Half-graphs, other non-stable degree sequences, and the switch Markov chain

Péter L. Erdős[*,†] Ervin Győri[*] Tamás Róbert Mezei[*,†]
István Miklós[†,‡] Dániel Soltész[†,§]

Alfréd Rényi Institute of Mathematics
Reáltanoda street 13–15,
H-1053 Budapest, Hungary

<erdos.peter,gyori.ervin,mezei.tamas.robert,
miklos.istvan,soltesz.daniel>@renyi.hu

Alfréd Rényi Institute of Mathematics (a Hungarian Academy of Sciences Centre of Excellence), Reáltanoda u. 13–15, 1053 Budapest, Hungary

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Abstract

One of the simplest methods of generating a random graph with a given degree sequence is provided by the Monte Carlo Markov Chain method using **switches**. The switch Markov chain converges to the uniform distribution, but generally the rate of convergence is not known. After a number of results concerning various degree sequences, rapid mixing was established for so-called P-stable degree sequences (including that of directed graphs), which covers every previously known rapidly mixing region of degree sequences.

In this paper we give a non-trivial family of degree sequences that are not P-stable and the switch Markov chain is still rapidly mixing on them. This family has an intimate connection to Tyshkevich-decompositions and strong stability as well.

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1 Introduction

1.1 Previous results on the rapid mixing of the switch Markov chain

An important problem in network science is to sample simple graphs with a given degree sequence (almost) uniformly. In this paper we study a **Markov Chain Monte Carlo** (MCMC) approach to this problem. The MCMC method can be successfully applied in many special cases. A vague description of this approach is that we start from an arbitrary graph with a given degree sequence and sequentially apply small random modifications that preserve the degree sequence of the graph. This can be viewed as a random walk on the space of **realizations** (graphs) of the given degree sequence. It is well-known that after sufficiently many steps the distribution over the state space is close to the uniform distribution. The goal is to prove that the necessary number of steps to take (formally, the mixing time of the Markov chain) is at most a polynomial of the length of the degree sequence.

In this paper we study the so-called **switch Markov chain** (also known as the swap Markov chain). For clarity, we refer to the degree sequence of a simple graph as an **unconstrained** degree sequence.

Throughout the paper, we work with finite graphs and finite degree sequences. For two graphs G_1, G_2 on the same labelled vertex set, we define their *symmetric difference* $G_1 \triangle G_2$ with $V(G_1 \triangle G_2) = V(G_1) = V(G_2)$ and $E(G_1 \triangle G_2) = E(G_1) \triangle E(G_2)$.

Definition 1.1 (switch). *For a bipartite or an unconstrained degree sequence* $\boldsymbol{d}$*, we say that two realizations* $G_1, G_2 \in \mathcal{G}(\boldsymbol{d})$ *are connected by a switch, if*

$$|E(G_1 \triangle G_2)| = 4.$$

A widely used alternative name for switch is *swap*. A switch (swap) can be seen in Figure 1; for the precise definition of the switch Markov chain, see Definition 3.1. Clearly, if G_1 and G_2 are two simple graphs joined by a switch, then $F = E(G_1) \triangle E(G_2)$ is a cycle of length four (a C_4), and $E(G_2) = E(G_1) \Delta F$. Hence, the term switch is also used to refer to the operation of taking the symmetric difference with a given C_4. It should be noted, though, that only a minority of C_4's define a (valid) switch. The majority of C_4's do not preserve the degree sequence (if the C_4 does not alternate between edges of G_1 and G_2), or they introduce an edge which violates the constraints of the model (say, an edge inside one of the color classes in the bipartite case).

The question whether the mixing time of the switch Markov chain is short enough is interesting from both a practical and a theoretical point of view (although short enough depends greatly on the context). The switch Markov chain is already used in applications, hence rigorous upper bounds on its mixing time are much needed, even for special cases.

The switch Markov chain uses transitions which correspond to minimal perturbations. There are many other instances where the Markov chain of the smallest perturbations have

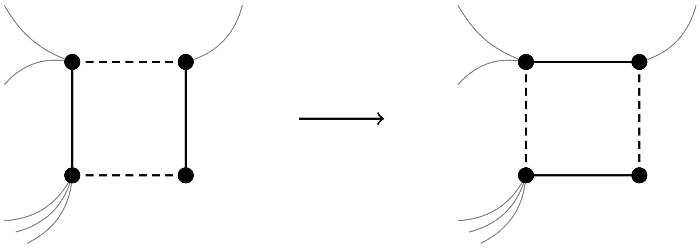

Figure 1: A switch (dashed lines emphasize missing edges)

polynomial mixing time, see [20]. However, it is unknown whether the mixing time of the switch Markov chain is uniformly bounded by a polynomial for every (unconstrained) degree sequence. Hence from a theoretical point of view, even an upper bound of $\mathcal{O}(n^{10})$ on the mixing time of the switch Markov chain would be considered a great success, even though in practice it is only slightly better than no upper bound at all.

The present paper is written from a theoretical point of view and should be considered as a step towards answering the following question.

Question 1.2 (Kannan, Tetali, and Vempala [17]). *Is the switch Markov chain rapidly mixing on the realizations of all graphic degree sequences?*

P-stability was introduced by Jerrum and Sinclair for the Jerrum-Sinclair chain, for whose rapid mixing the notion presents a natural boundary [23]. Jerrum, Sinclair, and McKay [15] already give an example for a non-P-stable degree sequences which has a unique realization (trivially rapidly mixing): take

$$(2n-1, 2n-2, \ldots, n+1, n, n, n-1, \ldots, 2, 1) \in \mathbb{N}^{2n}. \tag{1}$$

In its unique realization, the first n vertices form a clique, while the remaining vertices form an independent set.

We will denote graphs with upper case letters (e.g. G), degree sequences (which are non-negative integer vectors) with bold lower case letters (e.g. $\boldsymbol{d}$). Classes of graphs and classes of degree sequences are both denoted by upper case calligraphic letters (e.g. $\mathcal{H}$). We say that a graph G is a realization of a degree sequence $\boldsymbol{d}$, if the degree sequence of G is $\boldsymbol{d}$. For a degree sequence $\boldsymbol{d}$, we denote the set of all realizations of $\boldsymbol{d}$ by $\mathcal{G}(\boldsymbol{d})$. The ℓ^1-norm of a vector x is denoted by $\|x\|_1$.

Definition 1.3 (Greenhill and Gao [11]). *Let $\mathcal{D}$ be a set of graphic degree sequences and $k \in 2\mathbb{N}$. We say that $\mathcal{D}$ is **k-stable**, if there exists a polynomial $p \in \mathbb{R}[x]$ such that for any $n \in \mathbb{N}$ and any degree sequence $\boldsymbol{d} \in \mathcal{D}$ on n vertices, any degree sequence d' with $\|d' - d\|_1 \le k$ satisfies $|\mathcal{G}(\boldsymbol{d}')| \le p(n) \cdot |\mathcal{G}(\boldsymbol{d})|$. The term **P-stable** is an alias for 2-stable, which is the least restrictive non-trivial class defined here.*

There is a long line of results where the rapid mixing of the switch Markov chain is proven for certain degree sequences, see [2, 18, 13, 6, 7, 12]. Some of these results were unified, first by Amanatidis and Kleer [1], who established rapid mixing for so-called *strongly stable* classes of degree sequences of simple and bipartite graphs (definition given in Section 2.3).

The most general result at the time of writing is proved by Erdős, Greenhill, Mezei, Miklós, Soltész, and Soukup:

Theorem 1.4 ([4]). *The switch Markov chain is rapidly mixing on sets of unconstrained, bipartite, and directed degree sequences that are* ***P-stable*** *(see Definition 1.3).*

For the sake of being less redundant, the phrase "$\mathcal{D}$ is rapidly mixing" shall carry the same meaning as "switch Markov chain is rapidly mixing on $\mathcal{D}$".

Recently, Greenhill and Gao [11] presented elegant conditions which when satisfied ensure 8-stability of a class of degree sequences (8-stable degree sequence are by definition P-stable). In particular, they show that for $\gamma > 2$, power-law distributed degree sequences are 8-stable, hence rapidly mixing. They also give a proof that 8-stable sets of degree sequence are rapidly mixing.

In this paper we try to extend the set of rapidly mixing bipartite degree sequences beyond P-stability. The degree sequence (1) can naturally be turned into a bipartite one by assigning the role of the two color classes to the clique and the independent set, and then removing the edges of the clique.

Definition 1.5. *Let us define a bipartite degree sequence:*

$$\boldsymbol{h}_0(n) := \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ n & n-1 & n-2 & \cdots & 3 & 2 & 1 \end{pmatrix}$$
$$\mathcal{H}_0 := \left\{ \boldsymbol{h}_0(n) \;\middle|\; n \in \mathbb{N} \right\}$$

Let $A_n = \{a_1, \ldots, a_n\}$ and $B_n = \{b_1, \ldots, b_n\}$, often denoted simply A and B. We label the vertices of $\boldsymbol{h}_0(n)$ such that A is the first and B is the second color class, with $\deg_{\boldsymbol{h}_0(n)}(a_i) = n + 1 - i$ and $\deg_{\boldsymbol{h}_0(n)}(b_i) = i$ for $i \in [1, n]$. The unique realization $H_0(n)$, also known as the half-graph, is displayed on Figure 2.

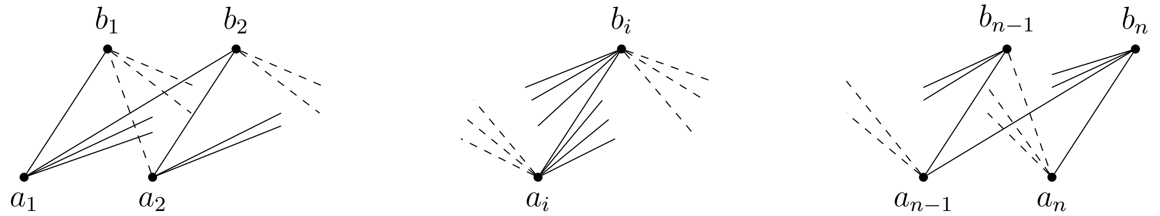

Figure 2: The unique realization $H_0(n)$ of $\boldsymbol{h}_0(n)$ is isomorphic to the half-graph.

In this paper, we conduct a detailed study of $\boldsymbol{h}_0(n)$ and its neighborhood. Before presenting our main results, let us get familiar with two interesting properties of $\boldsymbol{h}_0(n)$.

1.2 Simple examples for rapidly mixing non-stable bipartite classes

Let $\mathbb{1}_x$ be the vector which takes 1 on x and zero everywhere else. As solving an easy linear recursion in Corollary 6.2 shows, $\boldsymbol{h}_0(n) - \mathbb{1}_{a_1} - \mathbb{1}_{b_n}$ has

$$\Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$$

realizations, therefore $\mathcal{H}_0$ is not P-stable.

Although $\boldsymbol{h}_0(n)$ seems very pathological as an example for a non-stable degree sequence, it is a source of more interesting examples. As pointed out to us by an anonymous reviewer, one may replace a_ib_i by a pair of independent edges simultaneously for all i: let us define

$$\boldsymbol{g}(n) := \begin{pmatrix} 1 & 1 & 3 & 3 & \cdots & 2n-1 & 2n-1 \\ 2n-1 & 2n-1 & 2n-3 & 2n-3 & \cdots & 1 & 1 \end{pmatrix}.$$

The number of realizations of $\boldsymbol{g}(n)$ is 2^n, because the two independent edges replacing an edge a_ib_i can be switched with the two induced non-edges. In addition, every realization of $\boldsymbol{g}(n)$ can be obtained this way, so for any realization of $\boldsymbol{g}(n)$ the previously mentioned n switches are the complete set of switches. Because the random-walk on a hypercube is rapidly mixing, the switch Markov chain is rapidly mixing on $\{\boldsymbol{g}(n) \mid n \in \mathbb{N}\}$. Moreover, by solving yet another a linear recursion, one can verify that $\{\boldsymbol{g}(n) \mid n \in \mathbb{N}\}$ is not P-stable.

In Section 2.2, we will draw the curtain on the explanation behind the behavior of $\boldsymbol{h}_0(n)$ and $\boldsymbol{g}(n)$. In the meantime, we present the main results of the paper.

1.3 Results

If $\boldsymbol{d}$ is the degree sequence of the bipartite graph $G[A,B]$, then $\boldsymbol{d} = (\boldsymbol{d}^A; \boldsymbol{d}^B)$ is split across the bipartition as well, and it is called a splitted bipartite degree sequence. We say that $G[A,B]$ is the empty bipartite graph if both $A = B = \emptyset$.

Definition 1.6. *For a set $\mathcal{D}$ of bipartite degree sequences, let*

$$B_{2k}(\mathcal{D}) = \bigcup_{\boldsymbol{d}\in\mathcal{D}} \left\{ \boldsymbol{e} : \mathrm{Dom}(\boldsymbol{d}) \to \mathbb{N} \;\middle|\; \|d-e\|_1 \le 2k,\ \|e^A\|_1 = \|e^B\|_1 \right\}$$

$$\mathbb{S}_{2k}(\mathcal{D}) = \bigcup_{\boldsymbol{d}\in\mathcal{D}} \left\{ \boldsymbol{e} : \mathrm{Dom}(\boldsymbol{d}) \to \mathbb{N} \;\middle|\; \|d-e\|_1 = 2k,\ \|e^A\|_1 = \|e^B\|_1 \right\}$$

be the (closed) ball and sphere of radius $2k$ around $\mathcal{D}$ (w.l.o.g. $k \in \mathbb{N}$). The requirement that $\|e^A\|_1 = \|e^B\|_1$, i.e., that the sum of the degrees on the two sides be equal is necessary for graphicality.

We will show in Section 5 that neighborhoods of $\mathcal{H}_0 = \{\boldsymbol{h}_0(n) \mid n \in \mathbb{N}\}$ are rapidly mixing.

Theorem 1.7. *For any fixed k, the switch Markov chain is rapidly mixing on the bipartite degree sequences in $B_{2k}(\mathcal{H}_0)$.*

Next, we show that even though balls of constant size around $\mathcal{H}_0$ are rapidly mixing, $\mathbb{S}_{2k}(\mathcal{H}_0)$ contains a degree sequence which is not P-stable.

Definition 1.8. *For all $k, n \in \mathbb{N}$ where $k < n$ let*

$$\boldsymbol{h}_k(n) := \boldsymbol{h}_0(n) - k \cdot \mathbb{1}_{a_1} - k \cdot \mathbb{1}_{b_n}$$
$$\mathcal{H}_k := \left\{\boldsymbol{h}_k(n) \mid k \leq n \in \mathbb{N}^+\right\}$$

be a bipartite degree sequence and a class of bipartite degree sequences, respectively.

Theorem 1.9. *The class of degree sequences $\mathcal{H}_k$ is not P-stable for any $k \in \mathbb{N}$.*

1.4 Outline

The rest of the paper is organized as follows.

- As promised at the end of Section 1.2, we introduce the Tyshkevich-decomposition of bipartite graphs in Section 2. We also expose a connection to strong stability which provides further motivation to studying $\boldsymbol{h}_0(n)$.
- In Section 3 we introduce the switch Markov chains, some related definitions, and Sinclair's result on mixing time.
- Section 4 describes the structure of realizations of degree sequences from $B_{2k}(\boldsymbol{h}_0)$, which is then used by Sections 5 and 6 to prove Theorems 1.7 and 1.9, respectively.
- Section 7 describes how $\boldsymbol{h}_0(n)$ relates to previous research. Possible generalization of Theorem 1.7 are conjectured.

2 Properties of Tyshkevich-decompositions

2.1 Tyshkevich-decomposition of bipartite graphs

Let G be a simple graph. It is a split graph if there is a partition $V(G) = A \uplus B$ ($A \neq \emptyset$ or $B \neq \emptyset$) such that A is a clique and B is an independent set in G. Split graph were first studied by Földes and Hammer [10], who determined that being split is a property of the degree sequence $\boldsymbol{d}$ of G. Note, that the partition is not necessarily unique, but the size of A is determined up to a $+1$ additive constant, see [14]. A split graph endowed with the partition is called a splitted graph, denoted by (G, A, B). In addition to [10], Tyshkevich and Chernyak [22] also determined that being split is a property of the degree sequence, thus every realization of a split degree sequence is a split graph.

Tyshkevich and co-authors have extensively studied a composition operator $\circ$ on (split) graphs; these results are nicely collected in [21]. The composition $(G, A, B) \circ H$ takes the disjoint union of split graph and a simple graph, and joins every vertex in A to every vertex of H. A fundamental result on this operator is that any simple graph can be uniquely decomposed into the composition of split graphs and possibly an indecomposable simple graph as the last factor. During the writing of this paper, we were greatly saddened to learn that Professor Tyshkevich passed away November, 2019

Let us slightly change the conventional notation $G[A, B]$ to also signal that the color classes A and B are ordered (2-colored); to emphasize this, we may refer to such graphs as *splitted* bipartite graphs. Observe, that a function Ψ removing the edges of the clique on A from (G, A, B) produces a splitted bipartite graph $G[A, B]$. Erdős, Miklós, and Toroczkai [9] adapted the results about split graphs and the composition operator $\circ$ to splitted bipartite graphs via the bijection given by Ψ.

Definition 2.1. *Given two splitted bipartite graphs $G[A, B]$ and $H[C, D]$ with disjoint vertex sets, we define their (Tyshkevich-) composition $G[A, B] \circ H[C, D]$ as the bipartite graph*

$$G[A, B] \circ H[C, D] := G[A, B] \cup H[C, D] + \{ad \mid a \in A,\ d \in D\}.$$

The $\circ$ operator is clearly associative, but not commutative. We say that a bipartite graph is *indecomposable* if it cannot be written as a composition of two *non-empty* bipartite graphs.

Lemma 2.2 ([9], adapted from Theorem 2(i) in [21])**.** *Let $G[A, B]$ be a bipartite graph with degree sequence $d = (d^A, d^B)$, where both d^A and d^B are in non-increasing order. Then $G[A, B]$ is decomposable if and only if there exists $p, q \in \mathbb{N}$ such that $0 < p+q < |A|+|B|$, $0 \le p \le |A|$, $0 \le q \le |B|$, and*

$$\sum_{i=1}^{p} d_i^A = p(|B| - q) + \sum_{|B|-q+1}^{|B|} d_i^B. \tag{2}$$

Theorem 2.3 ([9], adapted from Corollaries 6 and 9 in [21])**.**

(i) *Any splitted bipartite degree sequence d can be uniquely decomposed in the form*

$$d = \alpha_1 \circ \cdots \circ \alpha_k,$$

where α_i is an indecomposable splitted bipartite degree sequence for $i = 1, \ldots, k$.

(ii) *Any realization G of d can be represented in the form*

$$G = G[A_1, B_1] \circ \cdots \circ G[A_k, B_k],$$

where $G[A_i, B_i]$ is a realization of α_i.

(iii) *Any valid bipartite switch of G is a valid bipartite switch of $G[A_i, B_i]$ for some i.*

It follows from the previous theorem that indecomposability is determined by the degree sequence. Lemma 2.2 gives an explicit characterization of such splitted bipartite degree sequences.

Definition 2.4. *Let $\overline{\mathcal{D}^\circ}$ be the closure of $\mathcal{D}$ under the composition operator $\circ$.*

The following theorem is a due to Erdős, Miklós, and Toroczkai.

Theorem 2.5 (Theorem 3.6 in [9])**.** *If $\mathcal{D}$ is rapidly mixing, then so is $\overline{\mathcal{D}^\circ}$.*

Theorem 2.5 is a simple consequence of [8, Theorem 5.1]. By Theorem 2.5, for a class of degree sequences $\mathcal{D}$ to be rapidly mixing it is sufficient that indecomp($\mathcal{D}$) is rapidly mixing, where

$$\text{indecomp}(\mathcal{D}) := \{\alpha \mid \alpha \text{ is an indecomposable component of some } d \in \mathcal{D}\}.$$

Because the number of realizations is independent of the internal order of the bipartition, we revert to using "bipartite degree sequence" instead of the cumbersome "splitted bipartite degree sequence". From now on, bipartite graphs and their degree sequences are assumed to be splitted.

2.2 Non-stability of Tyshkevich-compositions

As promised, we now revisit the two examples in Section 1.2. Observe, that

$$\boldsymbol{h}_0(n) = \overbrace{(1;1) \circ \ldots \circ (1;1)}^{n}$$
$$H_0(n) = \overbrace{K_2 \circ \ldots \circ K_2}^{n}$$

Note, that $(1;1) = (0;\emptyset) \circ (\emptyset;0)$, so the indecomposable decomposition of $\boldsymbol{h}_0(n)$ has $2n$ components. Theorem 2.3 implies that $H_0(n)$ is the only realization of $\boldsymbol{h}_0(n)$. This innocent looking example leads to the following result:

Lemma 2.6. *For any class $\mathcal{D}$ of bipartite degree sequences, $\overline{\mathcal{D}^\circ}$ is not P-stable (except if $\alpha^A = \emptyset$ for all $\alpha \in \mathcal{D}$ or $\beta^B = \emptyset$ for all $\beta \in \mathcal{D}$).*

Proof. Take $\alpha, \beta \in \mathcal{D}$ such that $\alpha^A \neq \emptyset$ and $\beta^B \neq \emptyset$. Let

$$\boldsymbol{d}(r) = \overbrace{(\alpha \circ \beta) \circ \ldots \circ (\alpha \circ \beta)}^{r}.$$

From Theorem 2.3 it follows that

$$|\mathcal{G}(\boldsymbol{d}(r))| = |\mathcal{G}(\boldsymbol{\alpha})|^r \cdot |\mathcal{G}(\boldsymbol{\beta})|^r.$$

Let $G = (G_1 \circ G_2) \circ \ldots \circ (G_{2r-1} \circ G_{2r})$ be an arbitrary realization of $\boldsymbol{d}(r)$ where $G_{2i-1} \in \mathcal{G}(\alpha)$ and $G_{2i} \in \mathcal{G}(\beta)$. Recall that $\boldsymbol{h}_0(r) - \mathbb{1}_{a_1} - \mathbb{1}_{b_r}$ has exponentially many realizations (Corollary 6.2).

Choose a vertex a_i from the first class of G_{2i-1} and b_i from the second class of G_{2i} (for $i \in [1, r]$). Observe, that $G[\{a_1, \ldots, a_r\}, \{b_1, \ldots, b_r\}]$ is an induced copy $H_0(r)$. By replacing this subgraph with a realization of $\boldsymbol{h}_0(r) - \mathbb{1}_{a_1} - \mathbb{1}_{b_r}$, an exponential number of realizations of $\boldsymbol{d}(r) - \mathbb{1}_{a_1} - \mathbb{1}_{b_r}$ are obtained; however, because the substitution does not change the components G_{2i-1} and G_{2i} for any i, G is recoverable from such realizations. In other words, every realization of some $\boldsymbol{d}' \in \mathbb{S}_2(\boldsymbol{d}(r))$ is obtained from at most one realization of $\boldsymbol{d}(r)$, so $\mathcal{D}$ cannot be P-stable. □

The degree sequence $\boldsymbol{g}(n)$ was obtained by replacing $a_i b_i$ with two independent edges. Therefore Lemma 2.6 applies to $\{\boldsymbol{g}(n) \mid n \in \mathbb{N}\}$:

$$\boldsymbol{g}(n) = \overbrace{(1,1;1,1) \circ \ldots \circ (1,1;1,1)}^{n}$$

Naturally, $\overbrace{2K_2 \circ \ldots \circ 2K_2}^{n}$ is a realization of $\boldsymbol{g}(n)$ and all 2^n realizations of $\boldsymbol{g}(n)$ are isomorphic to it (Theorem 2.3).

Theorem 1.9 is not, however, a simple consequence of Lemma 2.6:

Lemma 2.7. *The bipartite degree sequence $\boldsymbol{h}_k(n)$ is indecomposable for $0 < k < n$.*

Proof. Via Lemma 2.2. Suppose $\boldsymbol{h}_k(n)$ is decomposable. Substituting into (2), we get

$$\binom{n+1}{2} - k - \binom{n-p+1}{2} + \max\{k-p, 0\} =$$
$$= p(n-q) + \binom{q+1}{2} - \max\{k-n+q, 0\}$$
$$\max\{k-p, 0\} + \max\{k-n+q, 0\} - k = \binom{q-p+1}{2}$$

A short case analysis shows that the right hand side is larger than the left hand side. □

2.3 Strong stability and $\boldsymbol{H_0(\ell)}$

Strong stability is defined by Amanatidis and Kleer [1]. In their definition, they measure how stable a degree sequence is by measuring the maximum distance of a perturbed realization from the closest realization.

Definition 2.8 (adapted from [1])**.** *A degree sequence $\boldsymbol{d}$ is* distance-ℓ strongly stable *if for any realization G' of a degree sequence d' for which $\|d'-d\|_1 \le 2$ there exists a realization G such that $|E(G\triangle G')| \le \ell$. A set of degree sequences is called* strongly stable *if there exists an ℓ such that every degree sequence in the set is distance-ℓ strongly stable.*

The distance function $|E(G\triangle G')|$ used in Definition 2.8 differs from the function used in [1] up-to a factor of 2. Indeed, in one step, the Jerrum-Sinclair chain changes the size of the symmetric difference by at most 2. In the other direction, suppose G minimizes $|E(G\triangle G')|$. Take a vertex v where $\boldsymbol{d}(v) = \boldsymbol{d}'(v)$: $E(G)$ and $E(G')$ evenly contribute to the edges incident to v in $G\triangle G'$. For the two vertices where $\boldsymbol{d}$ and $\boldsymbol{d'}$ differ, there is an extra edge from G or G'. For this reason, if $G\triangle G'$ is not a path, then it contains a cycle C whose edges alternate between G and G', hence C is alternating (between edges and non-edges) in G as well. However,

$$|E((G\triangle C)\triangle G')| = |E(G\triangle G')| - |E(C)|,$$

which contradicts the minimality of G. If $G\triangle G'$ is path, the Jerrum-Sinclair chain needs at most $\lceil \frac{1}{2}|E(G\triangle G')|\rceil$ steps to transform G' into G.

The way we define strong stability immediately shows that strongly stable sets of degree sequences are also P-stable with $p(n) = n^{\ell+1}$.

Definition 2.9. *We say that a bipartite graph $G[A, B]$ is* covered by alternating cycles *if for any $x \in A$ and $y \in B$ there exists a cycle C which traverses (covers) xy and alternates between the vertex sets A and B, and also alternates between edges and non-edges of $G[A, B]$.*

Lemma 2.10. *The following statements are equivalent for a bipartite degree sequence $\boldsymbol{d}$.*

1. *$\boldsymbol{d}$ is indecomposable;*
2. *every $G \in \mathcal{G}(\boldsymbol{d})$ is covered by alternating cycles;*
3. *every $\boldsymbol{d}' \in \mathbb{S}_2(\boldsymbol{d})$ is graphic.*

Proof. Suppose $\boldsymbol{d}$ is decomposable; let $G \in \mathcal{G}(\boldsymbol{d})$ and say $G = G_1 \circ G_2$. Take $x \in V(G_1) \cap A$ and $y \in V(G_2) \cap B$ from distinct color classes, thus $xy \in E(G)$. If $\exists G' \in \mathcal{G}(\boldsymbol{d} + \mathbb{1}_x + \mathbb{1}_y)$, then take $G\triangle G'$: there x and y have one extra edge in G' compared to G, therefore there is an alternating path joining x to y in G starting on an non-edge, i.e., there is an alternating cycle on xy in G. This means that there is a realization of $\boldsymbol{d}$ in which xy is not an edge. Therefore $\boldsymbol{d} + \mathbb{1}_x + \mathbb{1}_y$ is not graphic. The proof is similar if $x \in V(G_1) \cap B$ and $y \in V(G_2) \cap A$ (take the complement).

In the other direction, suppose $\boldsymbol{d}$ is indecomposable. Let $G \in \mathcal{G}(\boldsymbol{d})$ and $\boldsymbol{d}' \in \mathbb{S}_2(\boldsymbol{d})$ be arbitrary. Suppose first, that $\boldsymbol{d}' = \boldsymbol{d} + \mathbb{1}_x + \mathbb{1}_y$ where $x \in A$ and $y \in B$. If $xy \notin E(G)$, then $G + xy$ is a realization of $\boldsymbol{d}'$. If $xy \in E(G)$ and there is an alternating cycle C on xy in G, then $G \triangle C + xy \in \mathcal{G}(\boldsymbol{d}')$.

If $xy \in E(G)$ is not contained in an alternating cycle in G, then let $A_1 \subset A$ and $B_1 \subset B$ be the set of vertices that are reachable from x on an alternating path starting on a non-edge. Define $A_2 = A \setminus A_1$ and $B_2 = B \setminus B_1$. We must have $y \in B_2$, otherwise there is an alternating cycle on xy. Observe, that $G = G[A_1, B_1] \circ G[A_2, B_2]$, a contradiction.

If $\boldsymbol{d}' = \boldsymbol{d} - \mathbb{1}_x - \mathbb{1}_y$ where $x \in A$ and $y \in B$, take the complement to arrive in the previous case.

Finally, we have $\boldsymbol{d}' = \boldsymbol{d} - \mathbb{1}_x + \mathbb{1}_y$ where $x, y \in A$ or $x, y \in B$. Without loss of generality, suppose that $x, y \in A$. Let $G \in \mathcal{G}(\boldsymbol{d})$ be arbitrary. If there is an alternating path P starting on a edge from x to y in G, then $G \triangle P \in \mathcal{G}(\boldsymbol{d}')$. If there is no such alternating path, take a z in B such that $xz \in E(G)$. Then $yz \in E(G)$, too. As before, there exists an alternating cycle C on yz in G, because $\boldsymbol{d}$ is indecomposable. Since C is an alternating cycle, $xz \notin E(G)$, thus $G \triangle C - xz + yz \in \mathcal{G}(\boldsymbol{d}')$. □

Lemma 2.11. *Suppose that the minimum length of an alternating cycle covering xy in G is $2\ell + 2$ and $G \in \mathcal{G}(\boldsymbol{d})$. Then a graphic element of $\mathbb{S}_2(\boldsymbol{d})$ is not distance-(2ℓ) strongly stable. Moreover, there is an induced copy of $H_0(\lceil \ell/3 \rceil)$ in G.*

Proof. Notice that all of the conclusions are invariant on complementing G. By taking the complement of G, we may suppose that $xy \notin E(G)$.

Take $\boldsymbol{d}' := \boldsymbol{d} + \mathbb{1}_x + \mathbb{1}_y$. For any realization $G' \in \mathcal{G}(\boldsymbol{d}')$ we have $|E(G \triangle G')| > 2\ell$, otherwise there is an alternating path of length at most $2\ell - 1$ in G which forms an alternating cycle of length 2ℓ with xy.

Let C be an alternating cycle of length $2\ell + 2$ on xy. Let $a_1 := x$ and $b_{\ell+1} := y$. Let a_i and b_i be the vertices at distance $2i - 2$ and $2i - 1$ from x on $C - xy$, respectively.

Notice, that $a_i b_j \in E(G)$ if $i + 1 \geq j$, and $a_i b_j \notin E(G)$ if $j \leq i - 2$, otherwise C is not the shortest alternating cycle on xy. Let

$$A' := \{a_{3i-2} \ : \ i = 1, \dots, \lceil \ell/3 \rceil\},$$
$$B' := \{a_{3i-1} \ : \ i = 1, \dots, \lceil \ell/3 \rceil\}.$$

We have

$$G[A', B'] = (a_1, \emptyset) \circ (\emptyset, b_2) \circ (a_4, \emptyset) \circ \cdots \circ (\emptyset, b_k) \simeq H_0(\lceil \ell/3 \rceil).$$

□

3 The switch Markov chain

For the precise definition of Markov chains and an introduction to their theory, the reader is referred to Durrett [3]. To define the unconstrained and bipartite switch Markov chains, it is sufficient to define their transition matrices.

Definition 3.1 (unconstrained/bipartite switch Markov chain)**.** *Let $\boldsymbol{d}$ be an unconstrained or bipartite degree sequence on n vertices. The state space of the switch Markov chain $\mathcal{M}(\boldsymbol{d})$ is $\mathcal{G}(\boldsymbol{d})$. The transition probability between two different states of the chain is nonzero if and only if the corresponding realizations are connected by a switch, and in this case this probability is $\frac{1}{6}\binom{n}{4}^{-1}$. The probability that the chain stays at a given state is one minus the probability of leaving the given state.*

It is well-known that any two realizations of an unconstrained or bipartite degree sequence can be transformed into one-another through a series of switches.

The switch Markov chains defined are irreducible (connected), symmetric, reversible, and lazy. Their unique stationary distribution is the uniform distribution $\pi \equiv |\mathcal{G}(\boldsymbol{d})|^{-1}$.

Definition 3.2. *The mixing time of a Markov chain $\mathcal{M}$ is*

$$\tau_{\mathcal{M}}(\varepsilon) = \min\left\{t_0 \ : \ \forall x \ \forall t \geq t_0 \ \|P^t(x,\cdot) - \pi\|_1 \leq 2\varepsilon\right\},$$

where $P^t(x,y)$ is the probability that when $\mathcal{M}$ is started from x, then the chain is in y after t steps.

Definition 3.3. *The switch Markov chain is said to be rapidly mixing on an infinite set of degree sequences $\mathcal{D}$ if there exists a fixed polynomial* $\mathrm{poly}(n, \log\varepsilon^{-1})$ *which bounds the mixing time of the switch Markov chain on $\mathcal{G}(\boldsymbol{d})$ for any $\boldsymbol{d} \in \mathcal{D}$ (where n is the length of $\boldsymbol{d}$).*

Sinclair's seminal paper describes a combinatorial method to bound the mixing time.

Definition 3.4 (Markov graph)**.** *Let $G(\mathcal{M}(\boldsymbol{d}))$ be the graph whose vertices are realizations of $\boldsymbol{d}$ and two vertices are connected by an edge if the switch Markov chain on $\mathcal{G}(\boldsymbol{d})$ has a positive transition probability between the two realizations.*

Let Γ be a set of paths in $\mathcal{M}(\boldsymbol{d})$. We say that Γ is a *canonical path system* if for any two realizations $G, H \in \mathcal{G}(\boldsymbol{d})$ there is a unique $\gamma_{G,H} \in \Gamma$ which joins G to H in the Markov graph. The load of Γ is defined as

$$\rho(\Gamma) = \max_{P(e)\neq 0} \frac{|\{\gamma \in \Gamma \ : \ e \in E(\gamma)\}|}{|\mathcal{G}(\boldsymbol{d})| \cdot P(e)}, \tag{3}$$

where $P(e)$ is the transition probability assigned to the edge e of the Markov graph (this is well-defined because the studied Markov chains are symmetric). The next lemma follows from Proposition 1 and Corollary 4 of Sinclair [19].

Lemma 3.5. *If* Γ *is a canonical path system for* $\mathcal{M}(\boldsymbol{d})$ *then*

$$\tau_{\mathcal{M}(\boldsymbol{d})}(\varepsilon) \le \rho(\Gamma) \cdot \ell(\Gamma) \cdot \left(\log(|\mathcal{G}(\boldsymbol{d})|) + \log(\varepsilon^{-1})\right),$$

where $\ell(\Gamma)$ *is the length of the longest path in* Γ*.*

Obviously, $\log(|\mathcal{G}(\boldsymbol{d})|) \le n^2$, henceforth we focus on bounding ρ by a polynomial of n.

4 Flow representation

In this section we introduce a flow representation of realizations of bipartite degree sequences defined on A_n and B_n as their first and second color classes, respectively.

Let $F = F_n = (A_n, B_n, \vec{E})$ be a directed bipartite graph such that

- $a_i b_j \in \vec{E}(F)$ if and only if $i \le j$,
- $b_j a_i \in \vec{E}(F)$ if and only if $j < i$.

Every edge in $uv \in \vec{E}(F)$ has capacity 1 in the direction from u to v. We will only consider integer flows, so any admissible flow in F is a subgraph of F. If the sum of the flow injected at the sources is $r \in \mathbb{N}$, then the flow is called an r-flow.

Definition 4.1. *The flow representation* $\vec{\nabla}(G)$ *of a bipartite graph* $G[A_n, B_n]$ *is the subgraph of* F_n *obtained as follows: take the symmetric difference* $\nabla(G) = G[A_n, B_n] \triangle H_0(n)$*, then make* $\nabla(G)$ *directed such that each edge in* $\nabla(G)$ *matches its orientation in* F_n*.*

Lemma 4.2. *The flow representation* $\vec{\nabla}(G)$ *is an admissible flow in* F*. Moreover,*

- *every* $a_i \in A$ *is a* ***source*** *of* $(\deg_G(a_i) - \deg_{H_0(n)}(a_i))^-$ *commodity,*
  every $b_i \in B$ *is a* ***source*** *of* $(\deg_G(b_i) - \deg_{H_0(n)}(b_i))^+$ *commodity,*
- *every* $a_i \in A$ *is a* ***sink*** *of* $(\deg_G(a_i) - \deg_{H_0(n)}(a_i))^+$ *commodity,*
  every $b_i \in B$ *is a* ***sink*** *of* $(\deg_G(b_i) - \deg_{H_0(n)}(b_i))^-$ *commodity.*

Conversely, such a flow is the flow representation of some $G[A_n, B_n]$*.*

Proof. Observe the structure of $H_0(n)$ on Figure 2. We have

$$\begin{aligned}
\deg_G(a_i) - d_{H_0(n)}(a_i) &= \deg_{\nabla(G)}(a_i, \{b_1, \ldots, b_{i-1}\}) - \deg_{\nabla(G)}(a_i, \{b_i, \ldots, b_n\}) \\
&= \varrho_{\vec{\nabla}(G)}(a_i) - \delta_{\vec{\nabla}(G)}(a_i), \\
\deg_G(b_i) - d_{H_0(n)}(b_i) &= \deg_{\nabla(G)}(b_i, \{a_{i+1}, \ldots, a_n\}) - \deg_{\nabla(G)}(b_i, \{a_1, \ldots, a_i\}) \\
&= \delta_{\vec{\nabla}(G)}(a_i) - \varrho_{\vec{\nabla}(G)}(a_i).
\end{aligned}$$

In the other direction, remove the orientation from the flow and take its symmetric difference with $H_0(n)$ to obtain the appropriate $G[A_n, B_n]$. ◻

Corollary 4.3. *For any $\boldsymbol{d} \in \mathbb{S}_{2k}(\mathcal{H}_0)$, the function $G \mapsto \vec{\nabla}(G)$ is a bijection between $\mathcal{G}(\boldsymbol{d})$ and such k-flows on F where the sources and sinks prescribed according to Lemma 4.2.*

For example: every flow representation of a realization of

$$\boldsymbol{h}_0(n) - \mathbb{1}_{a_1} + 2 \cdot \mathbb{1}_{b_2} + \mathbb{1}_{a_7} - 2 \cdot \mathbb{1}_{b_8}$$

is a 3-flow with sources at a_1 and b_2, and sinks as at a_7 and b_8; see Figure 3.

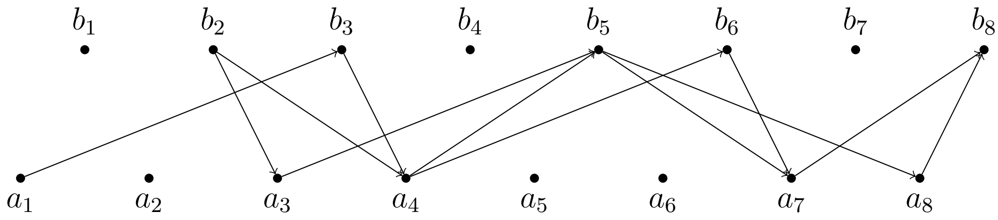

Figure 3: The flow representation of a realization of a degree sequence from $B_6(h_0(8))$.

5 Proof of Theorem 1.7: rapid mixing on $\boldsymbol{B_{2k}(\mathcal{H}_0)}$

5.1 Overview of the proof

Without loss of generality $\boldsymbol{d} \in \mathbb{S}_{2k}(\mathcal{H}_0)$. Let $X, Y \in \mathcal{G}(\boldsymbol{d})$ be two distinct realizations. We will define a switch sequence

$$\gamma_{X,Y} : X = Z_0, Z_1, \dots, Z_t = Y,$$

We will also define a set of corresponding encodings

$$L_0(X,Y), L_1(X,Y), \dots, L_t(X,Y).$$

The canonical path system $\Gamma := \{\gamma_{X,Y} \mid X, Y \in \mathcal{G}(\boldsymbol{d})\}$ on $G(\mathcal{M}(\boldsymbol{d}))$ will satisfy the following two properties:

- **Reconstructible:** there is an algorithm that for each i, takes Z_i and $L_i(X,Y)$ as an input and outputs the realizations X and Y.
- **Encodable in $\mathcal{G}(\boldsymbol{d})$:** the total number of encodings on each vertex of $G(\mathcal{M}(\boldsymbol{d}))$ is at most a $\text{poly}_k(n)$ factor larger than $|\mathcal{G}(\boldsymbol{d})|$.

The "Reconstructible" property ensures that the number of canonical paths traversing a vertex (and thus an edge) of the Markov graph $\mathcal{M}(\boldsymbol{d})$ is at most the size of the set of all possible encodings. Subsequently, by substituting into Equation (3), the "Encodable in $\mathcal{G}(\boldsymbol{d})$" property implies that $\rho(\Gamma) = \mathcal{O}(\text{poly}_k(n))$. According to Lemma 3.5, the last bound means that the bipartite switch Markov chain is rapidly mixing.

Now we give a description of how the $X = Z_0, Z_1, \dots, Z_{t+1} = Y$ canonical path is constructed. The main idea is to morph X into Y "from left to right": a region of constant width called the **buffer** is moved peristaltically through $A_n \cup B_n$, consuming X on its right and producing Y on its left; see Figure 4.

The encoding L_i will contain a realization whose structure is similar to Z_i, but the roles of X and Y are reversed. Furthermore, L_i will contain the position of the buffer and some additional information about the vertices in the buffer.

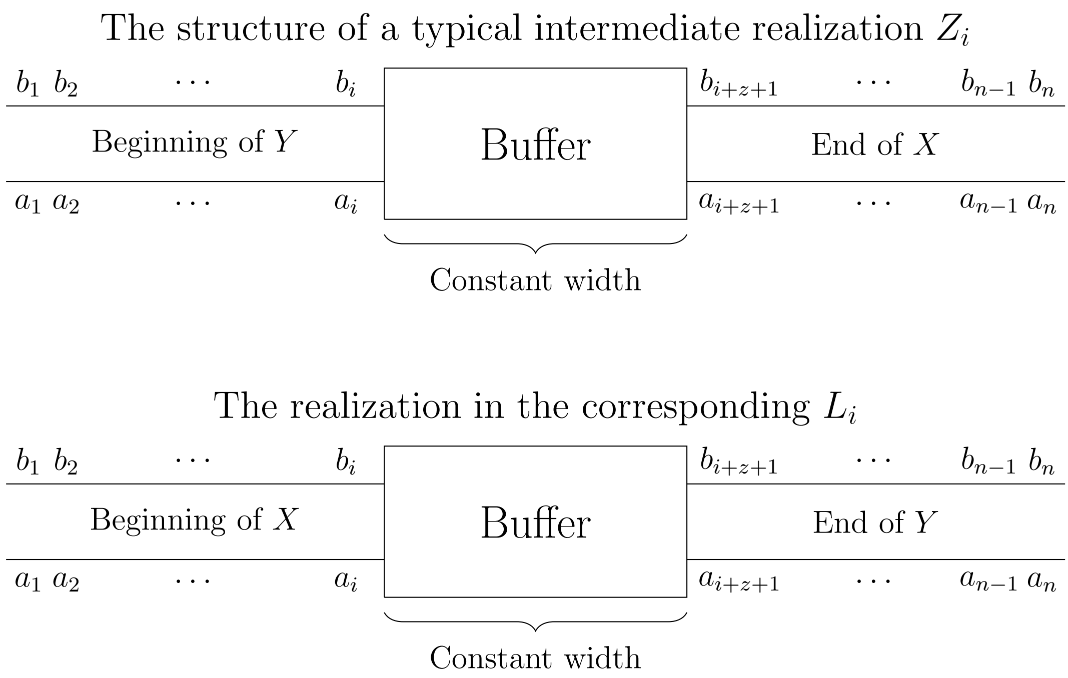

Figure 4: A realization along $\gamma_{X,Y}$ and the main part of the associated encoding.

Let $\overline{A}_i = A_n \setminus A_i$ and $\overline{B}_i = B_n \setminus B_i$. Also, let $U_i = A_i \cup B_i$ and $\overline{U}_i = \overline{A}_i \cup \overline{B}_i$. The following lemma shows the existence of a suitable buffer which can be used to interface two different realizations as displayed on Figure 4.

Lemma 5.1. *If $i, z \in \mathbb{N}$ satisfy $0 \le i \le n - z$ and $2k + \sqrt{2k} + 1 \le z$, then there is a realization $T_{X,Y}[i+1, i+z] \in \mathcal{G}(\boldsymbol{d})$ with the following properties:*

- *U_i induces identical subgraphs in $T_{X,Y}[i+1, i+z]$ and Y, and*
- *$\overline{U}_{i+z}$ induces identical subgraphs in $T_{X,Y}[i+1, i+z]$ and X.*

For $k = 1$, even $z = 1$ is sufficient.

Proof. We will work with the flow representation of X and Y. Since X and Y are the realizations of the same degree sequence, the source-sink distribution in their corresponding flow representation is identical. It is sufficient to design a flow which joins the flow $\vec{\nabla}(X)$ leaving U_i and redirects it to the vertices in $\overline{U}_{i+z}$ with the same distribution as $\vec{\nabla}(Y)$ flows into them from U_{i+z}.

The case $k = z = 1$ can be manually checked at this point.

To achieve the outlined goal for any k, we define an auxiliary network F' and prescribe the flow corresponding to the buffer on it. Let $e_D(W, Z)$ be the number of edges of D that are directed from W to Z.

$$\begin{aligned}
A_X &:= \{a_j \in A_i \mid e_{\vec{\nabla}(X)}(a_j, \overline{B}_i) > 0\} \\
B_X &:= \{b_j \in B_i \mid e_{\vec{\nabla}(X)}(b_j, \overline{A}_i) > 0\} \\
A_Y &:= \{a_j \in \overline{A}_{i+z} \mid e_{\vec{\nabla}(Y)}(B_{i+z}, a_j) > 0\} \\
B_Y &:= \{b_j \in \overline{B}_{i+z} \mid e_{\vec{\nabla}(Y)}(A_{i+z}, b_j) > 0\} \\
A' &:= A_X \cup (A_{i+z} \setminus A_i) \cup A_Y \\
B' &:= B_X \cup (B_{i+z} \setminus B_i) \cup B_Y
\end{aligned}$$

The underlying network F' is a subgraph of F:

$$F' := F[A', B'] - E(F[A_X \cup A_Y, B_X \cup B_Y]),$$

i.e., the flow cannot use edges between A_X, B_X, A_Y, B_Y. Note, that to prove the lemma for $k = z = 1$, one has to use edges of $F[A_X, B_Y]$ and $F[A_Y, B_X]$.

The flow in the buffer will be a subgraph $W \subset F'$. Let us define $f : A' \cup B' \to \mathbb{Z}$:

$$f(a_j) := \begin{cases} e_{\vec{\nabla}(X)}(a_j, \overline{B}_i), & \text{if } a_j \in A_X \\ -e_{\vec{\nabla}(Y)}(B_{i+z}, a_j), & \text{if } a_j \in A_Y \\ \deg_{H_0(n)}(a_j) - \deg_{\boldsymbol{d}}(a_j), & \text{if } a_j \in A_{i+z} \setminus A_i \end{cases}$$

$$f(b_j) := \begin{cases} e_{\vec{\nabla}(X)}(b_j, \overline{A}_i), & \text{if } b_j \in B_X \\ -e_{\vec{\nabla}(Y)}(A_{i+z}, b_j), & \text{if } b_j \in B_Y \\ \deg_{\boldsymbol{d}}(b_j) - \deg_{H_0(n)}(b_j), & \text{if } b_j \in B_{i+z} \setminus B_i \end{cases}$$

We prescribe sources and sinks in W as follows (recall Lemma 4.2):

$$\begin{aligned}
\delta_W(a_j) - \varrho_W(a_j) &= f(a_j) \quad \forall a_j \in A' \\
\delta_W(b_j) - \varrho_W(a_j) &= f(b_j) \quad \forall b_j \in B'
\end{aligned}$$

If such a W exists, then $\vec{\nabla}(X)[A_i, B_i] + W + \vec{\nabla}(Y)[\overline{A}_{i+z}, \overline{B}_{i+z}]$ is a k-flow which, according to Corollary 4.3, corresponds to a graph whose degree sequence is $\boldsymbol{d}$.

The existence of W is proved using Menger's theorem on the number of edge-disjoint directed st-paths. It is sufficient to show that any $S \subseteq A' \cup B'$ satisfies the cut-condition:

$$\delta_{F'}(S) \geq \sum_{s \in S} f(s) \tag{4}$$

Trivially, the right-hand side is at most k. Let us take an S for which $\delta_{F'}(S) - \sum_{s \in S} f(s)$ is minimal. We claim that the following four statements hold:

- If $|S \cap (A_{i+z} \setminus A_i)| > k$, then $B_Y \subset S$.
- If $|S \cap (B_{i+z} \setminus B_i)| > k$, then $A_Y \subset S$.
- If $|S \cap (A_{i+z} \setminus A_i)| < z - k$, then $B_X \cap S = \emptyset$.
- If $|S \cap (B_{i+z} \setminus B_i)| < z - k$, then $A_X \cap S = \emptyset$.

We only prove the first statement because the rest can be shown analogously. Suppose $|S \cap (A_{i+z} \setminus A_i)| > k$ and $b_j \in B_Y$, but $b_j \notin S$. Moving b_j into S changes the difference between the two sides of (4) by

$$-|S \cap (A_{i+z} \setminus A_i)| - f(b_j) < -k + e_{\vec{\nabla}(Y)}(A_{i+z}, b_j) \leq 0,$$

which contradicts the minimality of S.

Finally, we have four cases. In each case we show that (4) holds.

- **Case 1**: $|S \cap (A_{i+z} \setminus A_i)| \leq k$ and $|S \cap (B_{i+z} \setminus B_i)| \geq z - k$. We have

  $$\delta_{F'}(S) \geq e_{F'}\left(S \cap (B_{i+z} \setminus B_i), (A_{i+z} \setminus A_i) \setminus S\right) \geq \sum_{r=1}^{z-2k-1} r \geq \binom{z-2k}{2} \geq k,$$

  thus S satisfies (4).

- **Case 2**: $|S \cap (A_{i+z} \setminus A_i)| \leq k$ and $|S \cap (B_{i+z} \setminus B_i)| \geq z - k$: as in Case 1, we get that $\delta_{F'}(S) \geq k$.

- **Case 3**: $|S \cap (A_{i+z} \setminus A_i)| > k$ and $|S \cap (B_{i+z} \setminus B_i)| > k$. By our previous statements, we have $A_Y \cup B_Y \subseteq S$. Consequently,

  $$\begin{aligned}\delta_{F'}(S) &\geq \delta_{\vec{\nabla}(X) \cap F'}(S) = \delta_{\vec{\nabla}(X)}(S \cup \overline{U}_{i+z}) - \delta_{\vec{\nabla}(X) \cap F[A_i, B_i]}(S) = \\ &= \sum_{s \in S \cup \overline{U}_{i+z}} \left(\delta_{\vec{\nabla}(X)}(s) - \varrho_{\vec{\nabla}(X)}(s)\right) - \delta_{\vec{\nabla}(X) \cap F[A_i, B_i]}(S) = \\ &= -\sum_{s \in \overline{U}_{i+z}} e_{\vec{\nabla}(X)}(U_{i+z}, s) + \sum_{s \in S \cap (U_{i+z} \setminus U_i)} f(s) + \sum_{s \in U_i} e_{\vec{\nabla}(X)}(s, \overline{U}_i) = \\ &= -\sum_{s \in \overline{U}_{i+z}} e_{\vec{\nabla}(Y)}(U_{i+z}, s) + \sum_{s \in S \cap U_{i+z}} f(s) = \sum_{s \in S} f(s),\end{aligned}$$

  which is what we wanted to show.

- **Case 4**: $|S \cap (A_{i+z} \setminus A_i)| < z - k$ and $|S \cap (B_{i+z} \setminus B_i)| < z - k$: by our previous statements, we have $S \cap (A_X \cup B_X) = \emptyset$. Since $\delta_{F'}(S) = \varrho_{F'}(A' \cup B' \setminus S)$, the proof is practically the same as that of Case 3, we can use $\vec{\nabla}(Y)$ to demonstrate that (4) is satisfied by S.

□

5.2 Constructing the canonical path $\gamma_{X,Y}$.

We will explicitly construct $2(n-3k-3)+1$ intermediate realizations along the switch sequence $\gamma_{X,Y}$. Let X and Y be the two different realizations which we intend to connect. The switch sequence includes $T_{X,Y}[i+1, i+3k+1]$, $T_{X,Y}[i+1, i+3k+2]$, $T_{X,Y}[i+2, i+3k+2]$ for each $i = 1, \ldots, n-3k-3$ in increasing order. These realizations are called **milestones**. A roadmap is shown on Figure 5.

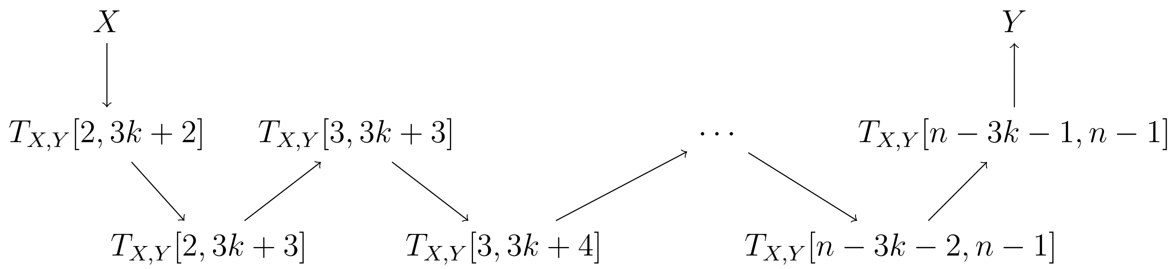

Figure 5: Roadmap of the switch sequence between X and Y. The existence of a short switch sequence between milestones of the sequence is guaranteed by Lemma 5.2.

Lemma 5.2. *There is a switch sequence of length $\mathcal{O}(k^2)$ that connects $T_{X,Y}[i+1, i+z]$ to $T_{X,Y}[i+1, i+z+1]$ and $T_{X,Y}[i+2, i+z]$ to $T_{X,Y}[i+1, i+z+1]$.*

Proof. According to [5], there is a switch sequence of length at most

$$\frac{|E(T_{X,Y}[i+1, i+z]) \triangle E(T_{X,Y}[i+1, i+z+1])|}{2} \le \frac{1}{2}(z+1+2k)^2 \le \frac{1}{2}(5k+2)^2,$$

between $T_{X,Y}[i+1, i+z]$ and $T_{X,Y}[i+1, i+z+1]$, since they induce identical graphs on U_i and $\overline{U}_{i+z+1}$, and the at most $k-k$ edges entering U_{i+1} and leaving U_{i+z+1} in the flow representations are incident on the same set of vertices in the two flows. □

Note that in Lemma 5.1, X satisfies the role of $T_{X,Y}[1, 3k+2]$ and Y satisfies the role of $T_{X,Y}[n-3k-1, n]$. By applying Lemma 5.2, the arrows in Figure 5 can be substituted with switch sequences of constant length. Concatenating these short switch sequences and pruning the circuits from the resulting trail (so that any realization is visited at most once by the canonical path) produces the switch sequence $\gamma_{X,Y}$ connecting X to Y in the Markov graph.

5.3 Assigning the encodings.

Each realization visited by $\gamma_{X,Y}$ receives an encoding that will be an ordered 4-tuple consisting of another realization, two graphs of constant size, and an integer in $\{1,\ldots,n\}$.

The closed neighborhood of a subset of vertices $U \subseteq V(G)$ in a graph G is denoted by $N_G[U] \supseteq U$. For the two graphs of constant size, we need the following definition.

Definition 5.3 (left-compressed induced subgraph)**.** *Let X be a realization and let $R \subset A \cup B$. Let us group the vertices $A \cup B$ into pairs: $\{(a_i,b_i)\}_{i=1}^n$. Remove the pairs that do not intersect R, and let the remaining pairs be $\{(a_{i_j},b_{i_j})\}_{j=1}^r$ for some $i_1 < \cdots < i_r$. For each edge of $E(X[R])$, map $a_{i_j} \mapsto a_j$ and $b_{i_j} \mapsto b_j$ for all j simultaneously. This changes the embedding of the vertices of $X[R]$, and we call this new graph the left-compressed copy of $X[R]$.*

To any realization on the canonical path $\gamma_{X,Y}$ we will assign an encoding

$$L_i(X,Y) := \Big(T_{Y,X}[i+1,i+3k+1], G_X[i], G_Y[i], i\Big)$$

for some $0 \le i \le n-3k-1$, where $G_X[i]$ is the left-compressed subgraph of X induced by $N_{\vec{\nabla}(X)}[U_{i+3k+1} \setminus U_i]$ and $G_Y[i]$ is the left-compressed copy of subgraph of Y induced $N_{\vec{\nabla}(Y)}[U_{i+3k+1} \setminus U_i]$. An encoding is assigned to each realization along the switch sequence $\gamma_{X,Y}$ as follows:

- The encoding $L_0(X,Y)$ (where $T_{Y,X}[1,3k+1] := Y$) is used from the beginning X of the switch sequence until it arrives at $T_{X,Y}[2,3k+3]$ (including this realization).
- For $1 \le i \le n-3k-1$, the encoding L_i is used between $T_{X,Y}[i+1,i+3k+2]$ (not included) and $T_{X,Y}[i+2,i+3k+2]$ (included).
- The encoding L_{n-3k-1} (where $T_{Y,X}[n-3k,n] := X$ is chosen) is used from $T_{X,Y}[n-3k-2,n-1]$ (not included) to Y.

5.4 Estimating the load $\boldsymbol{\rho}(\boldsymbol{\Gamma})$

The total number of possible encodings is at most

$$\mathcal{O}_k(|\mathcal{G}(\boldsymbol{d})| \cdot n)$$

(where the index k warns that this expression may depend on k), since the number of left-compressed graphs on at most $5k+2$ vertices is a constant depending only on k.

Lemma 5.4 (Reconstructability)**.** *Given $\boldsymbol{d}$, there is an algorithm that takes $Z_i \in \gamma_{X,Y}$ and $L_i(X,Y)$ as an input and outputs the realizations X and Y (for any i).*

Proof. The first coordinate of L_i is an realization, of the form $T_{Y,X}[i+1, i+3k+1]$ for an unknown X, Y. The index i is known, because it is the last component of L_i. W.l.o.g. we show how to recover X. From $T_{Y,X}[i+1, i+3k+1]$ and i, we know the induced subgraph of X on the vertices U_i. Similarly, the induced subgraph of Z_i on the vertices $\overline{U}_{i+3k+1}$ is identical to the induced subgraph of X on the same vertices. Hence the only unknown part of X is its induced subgraph on $N_X[U_{i+3k+1} \setminus U_i]$. The subgraph in the second component of $L_i(X,Y)$ is the left-compressed copy of $X[N_X[U_{i+3k+1} \setminus U_i]]$. Since left-compression preserves the order of the indices of $a_j \in A$ and $b_j \in B$, X can be fully recovered. □

Proof of Theorem 1.7. We have shown that $\rho(\Gamma) = \mathcal{O}(n \cdot n^4)$ and $\ell(\Gamma) = \mathcal{O}(n)$, thus $\tau(\varepsilon) \leq \mathcal{O}(n^8 \log \varepsilon^{-1})$, verifying that the switch Markov chain is rapidly mixing on $\mathbb{S}_{2k}(\mathcal{H}_0)$ and $B_{2k}(\mathcal{H}_0)$. □

6 Proof of Theorem 1.9: non-stability of $\mathcal{H}_k$

In this section we show that it is relatively straightforward to get the asymptotic growth rate of the number of realizations of $\boldsymbol{h}_k(n)$ when k is a constant and n tends to infinity. We first illustrate this for $k = 1$. Recall Corollary 4.3 and that $\boldsymbol{h}_1(n) = \boldsymbol{h}_0(n) - \mathbb{1}_{a_1} - \mathbb{1}_{b_1}$.

Lemma 6.1. *The number of all directed paths (integer 1-flows) from a_1 to b_n in F_n is*

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Proof. Let $S_1(\ell)$ be the number of paths in F_n that start at a_1 and end in B_ℓ. Similarly, let $S_2(\ell)$ be the number of paths in F_n that start at a_1 and end in one of the vertices in A_ℓ. We have

$$\begin{aligned} S_1(\ell+1) &= 2S_1(\ell) + S_2(\ell), \\ S_2(\ell+1) &= S_1(\ell) + S_2(\ell). \end{aligned} \tag{5}$$

Observe that $a_1 \to b_n$ paths in F_n are in bijection with paths starting at a_1 and ending in A_n: the corresponding paths are obtained by deleting the last edge incident to b_n. Since $S_1(1) = S_2(1) = 1$, from (5) we get that $S_2(n)$ is the quantity in the statement of the Lemma and the proof is complete. □

Corollary 6.2. *The number of realizations of $\boldsymbol{h}_1(n)$ is $\Theta\left(\left(\frac{3+\sqrt{5}}{2}\right)^n\right)$*

Proof. Neither $[1, 1]$ nor $[0, 1]^T$ is perpendicular to the eigenvector that belongs to the largest eigenvalue $\frac{3+\sqrt{5}}{2}$ of the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

□

The proof of Lemma 6.1 can be interpreted as follows. We count $a_1 \to b_n$ paths by looking at their induced subgraphs on the vertices in U_ℓ (the number of these is precisely $S_1(\ell) + S_2(\ell)$). The main observation is that the number of ways an $a_1 \to U_\ell$ path can be extended to an $a_1 \to U_{\ell+1}$ path only depends on whether the path's endpoint lies in A_ℓ or in B_ℓ.

Again, according to Corollary 4.3, realizations of $\boldsymbol{h}_k(n)$ are in a 1-to-1 correspondence with integer k-flows from a_1 to b_n. We shall mimic the argument of Lemma 6.1 with k-flows. The recursion will consider the beginning of a k-flow on U_ℓ and its "termination-type".

Definition 6.3 (set of types)**.** *Let $\mathcal{P}_k$ be the set of partitions of k (the set of multisets of positive integers whose sum of elements is exactly k) and $\mathcal{P}_0 := \{\emptyset\}$. For all positive integers k, we define* the set of types*:*

$$\mathcal{T}_k := \{(R, Q) \mid \exists\, 0 \leq m \leq k : R \in \mathcal{P}_m, Q \in \mathcal{P}_{k-m}\}.$$

Definition 6.4 (type of a flow)**.** *Let X be k-flow in $F_n[U_\ell]$ from a single source a_1, and the sinks are arbitrarily distributed in U_ℓ. We say that* the type of X *is $T = (R, Q) \in \mathcal{T}_k$ if there is an injective function $f : R \to A_\ell$ such that for every $a_i \in f(R)$ we have*

$$\varrho_X(a_i) - \delta_X(a_i) = f^{-1}(a_i),$$

and for all $a_i \in A_\ell \setminus f(R)$ we have $\varrho_X(a_i) = \delta_X(a_i)$. Similarly, there is an injective function $g : Q \to B_\ell$ such that for every $b_i \in g(Q)$

$$\varrho_X(b_i) - \delta_X(b_i) = g^{-1}(b_i),$$

and for all $b_i \in B_\ell \setminus g(Q)$ we have $\varrho_X(b_i) = \delta_X(b_i)$.

Informally, the type of X describes the multiplicities of the incidences of the endpoints of the k-flow on U_ℓ.

In the proof of Lemma 6.1, the functions $S_1(\ell), S_2(\ell)$ were actually the number of 1-flows on U_ℓ of type $(\emptyset, \{1\})$ and $(\{1\}, \emptyset)$, respectively. The next definition is the analogue of the matrix in the proof of Corollary 6.2 for large k.

Definition 6.5 (type matrix)**.** *For all k, let us fix an ordering of the types: $\mathcal{T}_k = (T_1, \ldots, T_{|\mathcal{T}_k|})$. Let ℓ and n be so large, that there exists a k-flow which has type T_i on U_ℓ. We define $p_{i,j}$ to be the number possible ways a k-flow on U_ℓ from the single source a_1 can be extended to a k-flow of type T_j on $U_{\ell+1}$. We define the type-matrix $\mathcal{P}_k$ to be the $|\mathcal{T}_k| \times |\mathcal{T}_k|$ matrix whose element in the i-th row and j-th column is $p_{i,j}$.*

It is not hard to see that $p_{i,j}$ is well-defined, in other words, $p_{i,j}$ does not depend on either ℓ, n, or the k-flow.

In the proof of Corollary 6.2, the type matrix

$$\mathcal{P}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

corresponds to the ordering $\mathcal{T}_1 = \big((\emptyset, \{1\}), (\{1\}, \emptyset)\big)$. Now we are ready to prove the analogue of Lemma 6.1 for k-flows where $k > 1$.

Lemma 6.6. *For every $k \geq 1$, the number of k-flows on F_n from the single source a_1 to the single sink b_n is*

$$v^T \mathcal{P}_k^{n-1} w$$

where:

- *v is the vector of length $|\mathcal{T}_k|$ which contains 1 at the coordinates which correspond to the types $(\{k-1\}, \{1\})$, $(\{k\}, \emptyset) \in \mathcal{T}_k$, and zero everywhere else,*
- *$\mathcal{P}_k$ is the type-matrix,*
- *w is the vector of length $|\mathcal{T}_k|$ that contains 1 at the coordinate that corresponds to the type*

$$(\overbrace{\{1, 1, \ldots, 1\}}^{k}, \emptyset) \in \mathcal{T}_k$$

and zero everywhere else.

Proof. With the appropriate substitutions, the proof is identical to the proof of Lemma 6.1. The type of a k-flow on U_1 emanating from a_1 is either $(\{1\}, \{k-1\})$ or $(\emptyset, \{k\})$. By the definition of $\mathcal{P}_k$, the vector $v^T \mathcal{P}^{n-1}$ contains the number of graphs on the vertices U_n with a given type. Of these, the k-flows from $a_1 \to b_n$ correspond to graphs with type $(\emptyset, \{1, 1, \ldots, 1\})$ (deleting b_n and the incident edges results in a k-flow of this type). Hence the statement of the lemma follows. □

The following simple property of the type matrix will be used.

Definition 6.7. *A matrix $\mathcal{P}$ is primitive, if $\exists m$ for which every entry of $\mathcal{P}^m$ is positive.*

Lemma 6.8. *The type matrix $\mathcal{P}_k$ is primitive for any k.*

Proof. For every type $t \in \mathcal{T}_k$ it is easy to design a k-flow X such that the type of $X[U_\ell]$ (for some ℓ) is t and the type of $X[U_{\ell+k}]$ is $(\{1, 1, \ldots, 1\}, \emptyset)$. Hence in $\mathcal{P}_k^k$ the row and column that correspond to the type $(\{1, 1, \ldots, 1\}, \emptyset)$ are strictly positive. Since $\mathcal{P}_k$ is non-negative, it also follows that $\mathcal{P}_k^{2k}$ is positive. □

Now we are ready to prove the key lemma to refute the P-stability of the class of degree sequences $\mathcal{H}_k$.

Lemma 6.9. *For every k, the largest eigenvalue of the type-matrix $\mathcal{P}_k$ is smaller than the largest eigenvalue of the type matrix $\mathcal{P}_{k+1}$.*

Proof. By Lemma 6.8, both $\mathcal{P}_k$ and $\mathcal{P}_{k+1}$ are primitive. By the Perron-Frobenius theory, they both have a real positive eigenvalue r_k and r_{k+1}, respectively, that is larger in absolute value than all of their other eigenvalues. Moreover, both limits

$$\lim_{n\to\infty}\frac{\mathcal{P}_k^n}{r_k^n} \quad \text{and} \quad \lim_{n\to\infty}\frac{\mathcal{P}_{k+1}^n}{r_{k+1}^n}$$

exist and are one dimensional projections. Let the set of types $S \subset \mathcal{T}_{k+1}$ be defined as follows:

$$S := \{(R,Q) \in \mathcal{T}_{k+1} \;:\; 1 \in R\}.$$

Let $M^{(n)}$ be the principal minor of $\mathcal{P}_{k+1}^n$ that is obtained by taking those rows and columns which correspond to types in S. Without loss of generality, we may assume that if the i-th row of $M^{(1)}$ corresponds to a type (R,Q), then the i-th row of $\mathcal{P}_k$ corresponds to the type $(R \setminus \{1\}, Q)$. Moreover, we may assume that the ordering of $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ is compatible in the following sense: if $T = \{R,Q\}$ and $T' = (R',Q')$ are types in S and $T < T'$ according to the ordering on $\mathcal{T}_{k+1}$, then $(R \setminus \{1\}, Q) < (R' \setminus \{1\}, Q')$ according to the ordering on $\mathcal{T}_k$.

First, we prove the following two properties of $M^{(1)}$.

1. The matrix $M^{(1)}$ is element-wise larger than or equal to $\mathcal{P}_k$.
2. The matrix $M^{(1)}$ is not equal to $\mathcal{P}_k$.

Since $|S| = |\mathcal{T}_k|$, the matrix $M^{(1)}$ is a $|\mathcal{T}_k| \times |\mathcal{T}_k|$ matrix. We start with proving the second statement. The entry of $\mathcal{P}_k$ in the intersection of the row and column that correspond to the type $(\{1,\ldots,1\},\emptyset) \in \mathcal{T}_k$ and $(\emptyset,\{k\}) \in \mathcal{T}_k$, respectively, is clearly 1. On the other hand, the value of $M^{(1)}$ in this row and column corresponds to the number of transitions from $(\{1,\ldots,1\},\emptyset) \in \mathcal{T}_{k+1}$ to $(\{1\},\{k\})$ which is $k+1$ (the number of ways one can choose one of the $k+1$ paths which will not be extended). Therefore $M^{(1)} \neq \mathcal{P}_k$.

For the first statement, for any two types $(R,Q),(R',Q') \in \mathcal{T}_k$, if a type (R,Q) subgraph of a k-flow on the vertices U_ℓ can be extended to an another type (R',Q') subgraph on the vertices $U_{\ell+1}$ in p ways, then clearly a type $(R \cup \{1\}, Q)$ subgraph of a $k+1$-flow on the vertices U_ℓ can be extended to a type $(R' \cup \{1\}, Q')$ subgraph on the vertices $U_{\ell+1}$ in at least p ways. Therefore the first property is also proven.

Suppose to the contrary that $r_{k+1} \le r_k$. Since the limit

$$\lim_{n\to\infty}\frac{\mathcal{P}_{k+1}^n}{r_{k+1}^n}$$

exists and is finite, both the limits

$$\lim_{n\to\infty} \frac{\mathcal{P}_{k+1}^n}{r_k^n} \quad \text{and} \quad \lim_{n\to\infty} \frac{M^{(n)}}{r_k^n}$$

exist and are finite. Since $M^{(1)}$ is a principal minor of $\mathcal{P}_{k+1}$, and every element of $\mathcal{P}_{k+1}$ is non-negative, for all k the matrix $M^{(k)}$ is element-wise larger than or equal to $\left(M^{(1)}\right)^k$. Hence the sequence

$$\left\{ \frac{\left(M^{(1)}\right)^n}{r_k^n} \right\}_{n=1}^{\infty}$$

is bounded. By the two properties of $M^{(1)}$ and the fact that $\mathcal{P}_k$ is primitive, it follows that there is an integer m such that $\left(M^{(1)}\right)^m$ is element-wise strictly larger than $\mathcal{P}_k^m$. Thus there is a positive ε such that $\left(M^{(1)}\right)^m$ is element-wise strictly larger than $(1+\varepsilon)\mathcal{P}_k^m$. Therefore the sequence

$$\left\{ \frac{\left((1+\varepsilon)\mathcal{P}_k^m\right)^n}{r_k^{mn}} \right\}_{n=1}^{\infty} = \left\{ (1+\varepsilon)^n \frac{\mathcal{P}_k^{mn}}{r_k^{mn}} \right\}_{n=1}^{\infty}$$

is bounded, but this clearly contradicts the fact that the limit

$$\lim_{n\to\infty} \frac{\mathcal{P}_{k+1}^n}{r_{k+1}^n}$$

is a one dimensional projection. □

Proof of Theorem 1.9. Observe, that $\|\boldsymbol{h}_{k+1}(n) - \boldsymbol{h}_k(n)\|_1 = 2$. However, according to Lemma 6.9

$$\frac{|\mathcal{G}(\boldsymbol{h}_{k+1}(n))|}{|\mathcal{G}(\boldsymbol{h}_k(n))|} = \Theta\left(\left(\frac{r_{k+1}}{r_k} \right)^n \right),$$

which grows exponentially as $n \to \infty$, so $\mathcal{H}_k$ is not P-stable. □

7 Concluding remarks

7.1 Relationship to prior results

Although the sets of degree sequences $B_{2k}(\mathcal{H}_0)$ (for some k) are certainly not diverse compared to the class of P-stable degree sequences, they are more numerous than, say, the regular degree sequences, for which rapid mixing of the switch Markov chain were first proven in [2, 18, 13]. Because $B_{2k}(\mathcal{H}_0)$ is not P-stable, the Jerrum-Sinclair chain [16] cannot produce a sample in polynomial expected time. Although in principle, the proof of rapid mixing on P-stable degree sequences [4] may be applicable to $B_{2k}(\mathcal{H}_0)$, we do not expect that it can be easily tweaked to accommodate it, for the following reasoning:

Let $\mathcal{T}$ be the set of (X, Y) pairs of realizations of $\boldsymbol{h}_1(n)$ such that the paths $\vec{\nabla}(X)$ and $\vec{\nabla}(Y)$ are edge disjoint. It is simple to show that $|\mathcal{T}| \geq \exp(cn) \cdot |\mathcal{G}(\boldsymbol{h}_1(n))|$, because for almost every realization X we have $|E(\vec{\nabla}(X))| \approx \frac{2n}{\sqrt{5}}$. For a pair $(X, Y) \in \mathcal{T}$, the edges $E(X) \triangle E(Y)$ form a cycle which traverses both a_1 and b_n. From this structure it follows that the multicommodity flow Γ described in [4] between a pair of realizations $(X, Y) \in \mathcal{T}$ is a single switch sequence that passes through $H_0(n) - a_1 b_n \in \mathcal{G}(\boldsymbol{h}_1(n))$. Consequently, the load $\rho(\Gamma) \geq |\mathcal{T}|/|\mathcal{G}(\boldsymbol{h}_1(n))| \geq \exp(cn)$ is exponential in n.

7.2 Unconstrained (simple) graphs

As mentioned in Section 2.1, Ψ^{-1} embeds splitted bipartite graphs into the space of simple graphs. The map Ψ^{-1} preserves switches, since the symmetric difference of the edge sets of two realizations does not change by adding a clique to both graphs. Consequently, Ψ^{-1} induces an isomorphism between the Markov-graphs $\mathcal{M}(\boldsymbol{d})$ and $\mathcal{M}(\boldsymbol{d})(\Psi^{-1}(\boldsymbol{d}))$.

Furthermore, through Ψ^{-1}, a set of canonical paths Γ on $G(\mathcal{M}(\boldsymbol{d}))$ are mapped to a set of canonical paths $\Psi^{-1}(\Gamma)$ on $G(\mathcal{M}(\Psi^{-1}(\boldsymbol{d})))$ satisfying

$$\rho(\Psi^{-1}(\Gamma)) \leq \rho(\Gamma).$$

In summary, Theorem 1.7 can be pulled back to simple graphs: the switch Markov chain is rapidly mixing on $\Psi^{-1}(B_{2k}(\mathcal{H}_0))$. Note, however, that

$$\Psi^{-1}(B_{2k}(\mathcal{H}_0)) \subset B_{2k}(\Psi^{-1}(\mathcal{H}_0)),$$

because the right hand side contains graphs that are not split.

7.3 Possible generalizations

The proof of Theorem 1.7 presented in Section 5 works verbatim up to $k = \Theta(\sqrt{\log n})$, one just has to check the dependence on k in Section 5.4. In other words, the switch Markov chain is rapidly mixing on

$$\bigcup_{n=1}^{\infty} B_{c\cdot\sqrt{\log n}}(\boldsymbol{h}_0(n))$$

for some $c > 0$. We have not proved nor refuted P-stability of $\bigcup_{n=1}^{\infty} \mathbb{S}_{2k}(\boldsymbol{h}_0(n))$ when $k = \Theta(\sqrt{\log n})$.

We hope that the proof of Theorem 1.7 can be generalized to even broader classes. A defining property of $\boldsymbol{h}_k(n)$ is that for any realization $G \in \mathcal{G}(\boldsymbol{h}_k(n))$ and $i \in [1, n]$, we have

$$\delta_{\vec{\nabla}(G)}(A_i \cup B_i) = \delta_{\vec{\nabla}(G)}(A_i \cup B_i \setminus \{b_i\}) = k.$$

Relax these constraints to requiring only that $\leq k$ edges leave $A_i \cup B_i$ and $A_i \cup B_i \setminus \{b_i\}$ for every $i \in [1, n]$: the set of graph satisfying these is the set of realizations of a set of degree sequences we will call $\mathcal{H}_{\leq k}$. Naturally, $B_{2k}(\mathcal{H}_0) \subseteq \mathcal{H}_{\leq k}$, because a k-flow needs at most k edges in any cut.

Conjecture 7.1. *For any fixed k, the switch Markov chain is rapidly mixing on $\mathcal{H}_{\leq k}$.*

We also put forward a conjecture inspired by the work Greenhill and Gao [11]. Recall Definition 1.3.

Conjecture 7.2. *Suppose $\mathcal{D}$ is $(2k+2)$-stable for some $k \in \mathbb{N}$. Then the switch Markov chain is rapidly mixing on $B_{2k}(\overline{\mathcal{D}^\circ})$.*

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