arXiv:2103.16543 [cond-mat.stat-mech].
[14] V. Alba, J. Dubail, and M. Medenjak, Operator Entanglement in Interacting Integrable Quantum Systems: The Case of the Rule 54 Chain, Phys. Rev. Lett. 122, 250603 (2019), arXiv:1901.04521 [cond-mat.stat-mech].
[15] K. Klobas and B. Bertini, Entanglement dynamics in

Rule 54: exact results and quasiparticle picture, SciPost Phys. 11, 107 (2021), arXiv:2104.04513 [cond-mat.statmech].
[16] K. Klobas and B. Bertini, Exact relaxation to Gibbs and non-equilibrium steady states in the quantum cellular automaton Rule 54, SciPost Phys. 11, 106 (2021), arXiv:2104.04511 [cond-mat.stat-mech].
[17] K. Klobas, B. Bertini, and L. Piroli, Exact thermalization dynamics in the "rule 54 " quantum cellular automaton, Phys. Rev. Lett. 126, 160602 (2021), arXiv:2012.12256 [cond-mat.stat-mech].
[18] D. Takahashi and J. Satsuma, A soliton cellular automaton, J. Phys. Soc. Jap. 59, 3514 (1990).
[19] A. Kuniba, G. Misguich, and V. Pasquier, Generalized hydrodynamics in box-ball system, J. Phys. A 53, 404001 (2020), arXiv:2004.01569 [math-ph].
[20] Z. Maassarani, The XXC models, Phys. Lett. A 244, 160 (1998), arXiv:solv-int/9712008 [nlin.SI].
[21] M. Medenjak, K. Klobas, and T. Prosen, Diffusion in Deterministic Interacting Lattice Systems, Phys. Rev. Lett. 119, 110603 (2017), arXiv:1705.04636 [cond-mat.statmech].
[22] M. Medenjak, V. Popkov, T. Prosen, E. Ragoucy, and M. Vanicat, Two-species hardcore reversible cellular automaton: matrix ansatz for dynamics and nonequilibrium stationary state, SciPost Physics 6, 074 (2019), arXiv:1903.10590 [cond-mat.stat-mech].
[23] T. Gombor and B. Pozsgay, Superintegrable cellular automata and dual unitary gates from Yang-Baxter maps, SciPost Phys. 12, 102 (2022), arXiv:2112.01854 [cond-mat.stat-mech].
[24] M. Medenjak, Operator spreading in quantum hardcore gases, arXiv e-prints (2022), arXiv:2201.00395 [cond-mat.stat-mech].
[25] B. Pozsgay, Quantum quenches and generalized Gibbs ensemble in a Bethe Ansatz solvable lattice model of interacting bosons, J. Stat. Mech. 10, 10045 (2014), arXiv:1407.8344 [cond-mat.stat-mech].
[26] B. Pozsgay and V. Eisler, Real-time dynamics in a strongly interacting bosonic hopping model: Global quenches and mapping to the XX chain, J. Stat. Mech. 5, 053107 (2016), arXiv:1602.03065 [cond-mat.stat-mech].
[27] E. Tartaglia, P. Calabrese, and B. Bertini, Real-time evolution in the Hubbard model with infinite repulsion, SciPost Phys. 12, 028 (2022), arXiv:2109.15305 [cond-mat.stat-mech].
[28] L. Zadnik and M. Fagotti, The Folded Spin-1/2 XXZ Model: I. Diagonalisation, Jamming, and Ground State Properties, SciPost Phys. Core 4, 10 (2021), arXiv:2009.04995 [cond-mat.stat-mech].
[29] L. Zadnik, K. Bidzhiev, and M. Fagotti, The Folded Spin$1 / 2$ XXZ model: II. Thermodynamics and hydrodynamics with a minimal set of charges, SciPost Phys. 10, 99 (2021), arXiv:2011.01159 [cond-mat.stat-mech].
[30] B. Pozsgay, T. Gombor, A. Hutsalyuk, Y. Jiang, L. Pristyák, and E. Vernier, An integrable spin chain with Hilbert space fragmentation and solvable real time dynamics, Phys. Rev. E 104, 044106 (2021), arXiv:2105.02252 [cond-mat.stat-mech].
[31] L. Zadnik, S. Bocini, K. Bidzhiev, and M. Fagotti, Measurement catastrophe and ballistic spread of charge density with vanishing current, arXiv e-prints (2021), arXiv:2111.06325 [quant-ph].
[32] G. Giudice, G. Giudici, M. Sonner, J. Thoenniss,
A. Lerose, D. A. Abanin, and L. Piroli, Temporal entanglement, quasiparticles and the role of interactions, arXiv e-prints (2021), arXiv:2112.14264 [cond-mat.stat-mech].
[33] T. M. Wright, M. Rigol, M. J. Davis, and K. V. Kheruntsyan, Nonequilibrium Dynamics of OneDimensional Hard-Core Anyons Following a Quench: Complete Relaxation of One-Body Observables, Phys. Rev. Lett. 113, 050601 (2014), arXiv:1312.4657 [cond-mat.quant-gas].
[34] Y. Hao, Y. Zhang, and S. Chen, Ground-state properties of hard-core anyons in one-dimensional optical lattices, Phys. Rev. A 79, 043633 (2009), arXiv:0901.1224 [cond-mat.str-el].
[35] P. Fendley and K. Schoutens, Cooper pairs and exclusion statistics from coupled free-fermion chains, J. Stat. Mech. 2007, 02017 (2007), arXiv:cond-mat/0612270 [cond-mat.stat-mech].
[36] A. Kundu, Exact solution of double $\delta$ function bose gas through an interacting anyon gas, Phys. Rev. Lett. 83, 1275 (1999).
[37] O. I. Patu, V. E. Korepin, and D. V. Averin, Correlation functions of one-dimensional lieb-liniger anyons, J. Phys. A 40, 14963-14984 (2007), arXiv:0707.4520 [cond-mat.stat-mech].
[38] P. Fendley, Free parafermions, J. Phys. A 47, 075001 (2014), arXiv:1310.6049 [cond-mat.stat-mech].
[39] D. Rossini, M. Carrega, M. Calvanese Strinati, and L. Mazza, Anyonic tight-binding models of parafermions and of fractionalized fermions, Phys. Rev. B 99, 085113 (2019), arXiv:1811.09422 [cond-mat.str-el].
[40] A. S. Mastiukova, D. V. Kurlov, V. Gritsev, and A. K. Fedorov, Free Fock parafermions in the tight-binding model with dissipation, arXiv e-prints (2022), arXiv:2203.03554 [cond-mat.quant-gas].
[41] M. Suzuki, Relationship among Exactly Soluble Models of Critical Phenomena. I*): 2D Ising Model, Dimer Problem and the Generalized XY-Model, Progr. Theor. Phys. 46, 1337 (1971).
[42] I. Titvinidze and G. I. Japaridze, Phase diagram of the spin- $1 / 2$ extended XY model, Eu. Phys. J. B 32, 383 (2003), arXiv:cond-mat/0211380 [cond-mat.str-el].
[43] R. Z. Bariev, Integrable spin chain with two- and threeparticle interactions, J. Phys. A 24, L549 (1991).
[44] M. Grabowski and P. Mathieu, Structure of the conservation laws in integrable spin chains with short range interactions, Annals Phys. 243, 299 (1995), arXiv:hepth/9411045 [hep-th].
[45] P. P. Kulish, Factorization of the classical and the quantum s matrix and conservation laws, Theor Math Phys 26, 132 (1976).
[46] G. Mussardo, Off-critical statistical models: Factorized scattering theories and bootstrap program, Phys. Rept. 218, 215 (1992).
[47] T. Gombor and B. Pozsgay, Integrable spin chains and cellular automata with medium-range interaction, Phys. Rev. E 104, 054123 (2021), arXiv:2108.02053 [nlin.SI].
[48] P. Fendley, Free fermions in disguise, Journal of Physics A Mathematical General 52, 335002 (2019), arXiv:1901.08078 [cond-mat.stat-mech].
[49] S. J. Elman, A. Chapman, and S. T. Flammia, Free fermions behind the disguise, Commun. Math. Phys. 388, 969-1003 (2021), arXiv:2012.07857 [quant-ph].
[50] P. Calabrese and J. Cardy, Quantum quenches in extended systems, J. Stat. Mech. 2007, P06008 (2007),
arXiv:0704.1880 [cond-mat.stat-mech].
[51] P. Calabrese and J. Cardy, Evolution of entanglement entropy in one-dimensional systems, J. Stat. Mech. 2005, 04010 (2005), arXiv:cond-mat/0503393 [cond-mat.statmech].
[52] M. Fagotti and P. Calabrese, Evolution of entanglement entropy following a quantum quench: Analytic results for the XY chain in a transverse magnetic field, Phys. Rev. A 78, 010306 (2008), arXiv:0804.3559 [cond-mat.statmech].
[53] V. Alba and P. Calabrese, Entanglement and thermodynamics after a quantum quench in integrable systems, PNAS 114, 7947 (2017), arXiv:1608.00614 [cond-mat.strel].
[54] V. Alba and P. Calabrese, Entanglement dynamics after quantum quenches in generic integrable systems, SciPost Phys. 4, 017 (2018), arXiv:1712.07529 [cond-mat.statmech].
[55] P. Calabrese, Entanglement spreading in non-equilibrium integrable systems, SciPost Phys. Lect. Notes, 20 (2020), arXiv:2008.11080 [cond-mat.stat-mech].
[56] Y. Chen, Z. D. Wang, and F. C. Zhang, Exploring quantum phase transitions with a sublattice entanglement scenario, Phys. Rev. B 73, 224414 (2006), arXiv:quantph/0512143 [quant-ph].
[57] J. P. Keating, F. Mezzadri, and M. Novaes, Comb entanglement in quantum spin chains, Phys. Rev. A 74, 012311 (2006), arXiv:quant-ph/0604016 [quant-ph].
[58] Y. Chen, P. Zanardi, Z. D. Wang, and F. C. Zhang, Sublattice entanglement and quantum phase transitions in antiferromagnetic spin chains, New J. Phys. 8, 97 (2006).
[59] D. Poilblanc, Out-of-equilibrium correlated systems: Bipartite entanglement as a probe of thermalization,

Phys. Rev. B 84, 045120 (2011), arXiv:1011.2147 [cond-mat.str-el].
[60] R. Rossignoli, N. Canosa, and J. M. Matera, Even-odd entanglement in boson and spin systems, Phys. Rev. A 83, 042328 (2011), arXiv:1104.4005 [quant-ph].
[61] T. He, J. M. Magán, and S. Vandoren, Entanglement entropy of periodic sublattices, Phys. Rev. B 95, 035130 (2017), arXiv:1607.07462 [quant-ph].
[62] F. Iglói and I. Peschel, On reduced density matrices for disjoint subsystems, Europhys. Lett. 89, 40001 (2010), arXiv:0910.5671 [cond-mat.stat-mech].
[63] R. I. Nepomechie, Bethe ansatz on a quantum computer?, arXiv e-prints (2020), arXiv:2010.01609 [quant-ph].
[64] J. S. Van Dyke, G. S. Barron, N. J. Mayhall, E. Barnes, and S. E. Economou, Preparing Bethe Ansatz Eigenstates on a Quantum Computer, PRX Quantum 2, 040329 (2021), arXiv:2103.13388 [quant-ph].
[65] J. S. Van Dyke, E. Barnes, S. E. Economou, and R. I. Nepomechie, Preparing exact eigenstates of the open XXZ chain on a quantum computer, J. Phys. A 55, 055301 (2022), arXiv:2109.05607 [quant-ph].
[66] W. Li, M. Okyay, and R. I. Nepomechie, Bethe states on a quantum computer: success probability and correlation functions, arXiv e-prints (2022), arXiv:2201.03021 [quant-ph].
[67] A. Sopena, M. H. Gordon, D. García-Martín, G. Sierra, and E. López, Algebraic Bethe Circuits, arXiv e-prints (2022), arXiv:2202.04673 [quant-ph].
[68] S. Santra, A. Agarwala, and S. Bhattacharjee, Statisticstuned entanglement of the boundary modes in coupled Su-Schrieffer-Heeger chains, Phys. Rev. B 103, 195134 (2021), arXiv:2010.07327 [cond-mat.str-el].

