Annales Mathematicae et Informaticae
55 (2022) pp. 189–195
DOI: https://doi.org/10.33039/ami.2022.12.009
URL: https://ami.uni-eszterhazy.hu

Passing the exam and not mastering the material in geometry*

Csilla Bereczky-Zámbó[bc†], Csaba Szabó[ac], Anna Muzsnay[dc], Janka Szeibert[bc]

[a]Eötvös Loránd University
szabo.csaba.mathdid@ttk.elte.hu

[b]Eötvös Loránd University, Faculty of Primary and Pre-School Education
csilla95@gmail.com
szeibert.janka@gmail.com

[c]MTA-ELTE Theory of Learning Mathematics Research Group

[d]University of Debrecen, Doctoral School of Mathematical and Computational Sciences
annamuzsnay@gmail.com

Abstract. It is a common assumption that taking a mathematics course and passing the exam means that one has mastered the course requirements and gained a sufficiently deep understanding of the course material. According to the communication part of the Van Hiele Theory, if someone does not reach the expected entry-level, they won't be able to develop during the course.

In our research, we investigated this contradiction in the field of geometry. We examined this phenomenon with mathematics major and pre-service mathematics teacher students during their first geometry course.

Keywords: van Hiele levels, understanding geometry, development

1. Introduction and problem statement

The role of communication is crucial in the teaching and the learning process [3]. According to Vygotsky's zone of proximal development (ZPD), during the teaching procedure, one should take into consideration the pupils' level of understanding and their knowledge of the terms [21, p. 86]. The theory about the zone of proximal

*This research was supported by the ÚNKP-19-2 New National Excellence Program of the Ministry for Innovation and Technology and by the ELTE Tehetséggondozási Tanács.
†The second author thanks the fund Mészáros Alapítvány for their support.

Submitted: May 31, 2022
Accepted: December 19, 2022
Published online: December 21, 2022

development was translated into the field of geometry by the van Hiele couple [17]. The van Hieles suggest a possible way of structuring and describing people's understanding of geometry: focusing on understanding geometrical shapes and structures. They distinguish five levels of geometrical understanding characterized as visual, descriptive, relational, formal deduction, and rigor. Their theory says that a student advances sequentially from the initial level (Visualization) to the highest level (Rigor). Students cannot achieve one level of thinking successfully without having passed through the previous levels. To move from a level to the next one, the teaching process has to start at the proper Van Hiele level.

It is a common assumption that taking a mathematics course and passing the exam means that one has mastered the course requirements and gained a sufficiently deep understanding of the course material. Parallelly, according to the communication part of the van Hiele theory, if someone does not reach the expected entry-level of a course, they will not be able to develop during the course in terms of understanding. However, it often happens that someone doesn't fulfill the prerequisites of a course and passes the exam. This is a contradiction arising the question which statement is true: "If one has completed the subject and passed the exam, one understands the material." or, "If one has arrived underprepared, one cannot gain a real understanding of the material and cannot pass the exam.". To investigate this question, we need to find some students who took a course underprepared (based on measurement at the beginning of the course), passed the exam, and we need to determine their level of understanding of the subject. In our research we measured mathematics major and pre-service mathematics teacher students' van Hiele level before and after taking a geometry course. The tool of measurement was the van Hiele Geometry Test [17]. The following research question guided our research: The van Hiele theory states that if during the teaching process the teacher's communication is not adequate considering students' actual geometric level, no real development can be achieved. Does this statement hold at higher van Hiele levels (levels 3, 4, and 5)?

2. Description of geometrical understanding in the National Core Curriculum

In order to investigate pupils' understanding of geometry van Hiele's framework have been used in over 40 countries [1, 2, 4, 6, 8, 9, 14, 16, 18–20, 24]. In these studies the test developed by Usiskin [17] was used as a measure. Investigations were typically carried out in primary schools and high schools, focusing mostly on van Hiele levels 1–3. A few studies have examined the level of pre-service teachers, where, surprisingly, in almost all cases, researchers have reported low performance [11, 12]. Pre-service teachers scored at level 3 or below. When it comes to the case of people with a higher level of geometric understanding, the number of studies is limited. The van Hiele theory is probably the best and most well-known theory for students' levels of thinking in the field of geometry, it is not obvious, whether or not the theory works efficiently at higher levels [4], especially on the fifth level [22].

This study explores the van Hiele level of Hungarian mathematics major students and pre-service mathematics teachers. The Hungarian National Core Curriculum is parallel to the van Hiele levels [15], here we present only the correspondence for levels 3–5. For lower levels see.

Level 3: Abstraction At level 2 students perceive relationships between properties and between figures, they are able to establish the interrelationships of properties both within figures (e.g., in a quadrilateral, opposite angels being equal necessitates opposite sides being equal) and among figures (a rectangle is a parallelogram because it has all the properties of a parallelogram). So, at this level, class inclusion is understood, and definitions are meaningful. They are also able to give informal arguments to justify their reasoning. However, a student at this level does not understand the role and significance of formal deduction.

Level 4: Deduction The 4th level is the level of deduction: students can construct smaller proofs (not just memorize them), understand the role of axioms, theorems, postulates and definitions, and recognize the meaning of necessary and sufficient conditions. The possibility of developing a proof in more than one way is also seen and distinctions between a statement and its converse can be made at this level.

Level 5: Rigor This level is the most abstract of all. A person at this stage can think and construct proofs in different kind of geometric axiomatic systems. So, students at this level can understand the use of indirect proof and proof by contra-positive and can understand non-Euclidean systems.

The logic of this structure is also confirmed by the observation that the Van Hiele levels can be recognized in the Hungarian National Core Curriculum [23] step by step. The following sentences and requirements connecting to different grades are from the NCC.

- Grade 5–8: "Triangles and their categories. Quadrilaterals, special quadrilateral (trapezoids, parallelograms, kites, rhombuses). Polygons, regular polygons. The circle and its parts. Sets of points that meet given criteria."
- Grade 9–12: "The classification of triangles and quadrilaterals. Altitudes, centroid, incircle and circumcircle of triangles. The incircle and circumcircle of regular polygons. Thales' theorem."
  "Remembering argumentation, refutations, deductions, trains of thought; applying them in new situations, remembering proof methods is important."
  "Generalization, concretization, finding examples and counterexamples (confirming general statements by deduction; proving, disproving: demonstrating errors by supplying a counterexample); declaring theorems and proving them (directly and indirectly) is also necessary."

The levels correspond to age group, an 8^{th} grader (14 years old) should be on at least level 3, and at grade 12 students (18 years old) have to reach level 4, which means they have to reach the level of deductions – students have to be able to construct smaller proofs, understand the role of axioms, theorems, postulates and definitions.

Our reseach was carried out at Eötvös Loránd University, Budapest. We chose the sampling procedure by convinience [10], 65 mathematics students and 46 mathematics pre-service teachers were involved in the study, all from Eötvös Loránd University. All 111 participants were starting their first geometry course in their second semester at the university and had had passed several mathematics exams before. According to the National Core Curriculum all students were on at least on the 4^{th} van Hiele level. Although the curriculums of pre-service teachers and of math majors differ, both courses require logical reasoning ability, understanding, and the ability of constructing proofs. We measured the Van Hiele levels of all students right before their first geometry courses, and two weeks after accomplishing the courses, as well. Mathematics students completed the van Hiele tests in paper, while pre-service teachers completed the test electronically. In this study, the 1–5 scheme was used for the levels, which is consistent with Pierre van Hiele's numbering of the levels. All references and all results from studies using the 0–4 scale have been translated to the 1–5 scheme.

3. Results and discussion

The results of the test can be seen in Table 1. Altogether 28 math major and all 66 presevice teachers students filled in the post-test. All preservice teachers filled in the post-test. They had a follow-up class with the researchers, hence they felt more oblidged to fill in the second round. Although at least level 4 is a prerequisite for both courses, in both of them more then 40% of students filled in the test at level 3. This is not a surprise, as earlier findings show that there is a gap between the knowledge of students entering the university and the expectations and prequisites of the universities' curriculum [5]. All other students filled in the test on level 5. The exam was an oral exam, where students had to explain a topic of the course with full proofs and had to answer the questions of the examiner. In both cases the examiner was the lecturer, different for the two courses. Hence, on the exam the student had to convince the professor about their understanding of the material. Such an exam lasts usually 20–40 minutes and is quite rigorous. On one hand, one would expect that passing this exam is a sign of understanding and mastering the material. And on the day of the exam it seemed to be true for all students. On the other hand we would expect that those who did not fulfill the course prequisites, namely who were on level 3, will not be able to develope on the course, and will remain on level 3. After two weeks of the exams it seemed to be true. At the same time, the expectation is that passing such an exam means being on level 4 or 5. So, it is not easy to decide, whether or not it is a surprise that all students who filled in the test on level 3 passed the exam and remained on level 3.

Table 1. Cumulative results.

	math stud.		pre. teach.	
test	pre	post	pre	post
level 3	27	11	24	24
level 4	0	0	0	0
level 5	38	17	22	22
3 → 5	0		0	
5 → 3	0	0	0	0

This result strengthens the theory of Vigotsky and its van Hiele version for higher level mathematics. Accordingly, the teacher has to be aware of the student's understanding and has to correctly determine their ZPD. As it is known, the ZPD is "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance, or in collaboration with more capable peers" [21]. The theory suggests that every act of teaching should start from the actual level of the student, taking into consideration that there is a maximum which they can achieve in one step. Moreover, if the teacher speaks in a too sophisticated language, then they permanently remain beyond students' ZPD and this way they do not provide scaffolding for proper development. In the geometry courses none of the two initial conditions were fulfilled. Both professors assumed level 4 form the students and presented the course in that manner.

Parallelly, by the van Hiele theory students cannot achieve one level of thinking successfully without having passed through the previous levels. The advancement of students from one van Hiele level to the next depends more on teaching than, for example, on the age of the student. To move from a level to the next one, the teaching process has to start at the proper Van Hiele level. The model also states that people reasoning at different levels may not understand each other. It means that a student on level n will not understand the thinking of level $n + 1$ or higher. It follows that a student at level 3 cannot understand the reasoning of a teacher who speaks in a way that is adequate for students at level 4 or level 5. The teacher should evaluate how the student is interpreting a topic to communicate effectively. Probably, in both cases the course was adjusted to a standard level 4 ZPD.

It sounds surprising that a big proportion of students are on level 3. The van Hiele level of Hungarian high school students is fairly well investigated. It is shown that the average van Hiele level in Hungary is between 2 and 3, independently of age [15]. Talented students and special math classes are exceptions. They reach level 4 as early as grade 10, as expected by the NCC [7]. Pre-service math teacher and math major students are supposed to be over the average in mathematics. In Hungary high school studies are finished with a final exam, and the score of this exam counts to the tertiary entrance points. A thorough analysis of geometry problems on the final exams show that the level and topic of geometry problems are predictable and require level 3 [13]. It is an easy conclusion that math teachers prepare their students to the final exam, and do not teach the full curriculum. Thus, students on level 3 enter universities on exactly that level that they were taught to.

It seems that students enter the university with a geometry knowledge that does not meet the expectations of the university curriculum. One cannot learn a subject without being ready for it, without having the prequisites. And if the teacher or professor explains on a higher level where the student is, the student cannot learn the material. Still, the result contradicts the fact that these students passed the exam. This suggests that the so called "exam memory" exists in case of higher mathematics, where not only lexical knowledge is needed. Unfortunately this knowledge is just a short term knowledge and it is not accompanied by a higher level of understanding geometry.

References

[1] A. H. Abdullah, E. Zakaria: *Enhancing Students' Level of Geometric Thinking through Van Hiele's Phase-based Learning*, Indian Journal of Science and Technology 6.5 (2013), pp. 4432–4446.

[2] R. Astuti, D. Suryadi, T. Turmudi: *Analysis on geometry skills of junior high school students on the concept congruence based on Van Hiele's geometric thinking level*, Journal of Physics Conference Series (2018), pp. 1–5, doi: https://doi.org/10.1088/1742-6596/1132/1/012036.

[3] J. Brendefur, J. Frykhlom: *Promoting Mathematical Communication in the Classroom: Two Preservice Teachers' Conceptions and Practices.* Journal of Mathematics Teacher Education 3 (2000), pp. 125–153, doi: https://doi.org/10.1023/A:1009947032694.

[4] W. F. Burger, J. M. Shaughnessy: *Characterizing the van Hiele Levels of Development in Geometry*, Journal for Research in Mathematics Education 17.1 (1986), pp. 31–48, doi: https://doi.org/10.2307/749317.

[5] É. Erdélyi, A. Dukán, C. Szabó: *The transition problem in Hungary: curricular approach*, Teaching Mathematics and Computer Science 17.1 (2019), pp. 1–16, doi: https://doi.org/10.5485/TMCS.2019.0454.

[6] T. Erdogan, S. Durmus: *The effect of the instruction based on Van Hiele model on the geometrical thinking levels of preservice elementary school teachers*, Procedia - Social and Behavioral Sciences 1.1 (2009), pp. 154–159, doi: https://doi.org/10.1016/j.sbspro.2009.01.029.

[7] Á. Győry, E. Kónya: *Development of high school students' geometric thinking with particular emphasis on mathematically talented students*, Teaching Mathematics and Computer Science (2018), pp. 93–110, doi: https://doi.org/10.5485/TMCS.2018.0448.

[8] K. Jones: *Issues in the teaching and learning of geometry*, in: Aspects of Teaching Secondary Mathematics: perspectives on practice. London, GB: Routledge, 2002, pp. 121–139, doi: https://doi.org/10.4324/9780203165874.

[9] G. Kospentaris, P. Spyrou: *Assessing the development of geometrical thinking from the visual towards the analytic-descriptive level*, Annales de didactique et de sciences cognitives 13.5 (2008), pp. 133–157.

[10] J. H. McMillan: *Educational Research. Fundamentals for the consumers (3rd ed.)* New York: Addison Wesley, 2000.

[11] T. O. Moyer: *Raising the Van Hiele Level of College Students,* PRIMUS (2021), pp. 1–15, doi: https://doi.org/10.1080/10511970.2021.1978603.

[12] D. Patkin: *Global van Hiele (GVH) Questionnaire as a Tool for Mapping Knowledge and Understanding of Plane and Solid Geometry,* Research in Mathematical Education 18(2) (2014), pp. 103–128, doi: https://doi.org/10.7468/jksmed.2014.18.2.103.

[13] A. Rékasi, C. Szabó: *Modification of the geometry curriculum in relation to the curriculum reform in the light of the Van Hiele levels*, in: Münster: WTM Verlag für wissenschaftliche Texte und Medien, 2021, pp. 93–110, doi: https://doi.org/10.37626/GA9783959872003.0.15.

[14] S. Senk: *Van Hiele levels and achievement in writing geometry proofs*, Journal for Research in Mathematics Education 20.3 (1989), pp. 309–321, doi: https://doi.org/10.2307/749519.

[15] C. Szabó, C. Zámbó, A. Muzsnay, J. Szeibert: *Students' non-development in high school geometry*, Annales Mathematicae et Informaticae (2020), pp. 309–319, doi: https://doi.org/10.33039/ami.2020.12.004.

[16] A. Ural: *Investigating 11th Grade Students' Van-Hiele Level 2 Geometrical Thinking*, Journal Of Humanities And Social Science 21.12 (2016), pp. 13–19, doi: https://doi.org/10.9790/0837-2112061319.

[17] Z. Usiskin: *Van Hiele Levels and Achievement in Secondary School Geometry. CDASSG Project.* Chicago: Chicago Univ, IL., 1982.

[18] M. D. de Villers: *Some reflections on the Van Hiele theory*, in: June 2010.

[19] M. D. de Villers: *The role and function of a hierarchical classification of the quadrilaterals*, For the Learning of Mathematics 14.1 (1994), pp. 11–18.

[20] I. Vojkuvkova, J. Haviger: *The van Hiele Levels at Czech Secondary Schools*, Procedia - Social and Behavioral Sciences 171 (2015), pp. 912–918, doi: https://doi.org/10.1016/j.sbspro.2015.01.209.

[21] L. Vygotsky: *Mind in Society: Development of Higher Psychological Processes*, Cambridge: Harvard University Press, Cambridge, 1978, doi: https://doi.org/10.2307/2955559.

[22] M. Wilson: *Measuring a van Hiele Geometry Sequence: A Reanalysis,* Research in Mathematical Education 21(3) (1990), pp. 230–237, doi: https://doi.org/10.2307/749376.

[23] www.ofi.hu: *The Hungarian National Core Curriculum*, Teaching Mathematics and Computer Science (2012).

[24] I. Zachos: *Register of Educational Research in the United Kingdom – Problem Solving in Euclidean Geometry in Greek Schools*, in: vol. 10, 0615, London and New York: Routledge, 1995.