

Padovan and Perrin numbers of the form $x^a \pm x^b + 1$

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Abstract. Let P_k be the k th Padovan number and E_k be the k th Perrin number. In this paper, we study the Padovan and Perrin numbers of the form $x^a \pm x^b + 1$. In particular, we first find an upper bound for a, b, n as a function of x . Moreover, we determine all Padovan numbers and Perrin numbers of the form $x^a \pm x^b + 1$ such that $0 \leq b < a$ and $2 \leq x \leq 20$.

Keywords: Padovan numbers, Perrin numbers Linear form in logarithms, reduction method

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1. Introduction

Let $\mathbf{U} = \{U_n\}_{n \geq 0}$ be some *interesting sequence* of positive integers. The problem of finding U_n in a particular form has a very rich history. In 2006, Bugeaud, Mignotte and Siksek [2] proved that the only perfect power Fibonacci numbers are 0, 1, 8, 144 and the only perfect powers among Lucas numbers are 1, 4. Luca and Szalay [6] showed that there are only finitely many Fibonacci numbers of the form $p^a \pm p^b + 1$, where p is a number and a and b are positive integers with $\max\{a, b\} \geq 2$. In [8], Marques and Togbé determined all the Fibonacci and Lucas numbers of the form $2^a + 3^b + 5^c$, where a, b and c are nonnegative integers with $c \geq \max\{a, b\}$. In [1], Bravo and Luca determined all the generalized Fibonacci numbers which are some

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powers of two. Very recently, Qu and Zeng [10] determined all the Pell and Pell-Lucas numbers that are of the form $-2^a - 3^b + 5^c$, where a, b and c are nonnegative integers with some restrictions. For more related results, one can see [3–5, 7, 12].

In this paper, we continue this discussion to the sequences of Padovan, and Perrin numbers, which we define below. The Padovan sequence $\{P_m\}_{m \geq 0}$ is defined by

$$P_{m+3} = P_{m+1} + P_m,$$

for $m \geq 0$, where $P_0 = P_1 = P_2 = 1$. This is the sequence A000931 in the OEIS [14]. A few terms of this sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, \dots$$

Let $\{E_m\}_{m \geq 0}$ be the Perrin sequence given by

$$E_{m+3} = E_{m+1} + E_m,$$

for $m \geq 0$, where $E_0 = 3, E_1 = 0$ and $E_2 = 2$. Its first few terms are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, 367, 486, 644, \dots$$

It is the sequence A001608 in the OEIS [14].

Now, we are interested in studying the Padovan and Perrin numbers which are in the form of $x^a \pm x^b + 1$. More precisely, we consider the following equations

$$P_n = x^a \pm x^b + 1 \tag{1.1}$$

$$E_n = x^a \pm x^b + 1 \tag{1.2}$$

and prove the following results.

Theorem 1.1. *All the solutions of Diophantine equation (1.1) satisfy*

$$a < n < 2.58 \cdot 10^{31}(\log x)^4. \tag{1.3}$$

Furthermore, the only solutions of Diophantine equation (1.1) in positive integers (n, x, a, b) with $0 \leq b < a$ and $2 \leq x \leq 20$ are

$P_3 = P_4 = 2^1 - 2^0 + 1$	$P_{10} = 12^1 - 12^0 + 1$
$P_5 = 2^2 - 2^1 + 1 = 3^1 - 3^0 + 1$	$P_{11} = 2^4 - 2^0 + 1 = 4^2 - 4^0 + 1 = 16^1 - 16^0 + 1$
$P_6 = 2^2 - 2^0 + 1 = 4^1 - 4^0 + 1$	
$P_7 = 2^3 - 2^2 + 1 = 5^1 - 5^0 + 1$	$P_{12} = 5^2 - 5^1 + 1$
$P_8 = 2^3 - 2^1 + 1 = 3^2 - 3^1 + 1 = 7^1 - 7^0 + 1$	$P_{15} = 2^6 - 2^4 + 1 = 4^3 - 4^2 + 1 = 7^2 - 7^0 + 1$
$P_9 = 2^4 - 2^3 + 1 = 3^2 - 3^0 + 1 = 9^1 - 9^0 + 1$	$P_{16} = 2^7 - 2^6 + 1$

and

$$P_6 = 2^1 + 2^0 + 1$$

$$P_7 = 3^1 + 3^0 + 1$$

$$P_8 = 2^2 + 2^1 + 1 = 5^1 + 5^0 + 1$$

$$P_9 = 7^1 + 7^0 + 1$$

$$P_{10} = 10^1 + 10^0 + 1$$

$$P_{12} = 2^4 + 2^2 + 1 = 4^2 + 4^1 + 1 = 19^1 + 19^0 + 1$$

$$P_{14} = 2^5 + 2^2 + 1 = 3^3 + 3^3 + 1$$

$$P_{15} = 2^5 + 2^4 + 1$$

$$P_{19} = 5^3 + 5^2 + 1.$$

Theorem 1.2. *All the solutions of Diophantine equation (1.2) satisfy*

$$n \leq 3.35 \cdot 10^{30} (\log x)^4. \quad (1.4)$$

Furthermore, the only solutions of Diophantine equation (1.2) in positive integers (n, x, a, b) with $0 \leq b < a$ and $2 \leq x \leq 20$ are

$$E_0 = 2^2 - 2^1 + 1 = 3^1 - 3^0 + 1$$

$$E_2 = 2^1 - 2^0 + 1$$

$$E_3 = 2^2 - 2^1 + 1 = 3^1 - 3^0 + 1$$

$$E_4 = 2^2 - 2^1 + 1$$

$$E_5 = E_6 = 2^3 - 2^2 + 1 = 5^1 - 5^0 + 1$$

$$E_7 = 2^2 - 2^1 + 1 = 3^2 - 3^1 + 1 = 7^1 - 7^0 + 1$$

$$E_8 = 10^1 - 10^0 + 1$$

$$E_9 = 12^1 - 12^0 + 1$$

$$E_{10} = 2^5 - 2^4 + 1 = 17^1 - 17^0 + 1$$

$$E_{12} = 2^5 - 2^2 + 1$$

and

$$E_5 = E_6 = 3^1 + 3^0 + 1$$

$$E_7 = 2^2 + 2^1 + 1 = 5^1 + 5^0 + 1$$

$$E_8 = 2^3 + 2^0 + 1 = 8^1 + 8^0 + 1$$

$$E_9 = 10^1 + 10^0 + 1$$

$$E_{10} = 2^2 + 2^1 + 1 = 15^1 + 15^0 + 1$$

$$E_{11} = 20^1 + 20^0 + 1$$

$$E_{12} = 3^1 + 3^0 + 1$$

$$E_{14} = 7^2 + 7^0 + 1.$$

The outline of this paper is as follows. In section 2, we recall some results that are useful for the proofs of Theorem 1.1 and Theorem 1.2. Particularly, we recall some of the properties of Padovan and Perrin numbers, a result of Matveev [9] that we will use to obtain lower bounds for linear forms in logarithms of algebraic numbers, de Weger reduction method [15]. In the last two sections, we will completely prove Theorem 1.1 and Theorem 1.2 using Baker method and the reduction method.

2. Auxiliary results

First, we recall some facts and properties of the Padovan and the Perrin sequences which will be used later. One can see [11]. The characteristic equation

$$\Psi(x) := x^3 - x - 1 = 0$$

has roots $\alpha, \beta, \gamma = \bar{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12},$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Let

$$\begin{aligned} c_\alpha &= \frac{(1 - \beta)(1 - \gamma)}{(\alpha - \beta)(\alpha - \gamma)} = \frac{1 + \alpha}{-\alpha^2 + 3\alpha + 1}, \\ c_\beta &= \frac{(1 - \alpha)(1 - \gamma)}{(\beta - \alpha)(\beta - \gamma)} = \frac{1 + \beta}{-\beta^2 + 3\beta + 1}, \\ c_\gamma &= \frac{(1 - \alpha)(1 - \beta)}{(\gamma - \alpha)(\gamma - \beta)} = \frac{1 + \gamma}{-\gamma^2 + 3\gamma + 1} = \bar{c}_\beta. \end{aligned}$$

Binet's formula for P_n is

$$P_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \quad \text{for all } n \geq 0 \quad (2.1)$$

and Binet's formula for E_n is

$$E_n = \alpha^n + \beta^n + \gamma^n, \quad \text{for all } n \geq 0. \quad (2.2)$$

Numerically, we have

$$\begin{aligned} 1.32 &< \alpha < 1.33, \\ 0.86 &< |\beta| = |\gamma| < 0.87, \\ 0.72 &< c_\alpha < 0.73, \\ 0.24 &< |c_\beta| = |c_\gamma| < 0.25. \end{aligned}$$

It is easy to check that

$$|\beta| = |\gamma| = \alpha^{-1/2}.$$

Further, using induction on n , we can prove that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1}, \quad \text{for all } n \geq 4 \quad (2.3)$$

and

$$\alpha^{n-2} \leq E_n \leq \alpha^{n+1}, \quad \text{for all } n \geq 2. \quad (2.4)$$

Let $\mathbb{K} := \mathbb{Q}(\alpha, \beta)$ be the splitting field of the polynomial Ψ over \mathbb{Q} . Then, $[\mathbb{K} : \mathbb{Q}] = 6$. The Galois group of \mathbb{K} over \mathbb{Q} is given by

$$\text{Gal}(\mathbb{K}/\mathbb{Q}) \cong \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\} \cong S_3.$$

The next tools are related to the transcendental approach to solve Diophantine equations. For any non-zero algebraic number γ of degree d over \mathbb{Q} , whose minimal polynomial over \mathbb{Z} is $a \prod_{j=1}^d (X - \gamma^{(j)})$, we denote by

$$h(\gamma) = \frac{1}{d} \left(\log |a| + \sum_{j=1}^d \log \max(1, |\gamma^{(j)}|) \right)$$

the usual absolute logarithmic height of γ .

To prove Theorems 1.1 and 1.2, we use lower bounds for linear forms in logarithms to bound the index n appearing in equations (1.1) and (1.2). We need the following general lower bound for linear forms in logarithms due to Matveev [9].

Lemma 2.1. *Let $\gamma_1, \dots, \gamma_s$ be a real algebraic numbers and let b_1, \dots, b_s be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \dots, \gamma_s)$ over \mathbb{Q} and let A_j be a positive real number satisfying*

$$A_j = \max\{Dh(\gamma), |\log \gamma|, 0.16\} \quad \text{for } j = 1, \dots, s.$$

Assume that

$$B \geq \max\{|b_1|, \dots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} \neq 1$, then

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-C(s, D)(1 + \log B)A_1 \cdots A_s),$$

where $C(s, D) := 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)$.

After getting the upper bound of n which is generally too large, the next step is to reduce it. For this reduction, we present a variant of the reduction method of Baker and Davenport due to de Weger [15].

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{2.5}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{2.6}$$

$$Y \leq X \leq X_0. \tag{2.7}$$

When $\beta = 0$ in (2.5), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \dots]$$

and let the k th convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \dots$. We have the following results.

Lemma 2.2 ([15, Lemma 3.2]). *Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1},$$

where

$$Y_0 = -1 + \frac{\log(\sqrt{5}X_0 + 1)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

If (2.6) and (2.7) hold for x_1, x_2 and $\beta = 0$, then

$$Y < \frac{1}{\delta} \log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right).$$

When $\beta \neq 0$ in (2.5), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x , we let $\|x\| = \min\{|x - n|, n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 2.3 ([15, Lemma 3.3]). *Suppose that*

$$\|q\psi\| > \frac{2X_0}{q}.$$

Then, the solutions of (2.6) and (2.7) satisfy

$$Y < \frac{1}{\delta} \log\left(\frac{q^2c}{|\vartheta_2|X_0}\right).$$

We conclude this section by recalling two lemmas that we need in the sequel:

Lemma 2.4 ([13, Lemma 7]). *If $m \geq 1$, $T > (4m^2)^m$ and $T > y/(\log y)^m$. Then,*

$$y < 2^m T (\log T)^m.$$

Lemma 2.5 ([15, Lemma 2.2, page 31]). *Let $a, x \in \mathbb{R}$ and $0 < a < 1$. If $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a} |x|$$

and

$$|x| < \frac{a}{1-e^{-a}} |e^x - 1|.$$

3. Padovan numbers of the form $x^a \pm x^b + 1$

In this section, we will prove Theorem 1.1.

3.1. The proof of inequality (1.3)

First of all, we find a relation between a and n . By combining (2.3) together with (1.1), we get

$$\alpha^{n-2} < P_n = x^a \pm x^b + 1 < x^a + x^b + 1 < 3x^a \tag{3.1}$$

and

$$\frac{x^a}{2} < x^a - x^b < x^a \pm x^b + 1 = P_n < \alpha^{n-1}. \tag{3.2}$$

Taking logarithms on both sides of inequalities (3.1) and (3.2) and combining them, we obtain

$$\left(\frac{\log x}{\log \alpha}\right)a - \frac{\log 2}{\log \alpha} + 1 < n < \left(\frac{\log x}{\log \alpha}\right)a + \frac{\log 3}{\log \alpha} + 2. \tag{3.3}$$

Particularly, using the fact that $x \geq 2$, we conclude that

$$a < n. \tag{3.4}$$

By using (2.1), we rewrite equation (1.1) as

$$|c_\alpha \alpha^n - x^a| \leq 2|c_\beta| |\beta|^n + x^b + 1 < x^{b+1}.$$

Dividing both sides of the last inequality by x^a , we get

$$|c_\alpha \alpha^n x^{-a} - 1| < \frac{1}{x^{a-b-1}}. \tag{3.5}$$

Now, we apply Matveev’s result (see Lemma 2.1) to the left-hand side of (3.5). First, the expression on the left-hand side of (3.5) is nonzero, since this expression being zero means that $c_\alpha \alpha^n = x^a \in \mathbb{Z}$, which is false since if we conjugate this relation by the automorphism of Galois $\sigma := (\alpha\beta)$ we would get $1 < x^a = |c_\beta \beta^n| < 1$. In order to apply Lemma 2.1, we take $s := 3$,

$$(\gamma_1, b_1) := (c_\alpha, 1), \quad (\gamma_2, b_2) := (\alpha, n) \quad (\gamma_3, b_3) := (x, -a).$$

For this choice we have $D = 3$, $h(\gamma_1) = (\log 23)/3$, $h(\gamma_2) = (\log \alpha)/3$, $h(\gamma_3) = \log x$ and $\max\{1, n, a\} \leq n$. In conclusion, $B := n$, $A_1 := 3.2$, $A_2 := 0.3$ and $A_3 := 3 \log x$ are suitable choices. By Lemma 2.1, we obtain the following estimate

$$|c_\alpha \alpha^n x^{-a} - 1| \geq \exp(-7.79 \cdot 10^{12} \cdot \log x \cdot (1 + \log n)). \tag{3.6}$$

We combine (3.5) and (3.6) to obtain

$$a - b < 7.8 \cdot 10^{12}(1 + \log n). \tag{3.7}$$

We now use a second linear form in logarithms by rewriting equation (1.1) in a different way. Using Binet formula (2.1), we get that

$$|c_\alpha \alpha^n - (x^{a-b} \pm 1)x^b| \leq 2|c_\beta| |\beta|^n + 1 < 1.5.$$

Dividing both sides of the above inequality by $x^a \pm x^b$, we obtain

$$\left| c_\alpha (x^{a-b} \pm 1)^{-1} \alpha^n x^{-b} - 1 \right| < \frac{1.5}{x^a \pm x^b} < \frac{1}{\alpha^{n-10}}, \quad (3.8)$$

where we have also used the fact that $x^a \pm x^b > x^a/2$, $\alpha^{n-2} < 3x^a$ and $9 < \alpha^8$.

We observe that the left-hand side of (3.8) is nonzero, otherwise we would get

$$x^a \pm x^b = c_\alpha \alpha^n. \quad (3.9)$$

Conjugating (3.9) in $\mathbb{Q}(\alpha, \beta)$ by the automorphism $\sigma := (\alpha\beta)$, we get

$$1 < x^a \pm x^b = |c_\beta \beta^n| < 1.$$

Now we again apply Lemma 2.1 as before but with $s := 3$,

$$\gamma_1 := c_\alpha (x^{a-b} \pm 1)^{-1}, \quad \gamma_2 := \alpha, \quad \gamma_3 := x, \quad b_1 := 1, \quad b_2 := n, \quad b_3 := -b.$$

Here, we can take $D = 3$, $B := n$, $A_2 := 0.3$, $A_3 := 3 \log x$ and since (using the properties of absolute logarithmic height and inequality (3.7))

$$h(\alpha_1) < h(c_\alpha) + h(x^{a-b}) + \log 2 < 7.81 \cdot 10^{12} \log x(1 + \log n),$$

then we can take $A_1 := 2.35 \cdot 10^{13} \log x(1 + \log n)$. We obtain the following estimate

$$\exp(-5.72 \cdot 10^{25} \cdot (\log x)^2 (1 + \log n)^2) \leq \frac{1}{\alpha^{n-10}},$$

which leads us to

$$\frac{n}{(\log n)^2} < 8.14 \cdot 10^{26} (\log x)^2. \quad (3.10)$$

By applying Lemma 2.4 in the inequality (3.10), we obtain

$$n < 2^2 \cdot (8.14 \cdot 10^{26} (\log x)^2) \cdot (\log(8.14 \cdot 10^{26} (\log x)^2))^2. \quad (3.11)$$

Finally, combining equations (3.4) and (3.11), and using the fact that

$$\log(8.14 \cdot 10^{26} (\log x)^2) < 89 \log x, \quad \text{for } x \geq 2,$$

we obtain

$$a < n < 2.58 \cdot 10^{31} (\log x)^4. \quad (3.12)$$

This proves the first part of Theorem 1.1. Next, we determine the solutions of equation (1.1) in the specified range.

3.2. The solutions of equation (1.1) for $2 \leq x \leq 20$

Let x be a fixed integer such that $2 \leq x \leq 20$. The inequality (3.12) gives

$$n < 2.08 \cdot 10^{33}. \quad (3.13)$$

The upper bound of n given by (3.13) is very large, so we will reduce it further. To do this, we will use several times Lemma 2.3. From inequality (3.5), we put

$$\Lambda_1 := n \log \alpha - a \log x + \log c_\alpha \quad \text{and} \quad \Gamma_1 := e^{\Lambda_1} - 1.$$

Then, for $a - b \geq 2$ and $2 \leq x \leq 20$, we have

$$|\Gamma_1| < \frac{1}{x^{a-b-1}} < \frac{1}{2^{a-b-1}} < \frac{1}{2}.$$

By Lemma 2.5 and the above inequality, we get

$$|\Lambda_1| = |\log(\Gamma_1 + 1)| < \frac{4 \log 2}{2^{a-b}} < 2.8 \exp(-0.69(a - b)).$$

Since $\max\{a, n\} = n$, then inequality (3.13) implies that we can take $X_0 := 2.08 \cdot 10^{33}$. Further, we choose

$$\begin{aligned} c &:= 2.8, \quad \delta := 0.69, \quad \beta := \log c_\alpha, \\ (\vartheta_1, \vartheta_2) &:= (\log \alpha, \log x), \quad \vartheta := -\log \alpha / \log x, \quad \psi := \log c_\alpha / \log x. \end{aligned}$$

Using Maple, we find that q_{80} satisfies the hypotheses of Lemma 2.3, for all $x \in [2, 20]$. Furthermore, Lemma 2.3 implies the inequality

$$a - b \leq 237$$

in all cases.

Now, assume that $a - b \leq 237$. Let us consider

$$\Lambda_2 := n \log \alpha - b \log x - \log(c_\alpha(x^{a-b} \pm 1)) \quad \text{and} \quad \Gamma_2 := e^{\Lambda_2} - 1.$$

Then for $n \geq 13$, we have

$$|\Gamma_2| < \frac{1}{\alpha^3} < \frac{1}{2},$$

(see (3.8)). By Lemma 2.5, we get

$$|\Lambda_2| = |\log(\Gamma_2 + 1)| < \frac{2 \log 2}{\alpha^{n-10}} < 23.1 \exp(-0.28n).$$

Since $\max\{b, n\} = n$, then inequality (3.13) implies that we can take $X_0 := 2.08 \cdot 10^{33}$. Further, we can choose

$$c := 23.1, \quad \delta := 0.28, \quad \beta_m := -\log(c_\alpha(x^m \pm 1)), \quad 1 \leq m \leq 237,$$

$$(\vartheta_1, \vartheta_2) := (\log \alpha, -\log x) \quad \vartheta := -\log \alpha / \log x, \quad \psi_m = -\log(c_\alpha(x^m \pm 1)) / \log x.$$

Again, we use Maple to find that q_{120} satisfies the hypotheses of Lemma 2.3 for all $x \in [2, 20]$ and $m \in [1, 237]$. Moreover, Lemma 2.3 implies that $n \leq 845$ in all cases.

We use Maple for the second time using the new upper bound of n with q_{20} , we get $n \leq 196$ which implies by (3.3) that $a \leq 82$.

Finally, we write a program in Maple to obtain P_n 's which are of the form $x^a \pm x^b + 1$ with $2 \leq x \leq 20$, $1 \leq n \leq 196$, and $1 \leq a \leq 82$. One can check that the only solutions of equation (1.1) are those cited in Theorem 1.1. This completes the proof of Theorem 1.1.

4. Perrin numbers of the form $x^a \pm x^b + 1$

In this section, we will prove Theorem 1.2 using the above method to prove Theorem 1.1. For the sake of completeness, we will give almost all of the details.

4.1. The proof of inequality (1.4)

First of all, we will explore a relation between a and n . By combining (2.4) together with (1.2), we get

$$\alpha^{n-1} < E_n = x^a \pm x^b + 1 < x^a + x^b + 1 < 3x^a, \quad (4.1)$$

and

$$\frac{x^a}{2} < x^a - x^b < x^a \pm x^b + 1 = E_n < \alpha^{n+1}. \quad (4.2)$$

By taking logarithms on both sides of inequalities (4.1) and (4.2) and putting them together, we obtain

$$\left(\frac{\log x}{\log \alpha}\right)a - \frac{\log 2}{\log \alpha} - 1 < n < \left(\frac{\log x}{\log \alpha}\right)a + \frac{\log 3}{\log \alpha} + 1.$$

Particularly, using the fact that $x \geq 2$, we conclude that

$$a < n.$$

By using (2.2), we rewrite equation (1.2) into the form of

$$|\alpha^n - x^a| \leq 2|\beta|^n + x^b + 1 < x^{b+2}.$$

Dividing both sides of the last inequality by x^a , we get

$$|\alpha^n x^{-a} - 1| < x^{-(a-b-2)}. \quad (4.3)$$

Now, we are in a situation to apply Matveev's result (see Lemma 2.1) to the left-hand side of (4.3). The expression on the left-hand side of (4.3) is nonzero, since

this expression being zero means that $x^a = \alpha^n$. So $\alpha^n \in \mathbb{Z}$ for some positive integer n , which is false. In order to apply Lemma 2.1, we take $s := 3$,

$$\gamma_1 := \alpha, \quad \gamma_2 := x, \quad b_1 := n, \quad b_2 := -a.$$

For this choice, we have $D = 3$, $h(\gamma_1) = (\log \alpha)/3$, $h(\gamma_2) = \log x$, and $\max\{1, n, a\} = n$. In conclusion, we take $B := n$, $A_1 := 0.3$ and $A_2 := 3 \log x$. By Lemma 2.1, we obtain the following estimate

$$|\alpha^n x^{-a} - 1| \geq \exp(-1.18 \cdot 10^{11} \cdot \log x \cdot (1 + \log n)). \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$a - b < 1.19 \cdot 10^{11} (1 + \log n). \tag{4.5}$$

We now use a second linear form in logarithms by rewriting equation (1.2) in a little different way. Using Binet formula (2.2), we get

$$|\alpha^n - (x^{a-b} \pm 1)x^b| \leq 2|\beta|^n + 1 < 2.$$

Dividing both sides of the above inequality by $x^a \pm x^b$, we obtain

$$\left| (x^{a-b} \pm 1)^{-1} \alpha^n x^{-b} - 1 \right| < \frac{2}{x^a \pm x^b} < \frac{1}{\alpha^{n-10}}, \tag{4.6}$$

where we have also used the fact that $x^a \pm x^b > x^a/2$, $\alpha^{n-1} < 3x^a$ and $12 < \alpha^9$.

We observe that the left-hand side of (4.6) is nonzero, otherwise we would get

$$x^a \pm x^b = \alpha^n. \tag{4.7}$$

Conjugating (4.7) in $\mathbb{Q}(\alpha, \beta)$ by using the automorphism of Galois $\sigma := (\alpha\beta)$, we get

$$1 < |x^a \pm x^b| = \beta^n < 1.$$

Now, let us apply again Lemma 2.1 as before but with $s := 3$,

$$\gamma_1 := x^{a-b} \pm 1, \quad \gamma_2 := \alpha, \quad \gamma_3 := x, \quad b_1 := -1, \quad b_2 := n, \quad b_3 := -b.$$

Again here, we take $D = 3$, $B := n$, $A_2 := 0.9$, $A_3 := 3 \log x$ and since (using the properties of absolute logarithmic height and inequality (4.5))

$$h(\alpha_1) < h(x^{a-b}) + \log 2 < 1.2 \cdot 10^{11} \log x (1 + \log n),$$

then we can take $A_1 := 3.6 \cdot 10^{11} \log x (1 + \log n)$. We obtain the following estimate

$$\exp(-7.89 \cdot 10^{24} \cdot (\log x)^2 (1 + \log n)^2) \leq \frac{1}{\alpha^{n-10}},$$

which leads us to

$$\frac{n}{(\log n)^2} < 1.13 \cdot 10^{26} (\log x)^2, \tag{4.8}$$

where we used $1 + \log n < 2 \log n$. Finally, by Lemma 2.4 and using fact that $\log(1.13 \cdot 10^{26} (\log x)^2) < 86 \log x$, for $x \geq 2$, inequality (4.8) gives us

$$n \leq 3.35 \cdot 10^{30} (\log x)^4. \tag{4.9}$$

This proves the first part of Theorem 1.2.

4.2. The solutions of equation (1.2) for $2 \leq x \leq 20$

Let x be a fixed integer such that $2 \leq x \leq 20$. The inequality (4.9) becomes

$$n < 2.7 \cdot 10^{32}. \quad (4.10)$$

Now, we will reduce the upper bound of n given by (4.10) as this bound is very large. To do this, we will use several times Lemma 2.2. From inequality (4.3), we put

$$\Lambda_3 := n \log \alpha - a \log x \quad \text{and} \quad \Gamma_3 := e^{\Lambda_3} - 1.$$

Then, for $a - b \geq 3$ and $2 \leq x \leq 20$, we have

$$|\Gamma_3| < \frac{1}{x^{a-b-1}} < \frac{1}{2^{a-b-2}} < \frac{1}{2}.$$

By Lemma 2.5 and the above inequality, we get

$$|\Lambda_3| = |\log(\Gamma_3 + 1)| < \frac{4 \log 2}{2^{a-b}} < 2.8 \exp(-0.69(a - b)).$$

As $\max\{a, n\} = n$, then inequality (4.10) implies that we take $X_0 := 2.7 \cdot 10^{32}$ and

$$Y_0 := 155.85544\dots, \quad c := 2.8, \quad \delta := 0.69, \\ (\vartheta_1, \vartheta_2) := (\log \alpha, -\log x), \quad \vartheta := -\log \alpha / \log x.$$

Using Maple, we find that

$$A = 1584.$$

So from Lemma 2.2, we deduce that

$$a - b \leq 121$$

in all the cases.

Suppose now that $a - b \leq 121$. Let us consider

$$\Lambda_4 := n \log \alpha - b \log x - \log(x^{a-b} \pm 1) \quad \text{and} \quad \Gamma_4 := e^{\Lambda_4} - 1.$$

Then for $n \geq 13$, inequality (4.6) give

$$|\Gamma_4| < \frac{1}{\alpha^3} < \frac{1}{2}.$$

By Lemma 2.5, we get

$$|\Lambda_4| = |\log(\Gamma_4 + 1)| < \frac{2 \log 2}{\alpha^{n-10}} < 23.1 \exp(-0.28n).$$

We know that $\max\{b, n\} = n$, then inequality (3.13) implies that we can take $X_0 := 2.7 \cdot 10^{32}$. Further, we choose

$$c := 23.1, \quad \delta := 0.28, \quad \beta_m := -\log(x^m \pm 1), \quad 1 \leq m \leq 101,$$

$$(\vartheta_1, \vartheta_2) := (\log \alpha, -\log x), \quad \vartheta := -\log \alpha / \log x, \quad \psi_m := \log(x^m \pm 1) / \log x.$$

With Maple, we find that q_{125} satisfies the hypotheses of Lemma 2.3 for all $x \in [2, 20]$ and $m \in [1, 121]$ except in the cases

$$(a, x) \in \{(1, 1), (1, 3), (1, 9), (2, 5), (2, 3), (2, 9), (3, 3), (3, 9), (3, 7), (4, 15), (4, 17)\}.$$

Furthermore, Lemma 2.3 gives us $n \leq 890$ in all the cases.

In the cases when

$$(a, x) \in \{(1, 1), (1, 3), (1, 9), (2, 5), (2, 3), (2, 9), (3, 3), (3, 9), (3, 7), (4, 15), (4, 17)\},$$

we use Lemma 2.2 and get $n \leq 309$. So, in all the cases we have $n \leq 890$.

Finally, we write a program in Maple to determine E_n 's which are of the form of $x^a \pm x^b + 1$ with $2 \leq x \leq 20$, $1 \leq n \leq 890$, $1 \leq b < a \leq 340$. We find that the only solutions of the equation (1.2) are the ones cited in Theorem 1.2. Hence, Theorem 1.2 is completely proved.

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