PERFECT PACKING OF SQUARES

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Abstract

It is known that $\sum_{i=1}^{\infty} 1/i^2 = \pi^2/6$. Meir and Moser asked what is the smallest ϵ such that all the squares of sides of length 1, 1/2, 1/3, ... can be packed into a rectangle of area $\pi^2/6 + \epsilon$. A packing into a rectangle of the right area is called perfect packing. Chalcraft packed the squares of sides of length 1, 2^{-t} , 3^{-t} , ... and he found perfect packing for $1/2 < t \leq 3/5$. We will show based on an algorithm by Chalcraft that there are perfect packings if $1/2 < t \leq 2/3$. Moreover we show that there is a perfect packing for all t in the range $\log_3 2 \leq t \leq 2/3$.

Key words: packing, square, rectangle AMS 2010 Subject Classification: 52C15, 52C20

1. INTRODUCTION

Meir and Moser [10] originally noted that since $\sum_{i=2}^{\infty} 1/i^2 = \pi^2/6 - 1$, it is reasonable to ask whether the set of squares with sides of length 1/2, 1/3, 1/4, ... can be packed into a rectangle of area $\pi^2/6 - 1$. Failing that, find the smallest ϵ such that the squares can be packed in a rectangle of area $\pi^2/6 - 1 + \epsilon$. The problem also appears in [6], [4], [3].

A packing into a rectangle of the right (resp. not the right) area is called *perfect* (resp. *imperfect*) packing. In [10], [7], [2], [11] can be found better and better imperfect packing.

Chalcraft [5] generalized this question. He packed the squares of side n^{-t} for n = 1, 2, ... into a square of the right area. He proved that for all t in the range [0.5964, 0.6] there is a perfect packing of the squares. In [5] can be read that "Other packings will work for other ranges of t. We can probably make the t_0 in Theorem 8 as close to 1/2 as desired in this way. The more interesting challenge, however, seems to be to increase the bound $t \leq 3/5$." Our aim is to increase this bound.

Wästlund [12] proved if 1/2 < t < 2/3, then the squares of side n^{-t} for n = 1, 2, ... can be packed into some finite collection of square boxes of the same area $\zeta(2t)$ as the total area of the tiles. This is an increase of the bound $t \leq 3/5$, but we have many enclosing rectangles.

We can find several papers in this topic e.g. [9], [1], [8].

2. PERFECT PACKING

THEOREM 1 For t = 2/3, the squares S_n^t $(n \ge 1)$ can be packed perfectly into the rectangle of dimensions $\zeta(2t) \times 1$. THEOREM 2 For all t in the range $\log_3 2 \le t \le 2/3$, the squares S_n^t $(n \ge 1)$ can be packed perfectly into the rectangle of dimensions $\zeta(2t) \times 1$.

3. NOTATION

We use the Chalcraft's algorithm in [5] and we modify the proof of Chalcraft. For the sake of simplicity we use the Chalcraft's notation. For the completeness, we recall these.

Throughout the paper the width of a rectangle will always refer to the shorter side and the height will always refer to the longer side. We use the constant $1/2 < t \le 2/3$. As usual, $\zeta(t) = \sum_{i=1}^{\infty} i^{-t}$. Let S_n^t denote the square of side length n^{-t} . A box is a rectangle of sides x, y > 0. Let $x \times y$ denote the box B of sides x and y. We define its area a(B) = xy, its semi-perimeter p(B) = x + y, its width $w(B) = \min(x, y)$ and its height $h(B) = \max(x, y)$. Given a set of boxes $\mathscr{B} = \{B_1, \ldots, B_n\}$, we define $a(\mathscr{B}) = \sum_{i=1}^n a(B_i), h(\mathscr{B}) = \sum_{i=1}^n h(B_i)$ and $w(\mathscr{B}) = \max_{i=1,\ldots,n} w(B_i)$. Let $a(\emptyset) = h(\emptyset) = w(\emptyset) = 0$.

4. CHALCRAFT'S ALGORITHM

For the completeness, we repeat the description of the Chalcraft's algorithm.

First we recall the subroutine of Chalcraft, which we call Algorithm \mathbf{b} as in [5].

Algorithm **b**

Input: An integer $n \ge 1$ and a box B, where $w(B) = n^{-t}$.

Output: If the algorithm terminates, then it defines an integer $m_{\mathbf{b}} = m_{\mathbf{b}}(n, B) > n$ and a set of boxes $\mathscr{B}_{\mathbf{b}} = \mathscr{B}_{\mathbf{b}}(n, B)$.

Action: If the algorithm terminates, then it packs the squares $S_n^t, \ldots, S_{m_b-1}^t$ into B, and \mathscr{B}_b is the set of boxes containing the remaining area. If it does not terminate, then it packs the squares S_n^t, S_{n+1}^t, \ldots into B.

- (b1) Let $n_1 = n + 1$, $x_1 = h(B) n^{-t}$ and $\mathscr{B}_1 = \emptyset$.
- (b2) Put the square S_n^t snugly at one end of B.
- (b3) If $x_1 > 0$, then let B_1 be the remainder of B, so that B_1 has dimensions $x_1 \times n^{-t}$.
- (b4) For i = 1, 2, ...
- (b5) (Note: At stage *i*, we have packed $S_n^t, \ldots, S_{n_i-1}^t$ into *B*. The remaining boxes are \mathscr{B}_i , which we never use again in this algorithm, and B_i (as long as $x_i > 0$), which has dimensions $x_i \times n^{-t}$.)
- (b6) If $x_i = 0$, then terminate with $m_{\mathbf{b}} = n_i$ and $\mathscr{B}_{\mathbf{b}} = \mathscr{B}_i$.
- (b7) If $x_i < n_i^{-t}$, then terminate with $m_{\mathbf{b}} = n_i$ and $\mathscr{B}_{\mathbf{b}} = \mathscr{B}_i \cup \{B_i\}$.

- (b8) Let $x_{i+1} = x_i n_i^{-t}$.
- (b9) If $x_{i+1} = 0$, then let $C_i = B_i$.
- (b10) If $x_{i+1} > 0$, then split B_i into two boxes: one called C_i with dimensions $n_i^{-t} \times n^{-t}$, and the other called B_{i+1} with dimensions $x_{i+1} \times n^{-t}$.
- (b11) Apply Algorithm **b** recursively with inputs n_i and C_i . If this terminates, let $n_{i+1} = m_{\mathbf{b}}(n_i, C_i)$ and $\mathscr{C}_i = \mathscr{B}_{\mathbf{b}}(n_i, C_i)$.
- (b12) Let $\mathscr{B}_{i+1} = \mathscr{B}_i \cup \mathscr{C}_i$.
- (**b**13) End For.

The subroutine \mathbf{b} is used in the Chalcraft's algorithm \mathbf{c} .

Algorithm \mathbf{c}

Input: An integer $n \ge 1$ and a set of boxes \mathscr{B} .

Action: If the algorithm does not fail, then it packs the squares S_n^t, S_{n+1}^t, \dots into \mathscr{B} .

- (c1) Let $n_1 = n + 1$ and $\mathscr{B}_1 = \mathscr{B}$.
- (c2) For i = 1, 2, ...
- (c3) (Note: At stage *i*, we have packed $S_n^t, \ldots, S_{n_i-1}^t$ into *B*. The remaining boxes are \mathscr{B}_i .)
- (c4) If $w(\mathscr{B}_i) < n_i^{-t}$, then fail.
- (c5) Let $w_i = \min\{w(C) | C \in \mathscr{B}_i, w(C) \ge n_i^{-t}\}.$
- (c6) Let $h_i = \min\{h(C) | C \in \mathscr{B}_i, w(C) = w_i\}.$
- (c7) Choose any $B_i \in \mathscr{B}_i$ which satisfies $w(B_i) = w_i$ and $h(B_i) = h_i$.
- (c8) If $w_i = h_i = n_i^{-t}$, then
- (c9) Put $S_{n_i}^t$ snugly into B_i .
- (c10) Let $\mathscr{B}_{i+1} = \mathscr{B}_i \setminus \{B_i\}.$
- (c11) Let $n_{i+1} = n_i + 1$.
- (c12) Else
- (c13) Cut B_i into two boxes: one called C_i of dimensions $w_i \times n_i^{-t}$ and the other called D_i of dimensions $w_i \times (h_i - n_i^{-t})$.
- (c14) Call Algorithm **b** with inputs n_i and C_i . If this terminates, then let $n_{i+1} = m_{\mathbf{b}}(n_i, C_i)$ and $\mathscr{C}_i = \mathscr{B}_{\mathbf{b}}(n_i, C_i)$.
- (c15) Let $\mathscr{B}_{i+1} = \mathscr{B}_i \setminus \{B_i\} \cup \mathscr{C}_i \cup \{D_i\}.$
- $(\mathbf{c}16)$ End If.
- (c17) End For.

5. THE PROOF

The key lemma of Chalcraft is Lemma 1 in [5]. We modify that in the following way.

LEMMA 1 If
$$\mathscr{B} = \{B_1, \ldots, B_n\}$$
 $(n \ge 1)$, then $a(\mathscr{B}) \le w(\mathscr{B})h(\mathscr{B})$.

Proof. We have

$$a(\mathscr{B}) = \sum_{i=1}^{n} a(B_i) = \sum_{i=1}^{n} w(B_i)h(B_i) \le \sum_{i=1}^{n} w(\mathscr{B})h(B_i)$$
$$= w(\mathscr{B})\sum_{i=1}^{n} h(B_i) = w(\mathscr{B})h(\mathscr{B}),$$

which completes the proof.

We prove the modified Chalcraft's lemmas in which we use the height instead of the semi-perimeter.

LEMMA 2 Suppose $w(B) = n^{-t}$ and Algorithm **b** with inputs n and B terminates with $m_{\mathbf{b}} = m_{\mathbf{b}}(n, B)$ and $\mathscr{B}_{\mathbf{b}} = B_{\mathbf{b}}(n, B)$. Therefore

$$h(\mathscr{B}_{\mathbf{b}}) \le \sum_{j=n}^{m_{\mathbf{b}}-1} j^{-t}$$

Proof. The proof is similar to the proof of Lemma 2 in [5]. For completeness, we write it again.

The proof is by induction on the number of squares packed. Of course, if **b** terminates with $m_{\mathbf{b}} = n + 1$, then $h(\mathscr{B}_{\mathbf{b}}) \leq n^{-t} = \sum_{j=n}^{m_{\mathbf{b}}-1} j^{-t}$.

We can assume that the lemma is true of all the recursive calls to Algorithm **b**. We can also assume that **b** and all the recursive calls to **b** terminated. Suppose Algorithm **b** terminates when i = k, so $m_{\mathbf{b}} = n_k$. Since Algorithm **b** terminated without placing the next square, $x_k < n_k^{-t} < n^{-t}$, so $h(B_k) = n^{-t}$. Now by induction,

$$h(\mathscr{C}_{i}) \leq \sum_{j=n_{i}}^{n_{i+1}-1} j^{-t} \quad \text{for } i < k,$$
$$\sum_{i=1}^{k-1} h(\mathscr{C}_{i}) \leq \sum_{j=n_{1}}^{n_{k}-1} j^{-t} = \sum_{j=n+1}^{m_{b}-1} j^{-t}.$$

If the condition in (b6) was true, then

$$h(\mathscr{B}_{\mathbf{b}}) = \sum_{i=1}^{k-1} h(\mathscr{C}_i) \le \sum_{j=n+1}^{m_{\mathbf{b}}-1} j^{-t} < \sum_{j=n}^{m_{\mathbf{b}}-1} j^{-t}.$$

If the condition in (b7) was true, then

$$h(\mathscr{B}_{\mathbf{b}}) = \sum_{i=1}^{k-1} h(\mathscr{C}_i) + h(B_k) \le \sum_{j=n+1}^{m_{\mathbf{b}}-1} j^{-t} + n^{-t} = \sum_{j=n}^{m_{\mathbf{b}}-1} j^{-t},$$

which completes the proof.

LEMMA 3 We have

(1)
$$(b+1)^{1-t} - a^{1-t} < (1-t) \sum_{j=a}^{b} j^{-t} < b^{1-t} - (a-1)^{1-t},$$

(2)
$$a^{1-2t} - (b+1)^{1-2t} < (2t-1) \sum_{j=a}^{b} j^{-2t} < (a-1)^{1-2t} - b^{1-2t}.$$

Proof. We omit the proof.

LEMMA 4 Consider step (c4) for some value of i. Suppose the following conditions hold.

(3)
$$a(\mathscr{B}_i) \ge \sum_{j=n_i}^{\infty} j^{-2t},$$

(4)
$$h(\mathscr{B}_i) \le \frac{n_i^{1-t}}{2t-1}.$$

Therefore step (c4) will not fail for this value of *i*.

Proof. We assume, that the algorithm fail. Therefore we have $w(\mathscr{B}_i) < n_i^{-t}$. By Lemma 1, (4), (2),

$$a(\mathscr{B}_i) \le w(\mathscr{B}_i)h(\mathscr{B}_i) < \frac{n_i^{1-2t}}{2t-1} \le \sum_{j=n_i}^{\infty} j^{-2t} \le a(\mathscr{B}_i),$$

a contradiction, which completes the proof of the lemma.

LEMMA 5 Given an integer $n \ge 1$ and a non-empty set of boxes \mathscr{B} , suppose the following conditions hold

(5)
$$a(\mathscr{B}) \ge \sum_{j=n}^{\infty} j^{-2t},$$

(6)
$$h(\mathscr{B}) \leq \frac{1}{1-t}(n-1)^{1-t},$$
$$t \leq \frac{2}{3}.$$

If we run Algorithm \mathbf{c} with the inputs n and \mathscr{B} , then the conditions

(7)
$$a(\mathscr{B}_i) \ge \sum_{j=n_i}^{\infty} j^{-2t},$$

(8)
$$h(\mathscr{B}_i) \le h(\mathscr{B}) + \sum_{j=n}^{n_i-1} j^{-t}.$$

hold at step (c4) for all $i \ge 1$ for which step (c4) is executed. Moreover, the algorithm will never fail.

Proof. First, we will show that (7) and (8) ensure that the algorithm will not fail. By (8), (1), and (6),

$$h(\mathscr{B}_i) \le h(\mathscr{B}) + \sum_{j=n}^{n_i-1} j^{-t}$$

< $h(\mathscr{B}) + \frac{1}{1-t} ((n_i-1)^{1-t} - (n-1)^{1-t})$
 $\le \frac{1}{1-t} (n_i-1)^{1-t}.$

Since $t \leq 2/3$,

$$\frac{1}{1-t} \le \frac{1}{2t-1}.$$

Thus

$$h(\mathscr{B}_i) < \frac{1}{1-t}(n_i-1)^{1-t} \le \frac{1}{2t-1}(n_i-1)^{1-t} < \frac{n_i^{1-t}}{2t-1}$$

By Lemma 4, (c4) will not fail.

Now we prove (7) and (8) by induction on i. Of course they hold for i = 1 and (7) holds for all i. Let i > 1 be the smallest i for which (8) is not true.

If the condition in (c8) was true for i - 1, then $h(\mathscr{B}_i) = h(\mathscr{B}_{i-1}) - n_{i-1}^{-t}$ and $n_i = n_{i-1} + 1$. Thus by induction,

$$h(\mathscr{B}_{i}) = h(\mathscr{B}_{i-1}) - n_{i-1}^{-t} \le h(\mathscr{B}) + \sum_{j=n}^{n_{i-1}-1} j^{-t} - n_{i-1}^{-t}$$
$$< h(\mathscr{B}) + \sum_{j=n}^{n_{i-1}-1} j^{-t} = h(\mathscr{B}) + \sum_{j=n}^{n_{i}-2} j^{-t} < h(\mathscr{B}) + \sum_{j=n}^{n_{i}-1} j^{-t}.$$

If the condition in (c8) was not true for i - 1, then we distinguish two cases. If $w_{i-1} \ge h_{i-1} - n_{i-1}^{-t}$ (that is $h(D_{i-1}) = w_{i-1}$), then

$$h(\mathscr{B}_i) = h(\mathscr{B}_{i-1}) + h(\mathscr{C}_{i-1}) - h(B_{i-1}) + h(D_{i-1})$$

$$= h(\mathscr{B}_{i-1}) + h(\mathscr{C}_{i-1}) - h_{i-1} + w_{i-1} \le h(\mathscr{B}_{i-1}) + h(\mathscr{C}_{i-1}).$$

If $w_{i-1} < h_{i-1} - n_{i-1}^{-t}$ (that is $h(D_{i-1}) = h_{i-1} - n_{i-1}^{-t}$), then similarly

$$h(\mathscr{B}_i) \le h(\mathscr{B}_{i-1}) + h(\mathscr{C}_{i-1}).$$

Figure 1: The squares S_1^t, S_2^t, S_3^t and the set of boxes \mathscr{B} .

By induction and Lemma 2,

$$h(\mathscr{B}_{i}) \leq h(\mathscr{B}_{i-1}) + h(\mathscr{C}_{i-1})$$
$$\leq h(\mathscr{B}) + \sum_{j=n}^{n_{i-1}-1} j^{-t} + \sum_{j=n_{i-1}}^{n_{i}-1} j^{-t} = h(\mathscr{B}) + \sum_{j=n}^{n_{i}-1} j^{-t},$$

which completes the proof.

Proof of Theorem 1. If the first three squares are packed in the box $B = \zeta(2t) \times 1$ as in Fig. 1 (this is the Paulhus's algorithm [11]), then the remaining boxes are

$$\mathscr{B} = \left\{ \left(\zeta(2t) - 1 - 2^{-t} - 3^{-t} \right) \times 1, 2^{-t} \times \left(1 - 2^{-t} \right), 3^{-t} \times \left(1 - 3^{-t} \right) \right\}$$

and

$$h(\mathscr{B}) = \zeta(2t) - 2 \cdot 3^{-t} = 2.639$$

< 4.327 = $\frac{1}{1-t}(4-1)^{1-t}$.

By Lemma 5, the Algorithm **c** pack perfectly the squares S_n^t $(n \ge 4)$ into \mathscr{B} , which completes the proof.

REMARK 1 The squares S_n^t $(n \ge 1)$ in Theorem 1 can be packed similarly in a square of the right area.

Proof of Theorem 2. If the first three squares are packed in the box $B = \zeta(2t) \times 1$ as in Fig. 1, then the remaining boxes are

$$\mathscr{B} = \left\{ \left(\zeta(2t) - 1 - 2^{-t} - 3^{-t} \right) \times 1, 2^{-t} \times \left(1 - 2^{-t} \right), 3^{-t} \times \left(1 - 3^{-t} \right) \right\}.$$

Observe $\zeta(2t) - 1 - 2^{-t} - 3^{-t} > 1$, $2^{-t} > 1 - 2^{-t}$ and $1 - 3^{-t} \ge 3^{-t}$ if $t \in [\log_3 2, 2/3]$. Let $f(t) = h(\mathscr{B})$. Thus

$$h(\mathscr{B}) = f(t) = \zeta(2t) - 2 \cdot 3^{-t}.$$

Since

$$g(t) = \frac{1}{1-t}3^{1-t}$$

is an increasing, f(t) is a decreasing function on the interval $[\log_3 2, 2/3]$ and

$$f(\log_3 2) = 3.41 < 4.06 = g(\log_3 2),$$

the Algorithm **c** pack perfectly the squares S_n^t $(n \ge 4)$ into \mathscr{B} , which completes the proof.

5. DISCUSSION

If we increase the number of the packed squares before we start the Algorithm **c** and do detailed analysis of the height of the boxes, then we can decrease the constant $\log_3 2$. It remains an interesting question to increase the bound 2/3.

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