

Non-synthesizable varieties

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Abstract

In 2004 a counterexample was given for a 1965 result of R. J. Elliott claiming that discrete spectral synthesis holds on every Abelian group. Here we present a ring-theoretical approach to this problem, and show that some varieties fail to have spectral synthesis. In particular, we give a new proof for the result of the second author that spectral synthesis does not hold on Abelian groups with infinite torsion free rank.

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1. Introduction

Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. One considers the space $\mathcal{C}(G)$ of all complex valued continuous functions on a locally compact Abelian group G , which is a locally convex topological linear space with respect to point-wise linear operations (addition, multiplication with scalars) and

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to the topology of uniform convergence on compact sets. The *translate* by y in G of an element f in $\mathcal{C}(G)$ is defined by $\tau_y f(x) = f(x + y)$ for each x in G . A subset in $\mathcal{C}(G)$ is called *translation invariant* if it contains every translates of all of its elements. A closed translation invariant linear subspace of the space $\mathcal{C}(G)$ is called a *variety* on G . Continuous homomorphisms of G into the additive [multiplicative] topological group of [nonzero] complex numbers are called *additive* [*exponential*] *functions*. A function is a *polynomial* if it belongs to the algebra generated by the additive functions and constants. Usually, the product of a polynomial and an exponential is called an *exponential monomial*. This is equivalent to the property that the function generates a finite dimensional indecomposable variety (see e.g. [12]). We shall use this latter definition here.

It turns out that exponential functions, or more generally, exponential monomials can be considered as basic building blocks of varieties. A given variety may or may not contain any exponential function or exponential monomial. If each nonzero subvariety of a given variety contains an exponential function, then we say that *spectral analysis* holds for the variety. Another property is if the variety is *synthesizable*, which means that all exponential monomials in this variety span a dense subspace in the variety. If each subvariety of a given variety is synthesizable, then we say that *spectral synthesis* holds for the variety. It can be shown that spectral synthesis for a variety implies spectral analysis, too (see [12], Theorem 1). If spectral analysis, respectively, spectral synthesis holds for every nonzero variety on an Abelian group, then we say that *spectral analysis*, respectively, *spectral synthesis holds on the group*. A famous and pioneer result of L. Schwartz [8] exhibits the situation in a classical case by stating that if the underlying group is the reals with the Euclidean topology, then every nonzero variety contains an exponential function, that is, spectral analysis holds on the reals. Moreover, spectral synthesis also holds: there are sufficiently many exponential monomials in each variety in the sense that their linear hull is dense in the variety.

In his 1958 result [7] M. Lefranc proved that spectral synthesis holds on the group \mathbb{Z}^n for each positive integer n , \mathbb{Z} being the integers. In his 1965 paper [1] R. J. Elliot presented a theorem on spectral synthesis for arbitrary Abelian groups. However, in his 1987 private communication [2] Z. Gajda called the second author's attention to the fact that the proof of Elliot's theorem had several gaps. Later on several efforts have been made to solve the problem of discrete spectral analysis and spectral synthesis on arbitrary Abelian groups. Finally, a counterexample for Elliot's theorem was presented at the 41st International Symposium on Functional Equations, Noszvaj, Hungary, 2003 (see [9]). For basics, further developments and references on spectral analysis and spectral synthesis the reader should refer to [3, 11, 12].

In this paper we give a new proof for the failure of spectral synthesis on some types of discrete Abelian groups, which was shown by a counterexample

in [9]. Our method is based on ring-theoretical results and uses the annihilator technique. The basics of this method have been worked out in [13].

2. Basic concepts

In this paper we consider discrete commutative groups only, and $\mathcal{C}(G)$ is the set of all functions from G to \mathbb{C} . Let G be an Abelian group and let $\mathbb{C}G$ denote its *group algebra*, which is identified with the set of all finitely supported complex valued functions on G . Moreover, this set can be identified with the set of all finitely supported complex measures $\mathcal{M}_c(G)$ on G , using the definition

$$\int_G f d\mu = \sum_{x \in G} f(x) \mu(x),$$

whenever μ is in $\mathbb{C}G$ and f is in $\mathcal{C}(G)$. This formula expresses the well-known fact about the dual $\mathcal{C}(G)^*$ of the topological vector space $\mathcal{C}(G)$: it is identified with $\mathcal{M}_c(G)$, the pairing given by the previous formula.

Via these identifications the multiplication in the complex algebra $\mathbb{C}G$ is given by the *convolution* of measures as

$$\mu * \nu(f) = \sum_{x, y \in G} f(x + y) \mu(x) \nu(y)$$

for each μ, ν in $\mathbb{C}G$ and f in $\mathcal{C}(G)$. With this operation $\mathbb{C}G$ is a commutative unital complex algebra with identity δ_0 , which is the point mass concentrated at 0, the zero element of G . More generally, we denote by δ_x the characteristic function of the singleton $\{x\}$ for each x in G : it takes the value 1 at the element x and 0 otherwise.

Convolution is also defined between elements of $\mathbb{C}G$ and $\mathcal{C}(G)$ in the following manner:

$$\mu * f(x) = \sum_{y \in G} f(x - y) \mu(y),$$

whenever μ is in $\mathbb{C}G$, f is in $\mathcal{C}(G)$ and x is in G . With this operation $\mathcal{C}(G)$ turns into a $\mathbb{C}G$ -module, closed submodules being exactly the varieties. The intersection of all varieties including a particular f in $\mathcal{C}(G)$ is called the *variety of f* and is denoted by $\tau(f)$.

For each subset H in $\mathcal{C}(G)$ the *annihilator* H^\perp of H in $\mathbb{C}G$ is defined by

$$H^\perp = \{\mu : \mu * f = 0 \text{ for each } f \text{ in } H\}.$$

It is easy to see that H^\perp is an ideal in $\mathbb{C}G$. If $H = \{f\}$ is a singleton, then $H^\perp = \tau(f)^\perp$ and we call it the *annihilator of f* .

Similarly, the *annihilator* K^\perp in $\mathbb{C}G$ of a subset K in $\mathbb{C}G$ is defined by

$$K^\perp = \{f : \mu * f = 0 \text{ for each } \mu \text{ in } K\}.$$

It is also easy to check that K^\perp is a variety in $\mathcal{C}(G)$.

The following two theorems are important technical tools (see [6, 13]).

Theorem 2.1. *Let G be an Abelian group, V a variety on G and I an ideal in $\mathbb{C}G$. Then we have*

$$V^{\perp\perp} = V, \quad I^{\perp\perp} = I.$$

Theorem 2.2. *Let G be an Abelian group, $(V_\gamma)_{\gamma \in \Gamma}$ a family of varieties on G and $(I_\gamma)_{\gamma \in \Gamma}$ a family of ideals in $\mathbb{C}G$. Then we have*

$$\left(\sum_{\gamma \in \Gamma} V_\gamma\right)^\perp = \bigcap_{\gamma \in \Gamma} V_\gamma^\perp, \quad \left(\bigcap_{\gamma \in \Gamma} I_\gamma\right)^\perp = \sum_{\gamma \in \Gamma} I_\gamma^\perp.$$

3. Exponentials and maximal ideals

The basic building blocks of spectral analysis and spectral synthesis are exponential monomials. We call the reader's attention that we shall use the word "exponential" in several different meanings in the sequel. The generalized characters of G , that is, the homomorphisms of G into the multiplicative group of nonzero complex numbers will be called *exponential functions*, or simply *exponentials*. Later on we shall also use the terms "exponential maximal ideal", "exponential monomial", and "generalized exponential monomial", which refer to different, however, related concepts.

Exponentials can be characterized by a number of properties. We shall use the following result (see [13, Theorems 3 and 4], and [13, Corollaries 1 and 2]).

Theorem 3.1. *Let G be an Abelian group and $m: G \rightarrow \mathbb{C}$ be an arbitrary function. Then the following conditions are equivalent:*

1. *m is an exponential.*
2. *The variety of m is one dimensional and $m(0) = 1$.*
3. *The annihilator $\tau(m)^\perp$ is a maximal ideal in $\mathbb{C}G$, $\mathbb{C}G/\tau(m)^\perp$ is isomorphic to \mathbb{C} and $m(0) = 1$.*

Maximal ideals M in $\mathbb{C}G$ with the property that $\mathbb{C}G/M \cong \mathbb{C}$ play an important role, and they will be called *exponential maximal ideals*. They are closely related to modified differences defined as follows. For each function $f: G \rightarrow \mathbb{C}$ and y in G we define

$$\Delta_{f;y} = \delta_{-y} - f(y)\delta_0.$$

The measure $\Delta_{f;y}$ is called *modified difference*. For products of modified differences we shall use the notation

$$\Delta_{f;y_1, y_2, \dots, y_{n+1}} = \prod_{i=1}^{n+1} \Delta_{f;y_i},$$

whenever y_1, y_2, \dots, y_{n+1} are in G . The product on the right side is meant as a convolution.

Given f in $\mathcal{C}(G)$ the ideal in $\mathbb{C}G$ generated by all modified differences of the form $\Delta_{f;y}$ with y in G is denoted by M_f . It is reasonable to ask whether M_f is proper. We have the following result.

Theorem 3.2. *Let G be an Abelian group and $f: G \rightarrow \mathbb{C}$ be a function. The ideal M_f is proper if and only if f is an exponential. In this case $M_f = \tau(f)^\perp$, hence M_f is an exponential maximal ideal.*

Proof. Suppose first that M_f is proper. Then M_f^\perp is a nonzero variety, by Theorem 2.1, hence there is a nonzero g annihilated by all modified differences of the form $\Delta_{f;y}$. For x, y in G we have

$$0 = \Delta_{f;y} * g(x) = g(x+y) - f(y)g(x). \quad (3.1)$$

Putting $x = 0$ we have $g(y) = g(0) \cdot f(y)$. In particular, $g(0) \neq 0$, $f \neq 0$ and we obtain $f(x+y) = f(x)f(y)$. As f is nonzero, it follows $f(0) = 1$, hence f is an exponential.

Conversely, suppose that $f = m$ is an exponential. Then m is in M_m^\perp , as obviously $\Delta_{m;y} * m(x) = m(x+y) - m(y)m(x) = 0$ for each x, y in G . Hence M_m is proper. Moreover, $\tau(m)^\perp$ is an exponential maximal ideal in $\mathbb{C}G$, by Theorem 3.1. If g is in M_f^\perp , then, by (3.1), it is a constant multiple of m , hence it belongs to $\tau(m)$. It follows $M_m^\perp \subseteq \tau(m)$, thus $\tau(m)^\perp \subseteq M_m$. As $\tau(m)^\perp$ is maximal and M_m is proper, we have $\tau(m)^\perp = M_m$, and the theorem is proved. \square

These latter two results have been used in [13] to prove the following characterization results.

Theorem 3.3. *Let G be an Abelian group and V be a variety on G . Then spectral analysis holds for V if and only if each maximal ideal containing V^\perp is exponential.*

Corollary 3.4. *Let G be an Abelian group. Then spectral analysis holds on G if and only if each maximal ideal in $\mathbb{C}G$ is exponential.*

Corollary 3.5. *Let G be an Abelian group and V be a variety on G . Then spectral analysis holds for V if and only if each maximal ideal in $\mathbb{C}G/V^\perp$ is exponential.*

4. Exponential monomials

Let G be an Abelian group. The variety V on G is called *decomposable*, if it is the sum of two proper subvarieties, which means that the algebraic sum of two proper subvarieties is a dense submodule in it. Otherwise it is called *indecomposable*. The following theorem is obvious, by Theorem 2.2.

Theorem 4.1. *Let G be an Abelian group. A variety on G is decomposable if and only if its annihilator is the intersection of two ideals, which are different from it.*

Let G be an Abelian group. The function $f: G \rightarrow \mathbb{C}$ is called a *generalized exponential monomial*, if there exists an exponential m and a natural number n such that for each y_1, y_2, \dots, y_{n+1} we have

$$\Delta_{m; y_1, y_2, \dots, y_{n+1}} * f = 0 \quad (4.1)$$

holds. We reformulate this definition in terms of the annihilator of f .

Theorem 4.2. *Let G be an Abelian group. The function $f: G \rightarrow \mathbb{C}$ is a generalized exponential monomial if and only if its annihilator contains some positive power of an exponential maximal ideal.*

Proof. The condition of the theorem is equivalent to the following condition: there exists an exponential m and a natural number n such that

$$M_m^{n+1} \subseteq \tau(f)^\perp. \quad (4.2)$$

As the modified differences $\Delta_{m; y}$ with y in G generate M_m , hence the modified differences $\Delta_{m; y_1, y_2, \dots, y_{n+1}}$ generate M_m^{n+1} , that is, (4.1) and (4.2) are equivalent for f . \square

It is easy to check (see [13, Theorem 7]) that condition (4.2) can hold for at most one exponential m .

Theorem 4.3. *Let G be an Abelian group and $f: G \rightarrow \mathbb{C}$ be a nonzero generalized exponential monomial. Then there is a unique exponential m satisfying (4.1) for some natural number n . In other words, there is a unique exponential maximal ideal M satisfying $M^{n+1} \subseteq \tau(f)^\perp$ for some natural number n .*

Now we have the following characterization results (see [13, Theorem 8]).

Theorem 4.4. *Let G be an Abelian group. The function $f: G \rightarrow \mathbb{C}$ is a nonzero generalized exponential monomial if and only if $\mathbb{C}G/\tau(f)^\perp$ is a local ring with nilpotent exponential maximal ideal.*

Theorem 4.5. *Let G be an Abelian group. The function $f: G \rightarrow \mathbb{C}$ is an exponential monomial if and only if $\mathbb{C}G/\tau(f)^\perp$ is a local Artin ring with exponential maximal ideal.*

Proof. By the previous theorem $\mathbb{C}G/\tau(f)^\perp$ is a local ring with exponential maximal ideal. Any descending chain of ideals in $\mathbb{C}G/\tau(f)^\perp$ induces a descending chain of ideals containing $\tau(f)^\perp$ in $\mathbb{C}G$, which induces an ascending chain of subvarieties in $\tau(f)$, hence, by finite dimensionality, it terminates.

For the converse see [13, Theorem 8]. \square

5. The failure of spectral synthesis

Theorem 5.1. *Let G be an Abelian group and V be a variety on G . If V is indecomposable, and spectral synthesis holds for V , then $\mathbb{C}G/V^\perp$ is a local Artin ring.*

Proof. If V is synthesizable, then it is the topological sum of all subvarieties generated by exponential monomials belonging to V , by definition. This means that we have, by Theorem 2.2,

$$V^\perp = \bigcap_{\varphi \in V} \tau(\varphi)^\perp, \quad (5.1)$$

where the intersection is extended to all exponential monomials φ in V . As V is indecomposable, hence, by Theorem 4.1, V^\perp is equal to one of the factors of the intersection on the right side, that is $V^\perp = \tau(\varphi)^\perp$ for some exponential monomial φ in V . By Theorem 4.5, $\mathbb{C}G/V^\perp$ is a local Artin ring. \square

Theorem 5.2. *Let G be an Abelian group, and let $f : G \rightarrow \mathbb{C}$ be a generalized exponential monomial. Then $\tau(f)$ is synthesizable if and only if f is an exponential monomial.*

Proof. The statement is obvious by the definition of exponential monomials and by the previous theorem. \square

The following theorem follows immediately.

Theorem 5.3. *Let G be an Abelian group and V be a variety on G . If there is a generalized exponential monomial in V , which is not an exponential monomial, then spectral synthesis does not hold for V .*

As a consequence we obtain the following result (see [9]).

Theorem 5.4. *Let G be an Abelian group with infinite torsion free rank. Then spectral synthesis fails to hold on G .*

Proof. Indeed, by [10, Theorem 3], the torsion free rank of an Abelian group is infinite if and only if there is a generalized exponential monomial on the group, which is not an exponential monomial. \square

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