Annales Mathematicae et Informaticae
55 (2022) pp. 125–136
DOI: https://doi.org/10.33039/ami.2022.01.001
URL: https://ami.uni-eszterhazy.hu

Enumeration of Fuss-skew paths

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Abstract. In this paper, we introduce the concept of a Fuss-skew path and then we study the distribution of the semi-perimeter, area, peaks, and corners statistics. We use generating functions to obtain our main results.

Keywords: Skew Dyck path, Fuss-Catalan numbers, generating function

AMS Subject Classification: 05A15, 05A19

1. Introduction

A *skew Dyck path* is a lattice path in the first quadrant that starts at the origin, ends on the x-axis, and consists of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and left-steps $L = (-1, -1)$, such that up and left steps do not overlap. The definition of skew Dyck path was introduced by Deutsch, Munarini, and Rinaldi [4]. Some additional results about skew Dyck path can be found in [2, 5, 8, 14].

Let s_n denote the number of skew Dyck path of semilength n, where the semilength of a path is defined as the number its up-steps. The sequence s_n is given by the combinatorial sum $s_n = \sum_{k=1}^{n} \binom{n-1}{k-1} c_k$, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. The sequence s_n appears in OEIS as A002212 [15], and its first few values are

$$1, \quad 1, \quad 3, \quad 10, \quad 36, \quad 137, \quad 543, \quad 2219, \quad 9285, \quad 39587.$$

One way to generalize the classical Dyck paths is to regard the length of an up-step U as a parameter. Given a positive number ℓ, an *ℓ-Dyck path* is a lattice path in the first quadrant from $(0, 0)$ to $((\ell + 1)n, 0)$ where $n \geq 0$ using up-steps

Submitted: December 22, 2021
Accepted: January 14, 2022
Published online: January 29, 2022

$U_\ell = (\ell, \ell)$ and down-steps $U = (1, -1)$. For $\ell = 1$, we recover the classical Dyck path. The total number of ℓ-Dyck path with length $(\ell + 1)n$ is given by $c_\ell(n) = \frac{1}{tn+1}\binom{(t+1)n}{n}$ (cf. [1]). We will refer to ℓ-Dyck paths here as the "Fuss" case because the sequence $c_\ell(n)$ was first investigated by N. I. Fuss (see, for example, [7, 16] for several combinatorial interpretations for both the Catalan and Fuss-Catalan numbers).

Our focus in this paper is to introduce a Fuss analogue of the skew Dyck path. Given a positive integer ℓ, an *ℓ-Fuss-skew path* is a path in the first quadrant that starts at the origin, ends on the x-axis, and consists of up-steps $U_\ell = (\ell, \ell)$, down-steps $D = (1, -1)$, and left steps $L = (-1, -1)$, such that up and left steps do not overlap. Given an ℓ-Fuss-skew path P, we define the semilength of P, denote by $|P|$, as the number of up-steps of P. For example, Figure 1 shows a 3-Fuss-skew path of semilength 6. It is clear that the 1-Fuss-skew paths coincide with the skew Dyck paths. Let $\mathbb{S}_{n,\ell}$ denote the set of all ℓ-Fuss-skew path of semilength n, and $\mathbb{S}_\ell = \bigcup_{n\geq 0} \mathbb{S}_{n,\ell}$. For example, Figure 4 shows all the paths in $\mathbb{S}_{2,2}$.

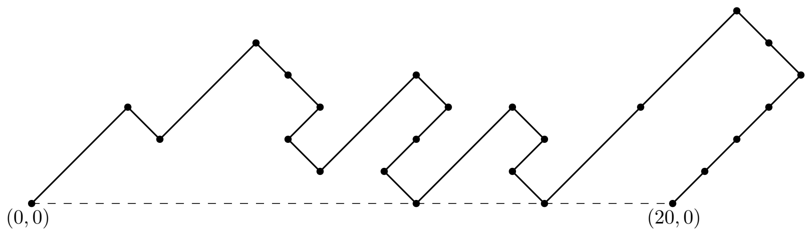

Figure 1. 3-Fuss-skew path of semilength 6.

2. Counting special steps

For a given path $P \in \mathbb{S}_\ell$, we use $u(P)$, $d(P)$, and $t(P)$ to denote the number of up-steps, down-steps, and left-steps of P, respectively. In this section, we study the distribution of these parameters over $\mathbb{S}_\ell$. Using these parameters, we define the generating function

$$F_\ell(x, p, q) := \sum_{P \in \mathbb{S}_\ell} x^{u(P)} p^{d(P)} q^{t(P)}.$$

For simplicity, we use F_ℓ to denote the generating function $F_\ell(x, p, q)$.

Theorem 2.1. *The generating function $F_\ell(x, p, q)$ satisfies the functional equation*

$$F_\ell = 1 + x(pF_\ell + q)^{\ell-1}(pF_\ell^2 + q(F_\ell - 1)). \tag{2.1}$$

Proof. Let $\mathcal{A}_i$ denote the ℓ-Fuss-skew paths whose last y-coordinate is i and let A_i denote the generating function defined by

$$A_i = \sum_{P \in \mathcal{A}_i} x^{u(P)} p^{d(P)} q^{t(P)}.$$

A non-empty ℓ-Fuss-skew path can be uniquely decomposed as either $U_\ell TDP$ or $U_\ell TL$, where $U_\ell T$ is a lattice path in $\mathcal{A}_1$ and P is an ℓ-Fuss-skew path (see Figure 2 for a graphical representation of this decomposition). From this decomposition, we obtain the functional equation (cf. [6])

$$F_\ell = 1 + x(pA_1F_\ell + qA_1). \tag{2.2}$$

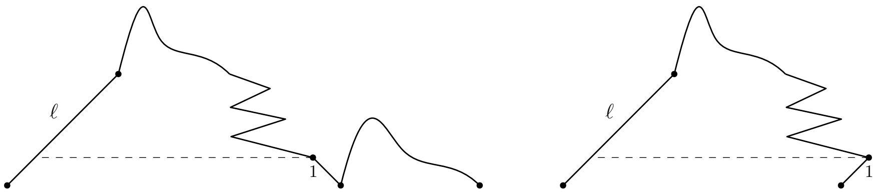

Figure 2. Decomposition of a ℓ-Fuss-skew path.

The paths of $\mathcal{A}_i$ can be decomposed as TDP or TL, where $T \in \mathcal{A}_{i+1}$ for $i = 1, \dots, \ell - 2$ and $P \in \mathbb{S}_\ell$ (see Figure 3 for a graphical representation of this decomposition). Moreover, the paths of $\mathcal{A}_{\ell-1}$ are decomposed as P_1DP_2 or $P'L$, where $P_1, P_2, P' \in \mathbb{S}_\ell$ and P' is non-empty.

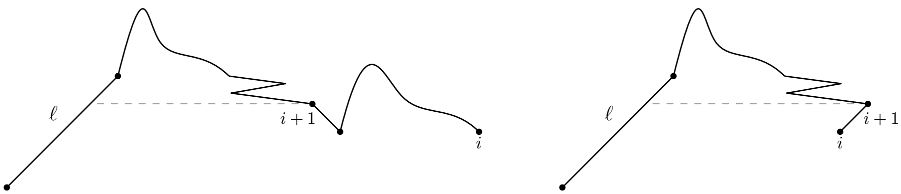

Figure 3. Decomposition of the paths in $\mathcal{A}_i$.

From the above decompositions, we obtain the functional equations

$$A_i = pA_{i+1}F_\ell + qA_{i+1}, \quad \text{for} \quad i = 1, \dots, \ell - 2, \quad \text{and} \quad A_{\ell-1} = pF_\ell^2 + q(F_\ell - 1).$$

Note that in these functional equations we do not consider the first up-step because it was considered in (2.2). Therefore, we have

$$\begin{aligned} F_\ell &= 1 + x(pF_\ell + q)A_1 = 1 + x(pF_\ell + q)^2 A_2 \\ &= \cdots = 1 + x(pF_\ell + q)^{\ell-1}(pF_\ell^2 + q(F_\ell - 1)). \end{aligned} \qquad \square$$

Let $s_\ell(n, p, q)$ denote the joint distribution over $\mathbb{S}_{n,\ell}$ for the number of down and left steps, that is,

$$s_\ell(n, p, q) = \sum_{P \in \mathbb{S}_{n,\ell}} p^{d(P)} q^{t(P)}.$$

It is clear that $F_\ell = \sum_{n \geq 0} s_\ell(n, p, q)x^n$. From the Lagrange inversion theorem (see for instance [13]), we give a combinatorial expression for the sequence $s_\ell(n, p, q)$.

Theorem 2.2. *For $n \geq 1$, the sequence $s_\ell(n,p,q)$ is given by*

$$\frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}p^{2n-1-2j}(2p+q)^k(p+q)^{n(\ell-2)+2j-k+1}.$$

In particular, the total number of ℓ-Fuss-skew paths of semilength n is

$$s_\ell(n) := s_\ell(n,1,1) = \frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}3^k 2^{n(\ell-2)+2j-k+1}.$$

Proof. The functional equation given in Theorem 2.1 can be written as

$$Q_\ell = x(p(Q_\ell+1)+q)^{\ell-1}(p(Q_\ell+1)^2+qQ_\ell),$$

where $Q_\ell = F_\ell - 1$. From the Lagrange inversion theorem, we deduce

$$\begin{aligned}
[x^n]H_\ell &= \frac{1}{n}[z^{n-1}](p(z+1)+q)^{(\ell-1)n}(p(z+1)^2+qz)^n\\
&= \frac{1}{n}[z^{n-1}]\sum_{s\geq 0}\binom{(\ell-1)n}{s}(pz)^s(p+q)^{(\ell-1)n-s}(pz^2+(2p+1)z+p)^n\\
&= \frac{1}{n}[z^{n-1}]\sum_{s\geq 0}\binom{(\ell-1)n}{s}(pz)^s(p+q)^{(\ell-1)n-s}\\
&\quad\times\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}p^{n-j}((2p+q)z)^k(pz^2)^{j-k}\\
&= \frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}p^{2n-1-2j}(2p+q)^k(p+q)^{n(\ell-2)+2j-k+1}. \qquad \square
\end{aligned}$$

For example, Figure 4 shows all 2-Fuss-skew paths of semilength 2 counted by the term $s_\ell(2,p,q) = 3p^4 + 6p^3q + 4p^2q^2 + pq^3$.

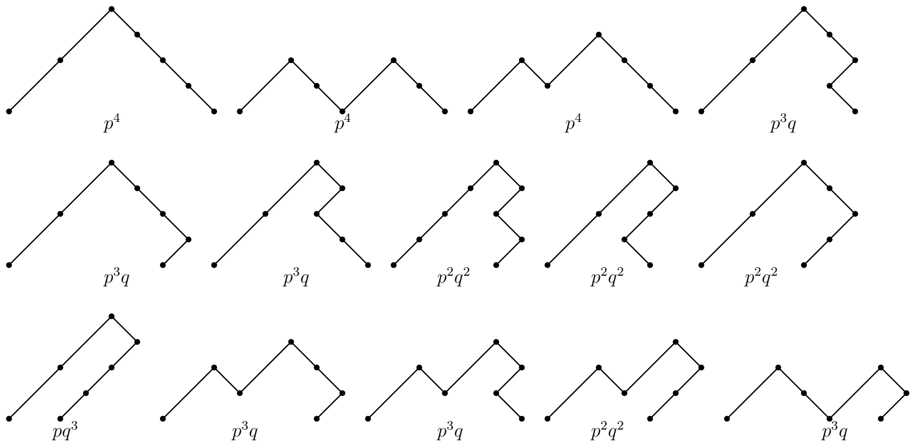

Figure 4. 2-Fuss-skew paths counted by $s_\ell(2,p,q)$.

From Theorem 2.2, we obtain that the total number of down-steps over the ℓ-Fuss-skew paths of semilength n is given by

$$\left.\frac{\partial s_\ell(n,p,1)}{\partial p}\right|_{p=1}$$
$$=\frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}2^{(\ell-2)n+2j-k}3^{k-1}(3n(\ell+2)+k-6j-3).$$

Moreover, the total number of left-steps over the ℓ-Fuss-skew paths of semilength n is

$$\left.\frac{\partial s_\ell(n,1,q)}{\partial q}\right|_{q=1}$$
$$=\frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1}2^{(\ell-2)n+2j-k}3^{k-1}(3n(\ell-2)-k+6j+3).$$

Equation (2.1) can be explicitly solved for $\ell=1$. In this case, we obtain the generating function

$$F_1(x,p,q)=\frac{1-qx-\sqrt{(1-qx)(1-(4p+q)x)}}{2px}.$$

Moreover, the generating functions for the total number of down-steps (A026388) and left steps (A026376) over the skew-Dyck paths are respectively

$$\frac{1-4x+3x^2-\sqrt{1-6x+5x^2}(1-x)}{2x\sqrt{1-6x+5x^2}}$$

and

$$\frac{1-3x-\sqrt{1-6x+5x^2}}{2\sqrt{1-6x+5x^2}}.$$

Notice that we recover some of the results of [5].

Finally, Table 1 shows the first few values of the total number of ℓ-Fuss-skew paths of semilength n.

Table 1. Values of $s_\ell(n,1,1)$ for $1 \leq \ell \leq 5$, $n = 1, \dots, 7$.

$\ell \backslash n$	1	2	3	4	5	6	7
$\ell = 1$	1	3	10	36	137	543	2219
$\ell = 2$	2	14	118	1114	11306	120534	1331374
$\ell = 3$	4	64	1296	29888	745856	19614464	535394560
$\ell = 4$	8	288	13568	734720	43202560	2681634816	172936069120

2.1. The width of a path

For a given path $P \in \mathbb{S}_\ell$, we define the *width* of P, denoted by $\nu(P)$, as the x-coordinate of the last point of P. For example, the width of the path given in Figure 1 is 20. We define the generating function

$$G_\ell(x,y) := G_\ell = \sum_{P \in \mathbb{S}_\ell} x^{u(P)} y^{\nu(P)}.$$

Note that each U_ℓ and D step of a path increases the width by ℓ units and 1 unit, respectively, while the left-step L decreases the width by 1 unit. Therefore, we have the functional equation

$$\begin{aligned} G_\ell &= 1 + xy^\ell (yG_\ell + y^{-1})^{\ell-1}(yG_\ell^2 + y^{-1}(G_\ell - 1)) \\ &= 1 + x(y^2G_\ell + 1)^{\ell-1}(y^2G_\ell^2 + (G_\ell - 1)). \end{aligned} \tag{2.3}$$

Let $g_\ell(n,y)$ denote the distribution over $\mathbb{S}_{n,\ell}$ for the width parameter, i.e.,

$$g_\ell(n,y) = \sum_{P \in \mathbb{S}_{n,\ell}} y^{\nu(P)}.$$

From the functional equation (2.3) and the Lagrange inversion theorem, we obtain the following theorem.

Theorem 2.3. *For $n \geq 1$, the sequence $g_\ell(n,y)$ is given by*

$$\frac{1}{n}\sum_{j=0}^{n}\sum_{k=0}^{j}\binom{n}{j}\binom{j}{k}\binom{n(\ell-1)}{n-2j+k-1} y^{4(n-j)-2}(y^2+1)^{n(\ell-2)+2j-k+1}(2y^2+1)^k.$$

For example, $g_2(2,y) = y^2 + 4y^4 + 6y^6 + 3y^8$. This polynomial can be found from the paths in Figure 4. For $\ell = 1$, we obtain the explicit generating function with respect to the width of a skew Dyck path.

$$G_1(x,y) = \frac{1 - x - \sqrt{(1-x)(1-x-4xy^2)}}{2xy^2}.$$

3. Number of peaks

For a given path $P \in \mathbb{S}_\ell$, we define the *peaks* of P, denoted by $\rho(P)$, as the number of subpaths of the form $U_\ell D$ (for counting peaks in a Dyck path, for example, see [9, 11]). For example, the number of peaks of the path given in Figure 1 is 5. We define the generating function

$$P_\ell(x,y) := P_\ell = \sum_{P \in \mathbb{S}_\ell} x^{u(P)} y^{\rho(P)}.$$

Theorem 3.1. *The generating function $P_\ell(x,y)$ satisfies the functional equation*

$$P_\ell = 1 + x(P_\ell + 1)^{\ell-1}((P_\ell - 1 + y)P_\ell + (P_\ell - 1)).$$

Proof. Let C_i denote the generating function defined by $C_i = \sum_{P \in \mathcal{A}_i} x^{u(P)} y^{\rho(P)}$. From the decomposition given for the ℓ-Fuss-skew paths, we have the equation $P_\ell = 1 + x(C_1 P_\ell + C_1)$. Moreover,

$$\begin{aligned} C_i &= C_{i+1}P_\ell + C_{i+1}, \quad \text{for } i = 1, \dots, \ell - 2, \text{ and} \\ C_{\ell-1} &= (P_\ell - 1 + y)P_\ell + (P_\ell - 1). \end{aligned}$$

From these relations, we obtain the desired result. □

Let $p_\ell(n,y)$ denote the distribution over $\mathbb{S}_n$ for the peaks statistic, i.e.,

$$p_\ell(n,y) = \sum_{P \in \mathbb{S}_n} y^{\rho(P)}.$$

From the Lagrange inversion theorem, we deduce the following result.

Theorem 3.2. *For $n \geq 1$, we have*

$$p_\ell(n,y) = \frac{1}{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \binom{n(\ell-1)}{n-2j+k-1} 2^{n(\ell-2)+2j-k+1} y^{n-j} (y+2)^k.$$

In particular, the total number of peaks in all ℓ-Fuss-skew paths of semilength n is

$$\begin{aligned} &\left.\frac{\partial p_\ell(n,y)}{\partial y}\right|_{y=1} \\ &= \frac{1}{n} \sum_{j=0}^{n} \sum_{k=0}^{j} \binom{n}{j} \binom{j}{k} \binom{n(\ell-1)}{n-2j+k-1} 2^{n(\ell-2)+2j-k+1} 3^{k-1} (3(n-j)+k). \end{aligned}$$

For example, $p_2(2,y) = 8y + 6y^2$. This polynomial can be found from the paths in Figure 4. For $\ell = 1$ we obtain the generating function

$$P_1(x,y) = \frac{1 - xy - \sqrt{(1-xy)^2 - 4(1-x)x}}{2x}.$$

Moreover, the generating function for the total number of peaks is

$$\frac{1 - x - \sqrt{1 - 6x + 5x^2}}{2\sqrt{1 - 6x + 5x^2}}.$$

Table 2 shows the first few values of the number of peaks in ℓ-Fuss-skew paths of semilength n.

Table 2. Total number of peaks in $\mathbb{S}_\ell$.

$\ell \setminus n$	1	2	3	4	5	6	7
$\ell = 1$	1	4	17	75	339	1558	7247
$\ell = 2$	2	20	226	2696	33138	415164	5270850
$\ell = 3$	4	96	2672	78848	2400896	74568704	2347934464
$\ell = 4$	8	448	29440	2054144	147986432	10878189568	810813030400

4. Number of corners

For a given path $P \in \mathbb{S}_\ell$, we define a *corner* of P as a right angle caused by two consecutive steps in the graph of P. For example, the path given in Figure 5 has 4 corners, depicted in red. This statistic has been studied in other combinatorial structures as integer partitions [3], compositions [10], and bargraphs [12].

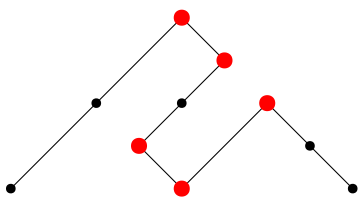

Figure 5. Corners of a path.

Let $\tau(P)$ denote the number of corners of P. We define the bivariate generating function

$$W_\ell(x,y) := W_\ell = \sum_{P \in \mathbb{S}_\ell} x^{u(P)} y^{\tau(P)}.$$

In this section, we analyze the cases $\ell = 1$ and $\ell = 2$. We leave as an open question the case $\ell \geq 3$.

Theorem 4.1. *The generating function* $W_1(x,y)$ *satisfies the functional equation*

$$xy(1+y)W_1^3 - (2-x(2-y^2))W_1^2 + 3(1-x)W_1 + x - 1 = 0.$$

Proof. Let $\mathcal{D}$ and $\mathcal{L}$ denote the skew Dyck paths whose last step is a down-step or a left-step, respectively. Let D and L denote the generating functions defined by

$$D = \sum_{P\in\mathcal{D}} x^{u(P)}y^{\tau(P)} \quad \text{and} \quad L = \sum_{P\in\mathcal{L}} x^{u(P)}y^{\tau(P)}.$$

A non-empty skew Dyck path can be uniquely decomposed as either UT_1L or UT_2DT_3, where T_1, T_2, and T_3 are lattice paths in $\mathbb{S}_1$ with T_1 non-empty. In the first case, T_1 has two options: the last step is a down-step or a left step, see Figure 6. Then, this case contributes to the generating function the term $x(yD+L)$.

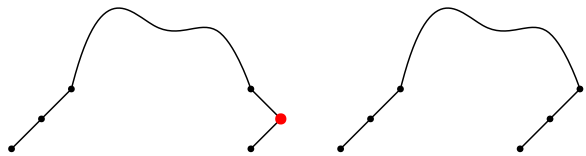

Figure 6. Decomposition of a skew Dyck path.

On the other hand, T_2 can be an empty path or a path in $\mathcal{D}$ or $\mathcal{L}$. If T_3 is empty, then this case contributes to the generating function the term $x(y+D+Ly)$. On the other hand, if the path T_3 is non-empty, then this case contributes to the generating function the term $x(y+D+yL)y(W_1-1)$, see Figure 7. Summarizing these cases, we obtain the functional equation

$$W_1 = 1 + x(yD+L) + x(y+D+yL)(1+y(W_1-1)).$$

From a similar argument, we obtain the equations

$$D = x(y+D+yL)(1+yD) \quad \text{and} \quad L = x(yD+L) + x(y+D+yL)(yL).$$

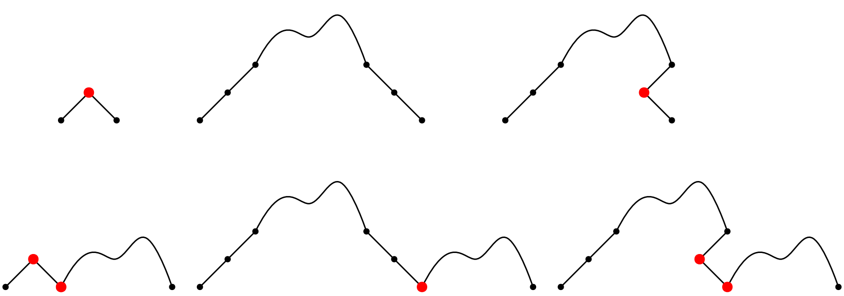

Figure 7. Decomposition of a skew Dyck path.

Using the Gröbner basis on the polynomial equations for W_1, D, and L, we obtain the desired result. □

We can use a symbolic software computation to obtain the first few terms of the formal power series of $W_1(x,y)$ as follows:

$$W_1(x,y) = 1 + xy + x^2(y + y^2 + y^3) + x^3(y + 2y^2 + 4y^3 + 2y^4 + y^5) + x^4(y + 3y^2 + 9y^3 + 9y^4 + 10y^5 + 3y^6 + y^7) + \cdots .$$

From the equation given in Theorem 4.1, we obtain

$$3xS^3(x) + 6xS^2(x)K(x) - 2xS^2(x) - 2(2-x)S(x)K(x) + 3(1-x)K(x) = 0,$$

where $K(x)$ is the generating function for the total number of corners in skew Dyck paths and $S(x) = (1 - x - \sqrt{1-6x+5x^2})/(2x)$ is the generating function for the number of the skew Dyck paths. Solving the above equation, we obtain the generating function

$$K(x) = \frac{2(1-x)(3+x)x}{(1-x)(3-2x)(1-5x) + (3-11x+4x^2)\sqrt{1-6x+5x^2}} = x + 6x^2 + 30x^3 + 145x^4 + 695x^5 + 3327x^6 + 15945x^7 + \cdots .$$

Theorem 4.2. *The generating function $W_2(x,y)$ satisfies the functional equation*

$$\begin{aligned}
&x^2y^4(1+y)^3W_2^6 - xy^2(1+y)^2(1 - x(1+6y+y^2-3y^3))W_2^5\\
&+ xy(-4 - 7y + 3y^3 + x(1+y)^2(4 + 9y - 11y^2 - 6y^3 + 3y^4))W_2^4\\
&+ (4 - 2x(1+y)^2(4 - 7y + y^2) - x^2(1+y)^2(-4 + 2y + 21y^2 - 8y^3 - 5y^4 + y^5))W_2^3\\
&+ (-12 - x^2(1+y)^2(8 + 4y - 18y^2 + 4y^3 + y^4) - 2x(-10 - 9y + 6y^2 + 6y^3 + y^4))W_2^2\\
&+ (12 + x^2(1+y)^2(5 + 4y - 7y^2 + y^3) + x(-17 - 16y + 2y^2 + 4y^3 + 3y^4))W_2\\
&+ (-4 + x^2(1+y)^2(-1 - y + y^2) + x(5 + 4y - y^4)) = 0.
\end{aligned}$$

Proof. Let $\mathcal{D}_2$ and $\mathcal{L}_2$ denote the 2-Fuss-skew paths whose last step is a down-step or a left-step, respectively. Let D_2 and L_2 denote the generating functions defined by

$$D_2 = \sum_{P\in\mathcal{D}_2} x^{u(P)}y^{\tau(P)} \quad \text{and} \quad L_2 = \sum_{P\in\mathcal{L}_2} x^{u(P)}y^{\tau(P)}.$$

From a similar argument as in the proof of Theorem 4.1, we obtain the system of polynomial equations

$$\begin{aligned}
W_2 &= 1 + x((y + yD_2 + L_2)(1 + y^2D_2 + yL_2)(1 + y(W_2 - 1)) + (D_2 + yL_2)\\
&\quad + (D_2 + yL_2)y(1 + y(W_2 - 1)) + (y + yD_2 + L_2)(y + yD_2 + y^2L_2)),\\
D_2 &= x((y + yD_2 + L_2)(1 + y^2D_2 + yL_2)yD_2 + (D_2 + yL_2) + (D_2 + yL_2)y(yD_2)\\
&\quad + (y + yD_2 + L_2)(y + yD_2 + y^2L_2)),\\
L_2 &= x((y + yD_2 + L_2)(1 + y^2D_2 + yL_2)(1 + yL_2) + (D_2 + yL_2)y(1 + yL_2)).
\end{aligned}$$

By using the Gröbner basis, we obtain the desired result. □

Expanding with *Mathematica* the functional equation for W_2, we find

$$W_2(x,y) = 1 + (y + y^2)x + (y + 3y^2 + 5y^3 + 4y^4 + y^5)x^2 \\ + (y + 5y^2 + 16y^3 + 27y^4 + 33y^5 + 25y^6 + 9y^7 + 2y^8)x^3 + \cdots .$$

Moreover, the first few terms of the total number of corners in $\mathbb{S}_2$ are

$$3x+43x^2+561x^3+7209x^4+92703x^5+1197151x^6+15532917x^7+202428373x^8+\cdots .$$

From Figure 4 one can verify that there are 43 corners over all paths in $\mathbb{S}_{2,2}$.

5. Other generalization

Let $\mathbb{H}_\ell$ denote the skew Dyck paths where left steps are below the line $y = \ell$. In particular, $\mathbb{H}_0$ are the Dyck path and $\mathbb{H}_\infty$ are the skew Dyck path. We define the generating function

$$H_\ell(x,p,q) := \sum_{P\in\mathbb{H}_\ell} x^{u(P)} p^{d(P)} q^{t(P)}.$$

For simplicity, we use H_ℓ to denote the generating function $H_\ell(x,p,q)$.

Theorem 5.1. *For $\ell \geq 1$, we have*

$$H_\ell = 1 + qx(H_{\ell-1} - 1) + pxH_{\ell-1}H_\ell, \tag{5.1}$$

with the initial value $H_0 = \frac{1-\sqrt{1-4px}}{2px}$.

Proof. A non-empty skew Dyck path in $\mathbb{H}_\ell$ can be decomposed as UT_1L or UT_2DT_3, where $T_1, T_2 \in \mathbb{H}_{\ell-1}$ with T_1 a non-empty path, and $T_3 \in \mathbb{H}_\ell$. From this decomposition follows the functional equation. □

Recall that the mth Chebyshev polynomial of the second kind satisfies the recurrence relation $U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$ with $U_0(t) = 1$ and $U_1(t) = 2t$. Thus by induction on ℓ and Theorem 5.1, we obtain the following result.

Theorem 5.2. *Let $t = \frac{1+qx}{2\sqrt{x(p+q-pqx)}}$ and $r = \sqrt{x(p+q-pqx)}$. The generating function H_ℓ is given by*

$$\frac{(qxU_{n-1}(t) - rU_{n-2}(t))C(px) + (1-qx)U_{n-1}(t)}{U_{n-1}(t) - rU_{n-2}(t) - pxU_{n-1}(t)C(px)},$$

where U_m is the mth Chebyshev polynomial of the second kind and $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.

The generating functions for the total number of skew Dyck path in $\mathbb{H}_\ell$ for $\ell = 1, 2, 3$ are

$$H_1(x,1,1) = \frac{3-2x-\sqrt{1-4x}}{1+\sqrt{1-4x}},$$
$$H_2(x,1,1) = \frac{1+2x-2x^2-(1-2x)\sqrt{1-4x}}{1-x-2(1-x)x+(1+x)\sqrt{1-4x}},$$
$$H_3(x,1,1) = \frac{1-3x+7x^2-4x^3+(1+x-3x^2)\sqrt{1-4x}}{1-4x+2x^3+(1+2x^2)\sqrt{1-4x}}.$$

Acknowledgements. We thanks Mark Shattuck for his comments on the previous version of the paper.

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