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Some identities of Gaussian binomial coefficients

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Abstract. In this paper, we present some identities of Gaussian binomial coefficients with respect to recursive sequences, Fibonomial coefficients, and complete functions by use of their relationships.

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1. Introduction

q-series are defined by

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n) \tag{1.1}$$

for integer $n > 0$ and $(q)_0 = 1$. Arising out of these are Gaussian binomial coefficients (or Gaussian coefficients as an abbreviation) for integers $n, k \geq 0$,

$$\binom{n}{k}_q = \begin{cases} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(q)_k}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

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$$= \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k \le n, \tag{1.2}$$

where the q-factorial $[m]_q!$ is defined by $[m]_q! = \Pi_{k=1}^{m}[k]_q = [1]_q[2]_q \cdots [m]_q$, and

$$[k]_q = \sum_{i=0}^{k-1} q^i = 1 + q + q^2 + \cdots + q^{k-1} = \begin{cases} \frac{1-q^k}{1-q} & \text{for } q \ne 1, \\ k & \text{for } q = 1. \end{cases}$$

From (1.2) we have $\binom{n}{0}_q = \binom{n}{n}_q = 1$, $\binom{n}{k}_q = \binom{n}{n-k}_q$,

$$(1-q^k)\binom{n}{k}_q = (1-q^n)\binom{n-1}{k-1}_q, \tag{1.3}$$

and for $0 < k < n$

$$\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q, \tag{1.4}$$

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q. \tag{1.5}$$

Identities (1.4) and (1.5) are analogs of Pascal's identities. Alternatively using (1.4) and (1.5), we obtain the identity

$$\binom{n}{k}_q = \binom{n-1}{k}_q + \binom{n-1}{k-1}_q - (1-q^{n-1})\binom{n-2}{k-1}_q \tag{1.6}$$

More precisely, by substituting (1.4) with the transformation $n \to n-1$ and $k \to k-1$ into (1.5), we have

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-1}\binom{n-2}{k-1}_q + q^{n-k}\binom{n-2}{k-2}_q.$$

Substituting (1.5) with the transformation $n \to n-1$ and $k \to k-1$ into the last term of the above identity, we have

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-1}\binom{n-2}{k-1}_q + \binom{n-1}{k-1}_q - \binom{n-2}{k-1}_q,$$

which implies (1.6).

In 1915 Georges Fontené (1848–1928) published a one page note [8] suggesting a generalization of binomial coefficients, replacing the natural numbers by an arbitrary sequence (A_n) of real or complex numbers, namely,

$$\binom{n}{k}_A = \frac{A_n A_{n-1} \cdots A_{n-k+1}}{A_k A_{k-1} \cdots A_1} \tag{1.7}$$

with $\binom{n}{0}_A = \binom{n}{n}_A = 1$, where A stands for (A_n). He gave the fundamental recurrence relation for these generalized coefficients and include the ordinary binomial coefficients as a special case for $A_n = n$, while for $A_n = q^n - 1$ we obtain the Gaussian binomial coefficients (or q-binomial coefficients) (1.6) studied by Gauss (as well as Euler, Cauchy, F . H, Jackson, and many others later). The history of Gaussian binomial coefficients can be seen in a recent paper by Shannon [18] and its references.

These generalized coefficients of Fontené were rediscovered by Morgan Ward (1901–1963) in a remarkable paper [23] in 1936 which developed a symbolic calculus of sequences without mentioning Fontené. In that paper, Ward posed the problem whether a suitable definition for generalized Bernoulli numbers could be framed so that a generalized Staudt-Clausen theorem [7] existed for them within the framework of the Jackson calculus [14]; the Staudt-Clausen theorem deals with the fractional part of Bernoulli numbers [20]. Rado [17] and Carlitz [4, 5] outlined partial generalizations of the theorem with the Jackson operators for q-Bernoulli numbers, and Horadam and Shannon completed this proof [13]. We shall follow Gould [10] and call the generalized coefficients (1.7) the Fontené-Ward generalized binomial coefficients.

Since 1964, there has been an accelerated interest in Fibonomial coefficients, which correspond to the choice $A_n = F_n$, where F_n are the Fibonacci numbers defined by $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0$, and $F_1 = 1$. For instance, see Trojovský [21] and its references. As far as we know, the first person to name them (not utilize them) was Stephen Jerbic, a research Master student of Verner Hoggatt, who completed his thesis in 1968 [15]. One of the authors of this paper read his MA thesis in 1975 when the author visited Verner Hoggatt in San Jose.

If the recursive number sequence $(U_n(a,b;p_1,p_2))$ that satisfies $U_{n+2} = p_1U_{n+1} - p_2U_n$ $(n \geq 0)$ and has initials $U_0 = a$ and $U_1 = b$ is used to replace (A_n) in the Fontené-Ward generalized binomial coefficients, then the corresponding Gaussian binomial coefficients are called the generalized Fibonacci binomial coefficients, which are shown in the recent paper [18] by Shannon. $U_n(0,1;p_1,p_2)$ can be represented by its Binet from $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ (cf. the authors [11]), where α and β are two distinct roots of the $(U_n)'s$ characteristic equation $x^2 - p_1x + p_2 = 0$. Throughout this paper, we always assume the characteristic equation $x^2 - p_1x + p_2 = 0$ has non-zero constant term p_2 and two distinct roots α and β. Since $\alpha\beta = p_2$, we have $\alpha, \beta \neq 0$. Shannon's paper starts from a nice relationship between the Gaussian binomial coefficients defined by (1.7) with $q = \beta/\alpha$ $(\alpha \neq 0, i.e., p_2 \neq 0)$ for $U_n(0,1;p_1,p_2)$ and the generalized Fibonacci binomial coefficients

$$\binom{n}{k}_U = \frac{U_nU_{n-1}\cdots U_{n-k+1}}{U_1U_2\cdots U_k}, \tag{1.8}$$

where U stands for $(U_n(0,1;p_1,p_2))$, represented by

$$\binom{n}{k}_q = \alpha^{-k(n-k)}\binom{n}{k}_U, \tag{1.9}$$

where $q = \beta/\alpha$, and $\alpha \neq 0$ and β are two distinct roots of the $(U_n)'s$ characteristic equation $x^2 - p_1 x + p_2 = 0$ assumed before. In fact, we have

$$\begin{aligned}\binom{n}{k}_q &= \frac{(1-(\beta/\alpha)^n)(1-(\beta/\alpha)^{n-1})\cdots(1-(\beta/\alpha)^{n-k+1})}{(1-\beta/\alpha)(1-(\beta/\alpha)^2)\cdots(1-(\beta/\alpha)^k)}\\ &= \frac{(\alpha^n-\beta^n)(\alpha^{n-1}-\beta^{n-1})\cdots(\alpha^{n-k+1}-\beta^{n-k+1})}{(\alpha-\beta)(\alpha^2-\beta^2)\cdots(\alpha^k-\beta^k)}\\ &\quad \frac{(1/\alpha^n)(1/\alpha^{n-1})\cdots(1/\alpha^{n-k+1})}{(1/\alpha^k)(1/\alpha^{k-1})\cdots(1/\alpha)}\\ &= \frac{U_nU_{n-1}\cdots U_{n-k+1}}{U_1U_2\cdots U_k}\left(\frac{1}{\alpha^{n-k}}\right)^k,\end{aligned}$$

which implies (1.9).

Based on the relationship (1.9), several interesting identities are established. For instance, [18] used (1.9) to establish the following identity.

$$\binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \frac{2-q^k-q^{n-k}}{1-q^n}\binom{n}{k}_q. \tag{1.10}$$

Obviously, identity (1.10) can also be proved by using (1.3) and

$$\begin{aligned}(1-q^{n-k})\binom{n}{k}_q &= (1-q^{n-k})\binom{n}{n-k}_q\\ &= (1-q^n)\binom{n-1}{n-k-1}_q = (1-q^n)\binom{n-1}{k}_q.\end{aligned}$$

Consequently, combining $(1-q^k)\binom{n}{k}_q = (1-q^n)\binom{n-1}{k-1}_q$ on the leftmost side and the rightmost side of the last equation yields

$$(1-q^n)\left(\binom{n-1}{k}_q + \binom{n-1}{k-1}_q\right) = (2-q^k-q^{n-k})\binom{n}{k}_q.$$

In this paper, we will continue Shannon's work to construct a few more identities.

The second part of this paper concerns complete homogenous symmetric functions, which have a natural connection with Gaussian coefficients. A good source of information for the early history of symmetric functions, such as the fundamental theorem of symmetric functions and the symmetry of the matrix, is [22] by Vahlen. In particular, the first published work on symmetric functions is due to Girard [9] in 1629, who gave an explicit formula expressing symmetric polynomials. The complete homogeneous symmetric polynomials are a specific kind of symmetric polynomials. Every symmetric polynomial can be expressed as a polynomial expression in complete homogeneous symmetric polynomials. The fundamental relation between the elementary symmetric polynomials and the complete homogeneous ones can be found in [16] by Macdonald. More historical context on the symmetric functions and the complete homogeneous symmetric polynomials can be found in [16] and Stanley [19]. The complete functions are q analogies of the complete homogenous symmetric polynomials.

The complete homogeneous symmetric polynomial of degree k in n variables $x_1, x_2, \ldots, x_n$, written h_k for $k = 0, 1, 2, \ldots$, is the sum of all monomials of total degree k in the variables. More precisely, for integers i_1, $i_2, \ldots$, i_k,

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{1.11}$$

or equivalently, for integers l_1, $l_2, \ldots$, l_k

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{l_1 + l_2 + \cdots + l_n = k,\, l_i \ge 0} x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}. \tag{1.12}$$

Here, l_p is the multiplicity of p in the sequence i_k. The first few of these polynomials are

$$\begin{aligned} h_0(x_1, x_2, \ldots, x_n) &= 1, \\ h_1(x_1, x_2, \ldots, x_n) &= \sum_{1 \le j \le n} x_j, \\ h_2(x_1, x_2, \ldots, x_n) &= \sum_{1 \le j \le k \le n} x_j x_k, \\ h_3(x_1, x_2, \ldots, x_n) &= \sum_{1 \le j \le k \le \ell \le n} x_j x_k x_\ell. \end{aligned}$$

Thus, for each nonnegative integer k, there exists exactly one complete homogeneous symmetric polynomial of degree k in n variables. Further results about complete homogeneous symmetric polynomials can be expressed in terms of their generating function (see, for example, Bhatnagar [1])

$$H(t) = \sum_{n \ge 0} h_n t^n = \Pi_{r=1}^{n} (1 - x_r t)^{-1}.$$

If $x_i = q^{i-1}$, from Cameron [3] (cf. P. 224), (1.11) defines the following relationship between $h_r(1, q, q^2, \ldots, q^{n-1})$ and Gaussian coefficients, where $h_r(1, q, q^2, \ldots, q^{n-1})$ is called the complete function of order (n, k).

$$h_r(1, q, q^2, \ldots, q^{n-1}) = \binom{n + r - 1}{r}_q. \tag{1.13}$$

From (1.9), we also have a relationship between $h_r(1, q, q^2, \ldots, q^{n-1})$ and generalized Fibonomial coefficients as follows:

$$\binom{n}{k}_U = \alpha^{k(n-k)} h_k(1, q, q^2, \ldots, q^{n-k}), \tag{1.14}$$

where $q = \beta/\alpha$ (recall that $\alpha \neq 0$ and β are two distinct roots of the equation $x^2 - p_1 x + p_2 = 0$), and U is referred to as recursive sequence $(U_n(a_0, a_1; p_1, p_2))_{n\geq 0}$.

In the next section, we give identities of Gaussian coefficients and generalized Fibonomial coefficients. In Section 3, by using formula (1.13) we will transfer the results between Gaussian coefficients and the complete functions.

2. Identities of Gaussian coefficients and Fibonomial coefficients

Theorem 2.1. *Let $\binom{n}{k}_q$ be the Gaussian binomial coefficients defined by (1.2). Then*

$$(1-q^k)(1-q^{n-k})\binom{n}{k}_q = (1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q \tag{2.1}$$

for $1 \leq k \leq n-1$.

Proof. By applying (1.3) we have

$$\begin{aligned}
&(1-q^k)(1-q^{n-k})\binom{n}{k}_q = (1-q^{n-k})(1-q^n)\binom{n-1}{k-1}_q \\
&= (1-q^n)(1-q^{n-k})\binom{n-1}{n-k}_q = (1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q.
\end{aligned}$$

An alternative proof may provides an example of the use of (1.6). Starting from (1.6) and noting (1.10), we have

$$\begin{aligned}
&(1-q^n)\binom{n}{k}_q \\
&= (1-q^n)\left(\binom{n-1}{k}_q + \binom{n-1}{k-1}_q\right) - (1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q \\
&= (2-q^k-q^{n-k})\binom{n}{k}_q - (1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q,
\end{aligned}$$

or equivalently,

$$(1-q^n-(2-q^k-q^{n-k}))\binom{n}{k}_q = -(1-q^n)(1-q^{n-1})\binom{n-2}{k-1}_q,$$

which implies (2.1). □

By applying mathematical induction to the recursive relation (2.1), we may prove the following formula.

Corollary 2.2. *Let* $\binom{n}{k}_q$ *be the Gaussian binomial coefficients defined by* (1.2)*. Then for* $0 \le j \le n$ *and* $j \le k \le n-j$

$$\left(\Pi_{\ell=0}^{j-1}(1-q^{k-\ell})(1-q^{n-k-\ell})\right)\binom{n}{k}_q = \left(\Pi_{\ell=0}^{2j-1}(1-q^{n-\ell})\right)\binom{n-2j}{k-j}_q. \tag{2.2}$$

Relationship (1.9) can be used to change an identity for Gaussian binomial coefficients to an identity for generalized Fibonomial coefficients and vice versa.

Corollary 2.3. *Let* $\binom{n}{k}_q$ *be the Gaussian binomial coefficients defined by* (1.2) *with* $q = \beta/\alpha$*, and let* $\binom{n}{k}_U$ *be the generalized Fibonomial coefficients defined by* (1.8)*. Then*

$$\alpha^k U_{n-k} + \alpha^{n-k} U_k = \frac{2 - q^k - q^{n-k}}{1-q^n} U_n. \tag{2.3}$$

Proof. Substituting

$$\begin{aligned}
\binom{n-1}{k}_q &= \alpha^{-k(n-k-1)}\binom{n-1}{k}_U = \alpha^{-k(n-k-1)}\frac{U_{n-1}U_{n-2}\cdots U_{n-k}}{U_1U_2\cdots U_k}\\
\binom{n-1}{k-1}_q &= \alpha^{-(k-1)(n-k)}\binom{n-1}{k-1}_U = \alpha^{-(k-1)(n-k)}\frac{U_{n-1}U_{n-2}\cdots U_{n-k+1}}{U_1U_2\cdots U_{k-1}}\\
\binom{n}{k}_q &= \alpha^{-k(n-k)}\binom{n}{k}_U = \alpha^{-k(n-k)}\frac{U_nU_{n-1}\cdots U_{n-k+1}}{U_1U_2\cdots U_k}
\end{aligned}$$

into (1.10), we have

$$\begin{aligned}
&\alpha^{-k(n-k-1)}\frac{U_{n-1}U_{n-2}\cdots U_{n-k}}{U_1U_2\cdots U_k} + \alpha^{-(k-1)(n-k)}\frac{U_{n-1}U_{n-2}\cdots U_{n-k+1}}{U_1U_2\cdots U_{k-1}}\\
&= \frac{2-q^k-q^{n-k}}{1-q^n}\alpha^{-k(n-k)}\frac{U_nU_{n-1}\cdots U_{n-k+1}}{U_1U_2\cdots U_k},
\end{aligned}$$

which implies (2.3). □

From [10], we have analogues of identities (1.4) and (1.5) for the generalized coefficients defined by (1.7).

Proposition 2.4. *Let* $\binom{n}{k}_q$ *be the Gaussian binomial coefficients defined by* (1.2)*, and let* $\binom{n}{k}_A$ *be the generalized coefficients defined by* (1.7)*. Then we have*

$$\binom{n}{k}_A - \binom{n-1}{k-1}_A = \binom{n-1}{k}_A \frac{A_n - A_k}{A_{n-k}} \quad \text{and} \tag{2.4}$$

$$\binom{n}{k}_A - \binom{n-1}{k}_A = \binom{n-1}{k-1}_A \frac{A_n - A_{n-k}}{A_k}, \tag{2.5}$$

which generate the identities (1.4) *and* (1.5)*, respectively, as the special cases for* $A_n = q^n - 1$.

Proof. From definition (1.7), we may write the left-hand side of (2.4) as

$$\frac{A_n A_{n-1}\cdots A_{n-k+1}}{A_1 A_2\cdots A_k}-\frac{A_{n-1}A_{n-2}\cdots A_{n-k+1}}{A_1A_2\cdots A_{k-1}}$$
$$=\frac{A_{n-1}A_{n-2}\cdots A_{n-k+1}A_{n-k}}{A_1A_2\cdots A_k}\frac{A_n-A_k}{A_{n-k}}=\binom{n-1}{k}_A\frac{A_n-A_k}{A_{n-k}},$$

which proves (2.4). Identity (2.5) can be proved similarly. To show (1.4) is a special case of (2.4) for$A_n=q^n-1$, we only need to notice that $\binom{n}{k}_A=\binom{n}{k}_q$ and

$$\frac{A_n-A_k}{A_{n-k}}=\frac{q^n-1-(q^k-1)}{q^{n-k}-1}=q^k,$$

which will convert identity (2.4) to (1.4). Similarly, the transformation $A_n=q^n-1$ will convert identity (2.5) to (1.5). □

Identities of Fibonomial coefficients can be changed to the identities of Fibonacci number sequence and vice versa. For instance, Hoggatt [12] (cf. formula (D)) gives the following identity for Fibonomial coefficients $\binom{n}{k}_F$, where $F=(F_n(0,1,1,-1))$ is the Fibonacci number sequence.

$$\binom{n}{k}_F=F_{k+1}\binom{n-1}{k}_F+F_{n-k-1}\binom{n-1}{k-1}_F. \tag{2.6}$$

By substituting $\binom{n}{k}_F=(F_nF_{n-1}\cdots F_{n-k+1})/(F_1F_2\cdots F_k)$ into the above identity and cancelling the same terms on the both sides of the equation, we obtain the following well-known identity for the Fibonacci number sequence:

$$F_n=F_{n-k}F_{k+1}+F_{n-k-1}F_k, \tag{2.7}$$

which presents a Fibonacci number in terms of smaller Fibonacci numbers. Conversely, from an identity of recursive number sequences, one may obtain identities of Fibonacci coefficients. For instance from Cassini's identity

$$F_{n+1}F_{n-1}-F_n^2=(-1)^n \tag{2.8}$$

we may obtain the following identity for Fibonacci coefficients:

$$F_nF_{n-k}\binom{n}{k}_F=(F_{n+1}F_{n-1}-(-1)^n)\binom{n-1}{k}_F, \tag{2.9}$$

which returns to Cassini's identity when $k=0$. Hence, we have the following results that can also be extended to other transformation between the identities of recursive sequences and the identities of Gaussian coefficients.

Proposition 2.5. *From Cassini's identity (2.8) and the identity (2.7) presenting Fibonacci numbers in terms of smaller Fibonacci numbers, we may derive the corresponding Gaussian Coefficient identities (2.9) and (2.6), respectively, and vice versa.*

3. Identities of the complete functions

Using the relationship (1.13), $h_r(1, q, q^2, \ldots, q^{n-1}) = \binom{n+r-1}{r}_q$, we may re-write the identities of Gaussian coefficients in terms of the complete functions. For instance, from the property of Gaussian coefficients $\binom{n+r-1}{r}_q = \binom{n+r-1}{n-1}_q$ and identities (1.3)-(1.6), we immediately have the following results.

Proposition 3.1. *Let $h_r(1, q, q^2, \ldots, q^{n-1})$ and $\binom{n}{k}_q$ be defined as before. Then*

$$h_r(1, q, q^2, \ldots, q^{n-1}) = h_{n-1}(1, q, q^2, \ldots, q^r), \tag{3.1}$$

$$(1 - q^k)h_k(1, q, q^2, \ldots, q^{n-k}) = (1 - q^n)h_{k-1}(1, q, q^2, \ldots, q^{n-k}), \tag{3.2}$$

$$\begin{aligned} h_k(1, q, q^2, \ldots, q^{n-k}) = q^k h_k(1, q, q^2, \ldots, q^{n-k-1}) \\ + h_{k-1}(1, q, q^2, \ldots, q^{n-k}), \end{aligned} \tag{3.3}$$

$$\begin{aligned} h_k(1, q, q^2, \ldots, q^{n-k}) = h_k(1, q, q^2, \ldots, q^{n-k-1}) \\ + q^{n-k}h_{k-1}(1, q, q^2, \ldots, q^{n-k}), \end{aligned} \tag{3.4}$$

$$\begin{aligned} h_k(1, q, q^2, \ldots, q^{n-k}) &= h_k(1, q, q^2, \ldots, q^{n-k-1}) \\ + h_{k-1}(1, q, q^2, \ldots, q^{n-k}) &+ (q^{n-1} - 1)h_{k-1}(1, q, q^2, \ldots, q^{n-k-1}). \end{aligned} \tag{3.5}$$

From (3.1), we have

$$h_1(1, q, q^2, \ldots, q^{n-1}) = h_{n-1}(1, q).$$

Then, by using (1.9) the recursive sequence $U_n = U_n(a_0, a_1; p_1, p_2) = (\alpha^n - \beta^n)/(\alpha - \beta)$, where α and β are two distinct roots of the equation $x^2 - p_1 x + p_2 = 0$, can be written as

$$\begin{aligned} U_n &= \alpha^{n-1}\frac{1 - q^n}{1 - q} = \alpha^{n-1}\binom{n}{1}_q \\ &= \alpha^{n-1}h_1(1, q, q^2, \ldots, q^{n-1}) = \alpha^{n-1}h_{n-1}(1, q), \end{aligned} \tag{3.6}$$

where $q = \beta/\alpha$. From the definition of $h_r(1, q, \ldots, q^{n-1})$ given by (1.12), we obtain

$$\begin{aligned} U_n = \alpha^{n-1}h_{n-1}(1, q) &= \alpha^{n-1}\sum_{l_1+l_2=n-1,\, l_1,l_2\geq 0} 1^{l_1}q^{l_2} \\ &= \alpha^{n-1}\sum_{l_1+l_2=n-1,\, l_1,l_2\geq 0} 1^{l_1}\left(\frac{\beta}{\alpha}\right)^{l_2} \\ &= \sum_{l_1+l_2=n-1,\, l_1,l_2\geq 0} \alpha^{l_1}\beta^{l_2} = h_{n-1}(\alpha, \beta). \end{aligned}$$

For Fibonacci numbers

$$F_{k+1} = h_k(\alpha, \beta),$$

where $\alpha = (1+\sqrt{5})/2$ and $q = (1-\sqrt{5})/(1+\sqrt{5})$, from (1.9) and (2.6) we obtain

$$\begin{aligned}
&h_k(1, q, \dots, q^{n-k}) \\
&= \alpha^{-k} h_k(\alpha, \beta) h_k(1, q, \dots, q^{n-k-1}) \\
&\quad + \alpha^{-n+k} h_{n-k+2}(\alpha, \beta) h_{k-1}(1, q \dots, q^{n-k}).
\end{aligned}$$

From (3.6) we may establish the following theorem.

Theorem 3.2. *Let ($U_n = U_n(a, b; p_1, p_2)$) be the recursive sequence defined by $U_{n+2} = p_1 U_{n+1} - p_2 U_n$ ($n \geq 0$) with the initials $U_0 = a$ and $U_1 = b$, and let α and β be two distinct roots of the characteristic equation $x^2 - p_1 x + p_2 = 0$. Then*

$$\alpha^2 \binom{n+2}{1}_q = \alpha p_1 \binom{n+1}{1}_q - p_2 \binom{n}{1}_q, \tag{3.7}$$

or equivalently,

$$\alpha^2 h_{n+1}(1, q) = \alpha p_1 h_n(1, q) - p_2 h_{n-1}(1, q). \tag{3.8}$$

Proof. Noting $p_1 = \alpha + \beta$, $p_2 = \alpha\beta$, and $q = \beta/\alpha$, where $\alpha \neq 0$ (i.e., $p_2 \neq 0$), the right-hand side of (3.7) can be re-written as

$$\begin{aligned}
&\alpha p_1 \binom{n+1}{1}_q - p_2 \binom{n}{1}_q \\
&= \alpha p_1 \frac{1-q^{n+1}}{1-q} - p_2 \frac{1-q^n}{1-q} \\
&= \frac{1}{1-q} \left(\alpha p_1 (1-q^{n+1}) - p_2 (1-q^n) \right) \\
&= \frac{1}{1-q} \left((\alpha p_1 - p_2) - q^n (\alpha p_1 q - p_2) \right) \\
&= \frac{1}{1-q} \left(\alpha^2 - \beta^2 q^n \right) = \alpha^2 \frac{1-q^{n+2}}{1-q},
\end{aligned}$$

which implies (3.7). Consequently, we obtain (3.8) by substituting

$$\binom{m}{1}_q = h_1(1, q, \dots, q^{m-1}) = h_{m-1}(1, q) \tag{3.9}$$

into (3.7) for $m = n, n+1$, and $n+2$, respectively. $\square$

Chen and Louck [6] and Bhatnagar [1] present different approaches to Sylvester's identity related to the complete homogeneous symmetric functions.

Theorem 3.3 (Sylvester's identity)**.** *For each integer $m \geq 0$, we have*

$$\sum_{i=1}^{n} \frac{x_i^m}{\Pi_{j \neq i}(x_i - x_j)} = h_{m-n+1}(x_1, x_2, \dots, x_n), \tag{3.10}$$

where h_k is the kth homogeneous symmetric function, which is defined to be zero for $k < 0$.

Divided differences is a recursive division process. The method can be used to calculate the coefficients of the interpolation polynomial in the Newton form. The divided difference of a function f at knots $x_1, x_2, \ldots, x_n$ has the formula (see, for example, Burden and Faires [2])

$$[x_1, x_2, \ldots, x_n]f = \sum_{i=1}^{n} \frac{f(x_i)}{\Pi_{j\neq i}(x_i - x_j)} = \sum_{i=1}^{n} \frac{f(x_i)}{g'(x_i)}. \tag{3.11}$$

where $g(t) = (t - x_1)(t - x_2)\cdots(t - x_n)$. Thus, from formulas (3.10) and (3.11) we obtain a corollary of Theorem 3.3.

Corollary 3.4. *The value of the complete homogeneous symmetric polynomial, $h_{m-n+1}(x_1, x_2, \ldots, x_n)$, of degree $m-n+1$ at n distinct points $x_1, x_2, \ldots, x_n$ is the coefficient of the highest power term of the Newton interpolation of function $f(x) = x^m$ at points $x_1, x_2, \ldots, x_n$. Particularly, if $m = n$, then the coefficient of power n in the Newton interpolation of $f(x) = x^n$ is $h_1(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$.*

Corollary 3.5. *If evaluating points of an interpolation are arranged geometrically as $x_i = q^{i-1}$, $i = 1, 2, \ldots, n$, then the coefficient of power n in the Newton interpolation of $f(x) = x^n$ is the Gaussian coefficient $h_1(1, q, \ldots, q^{n-1}) = \binom{n}{1}_q = (1 - q^n)/(1 - q)$.*

Corollary 3.6. *If evaluating points of an interpolation are arranged geometrically as $x_i = q^{i-1}$, $i = 1, 2, \ldots, n$, where $q = \beta/\alpha$ and $\alpha \neq 0$ and β are two distinct roots of the equation $x^2 - p_1 x + p_2 = 0$, then the coefficient of power n in the Newton interpolation of $f(x) = x^n$ is the $\alpha^{-(n-1)}$ multiple of the Fibonacci binomial coefficient $\binom{n}{1}_U$, i.e., $h_1(1, q, \ldots, q^{n-1}) = \alpha^{-(n-1)}\binom{n}{1}_U$. Here, U is referred to as recursive sequence $(U_n(a_0, a_1; p_1, p_2))_{n\geq 0}$.*

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