

Saturation problems about forbidden 0-1 submatrices

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Abstract

A 0-1 matrix M is *saturating* for a 0-1 matrix P if M does not contain a submatrix that can be turned into P by changing some 1 entries to 0 entries, and changing an arbitrary 0 to 1 in M introduces such a submatrix in M . In saturation problems for 0-1 matrices we are interested in estimating the minimum number of 1 entries in an $m \times n$ matrix that is saturating for P , in terms of m and n . In other words, we wish to give good estimates for the *saturation function* of P . Recently, Brualdi and Cao initiated the study of saturation problems in the context of 0-1 matrices.

We extend their work in several directions. We prove that every 0-1 forbidden matrix has its saturation function either in $\Theta(1)$ or $\Theta(n)$ in the case when we restrict ourselves to square saturating matrices. Then we give a partial answer to a question posed by Brualdi and Cao about the saturation function of J_k , which is obtained from the identity matrix I_k by putting the first row after the last row. Furthermore, we exhibit a 5×5 permutation matrix with the saturation function bounded from the above by a fixed constant. We complement this result by identifying large classes of 0-1 matrices with linear saturation function. Finally, we completely resolve the related semisaturation problem as far as the constant vs. linear dichotomy is concerned.

1 Introduction

The study of extremal questions is a very popular theme in the area of combinatorics. Broadly speaking, in an extremal problem we are usually interested in an estimation of the maximum density of combinatorial objects, such as graphs or matrices, avoiding a given substructure that we call the *forbidden subgraph or submatrix*. In graphs, the density is simply the number of edges. For several extremal problems on (hyper)graphs their so-called saturation counterpart

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was also investigated [2, 3, 4, 5, 6, 12, 14, 17]. In a *saturation* problem, instead of the maximum density we are interested in estimating the smallest (*critical*) number of (hyper)edges such that adding an arbitrary (hyper)edge introduces the forbidden sub(hyper)graph. A variant of saturation problems is the so-called *semisaturation* (sometimes called strong saturation or oversaturation) problem [7], in which we are interested in the smallest (*critical*) number of (hyper)edges such that adding an arbitrary (hyper)edge introduces a new copy of the forbidden sub(hyper)graph.

Extremal problems on ordered bipartite graphs translate to extremal problems on 0-1 matrices, which we study in the present note. Before giving a small history of this area, we first define the three functions that we are concerned with.

Extremal function. A 0-1 matrix M contains a pattern P , which is a not-all-zeros 0-1 matrix, if M contains a submatrix P' that can be transformed into the matrix P by replacing some (potentially none) 1 entries with 0 entries. The matrix P' is an *occurrence* of P in M . A 0-1 matrix M avoids a pattern P if M does not contain P . For a 0-1 matrix M , the weight of M is the number of 1 entries of M . Let $ex(P, m, n)$ denote the maximal weight of a 0-1 matrix of size $m \times n$ that avoids P as a submatrix. We denote $ex(P, n) = ex(P, n, n)$.

Usually, P , the matrix to be avoided, is referred to as a forbidden or excluded pattern. Square-shaped patterns that have exactly one 1 entry in every row and column are called permutation patterns.

Saturation function. A matrix M is *saturating* for a pattern P if it avoids P as a submatrix and is maximal with this property, that is, if one changes any 0 entry to a 1 entry in the matrix M then the resulting matrix M' contains P . Let $sat(P, m, n)$ denote the minimal weight of a 0-1 matrix M of size $m \times n$ saturating for P . We denote $sat(P, n) = sat(P, n, n)$. The matrix M is *saturating* for P .

Semisaturation function. Let $ssat(P, m, n)$ denote the minimal weight of a 0-1 matrix M of size $m \times n$ such that if one changes any 0 entry to a 1 entry in the matrix M then the resulting matrix M' contains a new copy of P . Note that we do not require that M avoids P as a submatrix. We denote $ssat(P, n) = ssat(P, n, n)$. The matrix M is *semisaturating* for P .

Observation 1.1. *By definition it always holds that*

$$ssat(P, m, n) \leq sat(P, m, n) \leq ex(P, m, n)$$

and if $sat(P, m, n) = ex(P, m, n)$ then every maximal matrix avoiding P has the same weight.

History of the extremal problem. The study of the asymptotic behaviour of $ex(P, n)$ arose in the running time analysis of a shortest path geometric algorithm due to Mitchell [19]. Soon after a strong impetus for the development of this theory came from discrete geometry [9,

10]. The main highlight of the subsequent developments is a proof of the famous Stanley-Wilf conjecture [18] by Marcus and Tardos via Klazar's argument [15] showing that the conjecture holds once we prove that $ex(P, n) = O(n)$ for all permutation matrices P . Although the order of magnitude of $ex(P, n)$ was researched extensively for many families of patterns P , see e.g., [8, 11, 13, 18, 20, 21, 22], the asymptotic behavior of $ex(P, n)$ for natural families of patterns such as trees is still far from being well-understood.

In the study of $ex(P, n)$ two (overlapping) main tendencies can be pinpointed, first determining $ex(P, n)$ for specific matrices, e.g., in [22] all patterns with weight at most 4 and in [8, 13] further patterns of weight 5 are considered. It turns out that the extremal function is always at least linear but can have various bigger growth as well, such as $\Theta(n \log n)$ and $\Theta(n^{3/2})$. Second, it is easy to see that the growth rate is at least linear and at most quadratic and thus it gained a considerable attention to determine the patterns with linear or quasi-linear extremal functions. In this quest the most notable result is that $ex(P, n) = \Theta(n)$ for all permutation patterns [18], which was generalized to patterns that we get from permutation patterns by multiplying columns [11, 13], where they also show that this implies that there exist infinite many containment-minimal non-linear patterns. Other useful tools are various operations which can extend or combine patterns with linear extremal functions to get new such patterns [9, 22, 13]. Recently, patterns with quasi-linear extremal functions gained attention, it is conjectured that all patterns that are incidence matrices of bipartite trees are such [20, 16].

History of the saturation problem. While extremal problems on 0-1 matrices have a long history, only recently Brualdi and Cao [1] initiated the study of the saturation problems for 0-1 matrices, that is, the study of $sat(P, m, n)$ for various patterns P . In [1] it is proved that all maximal matrices avoiding the identity matrix I_k have the same weight:

Theorem 1.2. [1] $sat(I_k, m, n) = ex(I_k, m, n) = (k - 1)(m + n - (k - 1))$ (if $m, n \geq k$).

They also give a structural description of maximal families. Note that if $m < k$ or $n < k$ then no $m \times n$ matrix can contain I_k and thus the only saturating matrix is the all-1 matrix and so $sat(I_k, m, n) = ex(I_k, m, n) = mn$ trivially. Further, they consider the permutation pattern J_k which we get from I_k by putting its first row after the last row (see Figure 2). (Note that we do not denote by J_k an all 1 matrix which is quite common.) They show $sat(J_3, m, n) = ex(J_3, m, n) = 2(m + n - 2)$ and that $sat(J_k, m, n) \leq (k - 1)(m + n - (k - 1))$ and conjecture that $sat(J_k, m, n) = ex(J_k, m, n) = (k - 1)(m + n - (k - 1))$ for $k > 3$ as well.

Our results about saturation. We continue this line of research. While the above specific results would perhaps give the impression that $sat(\cdot)$ behaves always the same way or very similar to $ex(\cdot)$, we show multiple evidence that this is not the case. The following theorem shows that compared to the growth rate of $ex(P, n)$, $sat(P, n)$ is always at most linear and exhibits a much simpler dichotomic behaviour:

Theorem 1.3. *For any $k \times l$ pattern P , and for any fixed n_0, m_0 ,*

$$\text{sat}(P, m, n) \leq (k-1)n + (l-1)m - (k-1)(l-1),$$

$$\text{sat}(P, n) = O(1) \text{ or } \text{sat}(P, n) = \Theta(n),$$

$$\text{sat}(P, m_0, n) = O(1) \text{ or } \text{sat}(P, m_0, n) = \Theta(n),$$

$$\text{sat}(P, m, n_0) = O(1) \text{ or } \text{sat}(P, m, n_0) = \Theta(m).$$

In addition, if $\text{sat}(P, m_0, n) = O(1)$ then $\text{sat}(P, m_1, n) = O(1)$ for every $m_1 > m_0$ and similarly if $\text{sat}(P, m, n_0) = O(1)$ then $\text{sat}(P, m, n_1) = O(1)$ for every $n_1 > n_0$.

Note that when $n < k$ or $m < l$ then even the $m \times n$ matrix with all 1 entries avoids P , thus $\text{sat}(P, m, n) = nm$, which shows that $n_0 < k$ and $m_0 < l$ are trivial cases of Theorem 1.3 where $\text{sat}(P, m_0, n) = \Theta(n)$ and $\text{sat}(P, m, n_0) = \Theta(m)$. Note also that the upper bound of Theorem 1.3 is attained by the identity matrix I_k [1].

From now on for simplicity we concentrate only on $\text{sat}(P, n)$ and most of the times we omit variants of our results for $\text{sat}(P, m, n)$, although in many cases we could state such variants.

Even though there are only two different classes of patterns with respect to the growth rate, it seems to be challenging to understand when does a pattern have a bounded saturation function and when a linear one. Our aim is to take the first steps in this direction.

First, it is already non-trivial to come up with a pattern that has a bounded saturation function, our first result is that there exists such a permutation pattern of weight 5. Let

$$Q = \begin{pmatrix} & \bullet & & \\ & & \bullet & \\ \bullet & & & \\ & & & \bullet \end{pmatrix}$$

where 1 entries are shown as dots and 0 entries are not shown. We know that $\text{ex}(Q, n) = \Theta(n)$.

We show that $\text{sat}(Q, n)$ differs substantially from $\text{ex}(Q, n)$, as it is bounded.

Theorem 1.4. *The pattern Q has $\text{sat}(Q, n) < 400$.*

In contrast, we exhibit a wide class of patterns for which the saturation function is linear:

Claim 1.5. *If every row (or column) of a pattern P contains at least two 1-entries then $\text{sat}(P, n) = \Theta(n)$.*

Theorem 1.6. *If the first (or last) row (or column) of a pattern P is all-0 then $\text{sat}(P, n) = \Theta(n)$.*

Note that it's an easy statement that adding empty rows and columns to P does not change the order of magnitude of $ex(P, n)$, while for $sat(P, n)$ the previous theorem shows all that we know about the effect of such an operation on the saturation function. Thus even for such a seemingly harmless operation we can see a non-trivial behaviour, which is many times the case when considering saturation problems.

Theorem 1.7. *If $P = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$, for some 0-1 submatrices¹ $A, B \neq \mathbf{0}$, then*

$$sat(P, n) = \Theta(n).$$

It is an easy observation that the function $ex(\cdot)$ is monotone for pattern containment, that is, if from a pattern P we get pattern P' by replacing some 0 entries with 1 entries then $ex(P, n) \leq ex(P', n)$. However, the function $sat(\cdot)$ shows again a different behaviour, as it does not have this natural property. Indeed, $sat(Q, n) = O(1)$, while if we delete, e.g., the leftmost 1 entry from Q , we get a pattern Q' which has $sat(Q', n) = \Theta(n)$ by Theorem 1.6. Also, if we replace all 0-entries in Q by a 1 entry then we get the full-1 5×5 pattern Q'' which is easy to see that has $sat(Q'', n) = \Theta(n)$. Together, these imply that $sat(\cdot)$ cannot be monotone increasing nor decreasing for pattern containment.

We also make progress about the saturation function of J_k , which was asked in [1]:

Theorem 1.8. $sat(J_k, m, n) \geq (k-2) \max(m, n) + m + n - 1 - \frac{(k-2)(k-1)}{2}$ (if $m, n \geq k$).

Notice that similar to the identity matrix, if $m < k$ or $n < k$ then $sat(J_k, m, n) = mn$. Also, $sat(J_k, m, n) \leq (k-1)m + (k-1)n - (k-1)^2$ by the trivial construction of putting 1 entries in the first $k-1$ rows and last $k-1$ rows [1], which for the case $n = m$ is about twice as much as our lower bound². Due to the same construction $ex(J_k, m, n) \geq (k-1)m + (k-1)n - (k-1)^2$ while $ex(J_k, m, n) = O(m+n)$ by [18]. We did not try to get an explicit upper bound from their proof as anyway it is conjectured that the lower bound is the right answer [1].

Theorem 1.9. *Let $P' = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$ and $P = \begin{pmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix}$, for some 0-1 submatrix A (that is, we get P' from A by adding a new column and row at the end with a single 1 entry in their intersection and we get P from P' the same way), then*

$$ex(P, m, n) = ex(P', m-1, n-1) + m + n - 1,$$

$$sat(P, m, n) = sat(P', m-1, n-1) + m + n - 1.$$

¹With a slight abuse of notation, throughout the paper $\mathbf{0}$ denotes an all-0 matrix of any size.

²More precisely, if we subtract $m+n-1$ from both the lower bound and the upper bound, which is the size of the maximal staircase present in every matrix saturating for J_k (see the proof of Theorem 1.8), then the lower bound becomes exactly half of the upper bound.

First, applying this theorem repeatedly to $P' = I_1$ (note that $\text{sat}(I_1, m, n) = \text{ex}(I_1, m, n) = 0$ trivially) we get that $\text{sat}(I_k, m, n) = \text{ex}(I_k, m, n) = (k-1)(m+n-(k-1))$ (if $m, n \geq k$), which gives another proof of Theorem 1.2.

By repeated application of Theorem 1.9 we can also generalize Theorem 1.8. Let I'_k be the matrix we get from I_k by reversing the order of its rows, that is, I'_k has 1 entries in its main skew diagonal and nowhere else (see Figure 2).

Corollary 1.10. *If P has form $\begin{pmatrix} \mathbf{0} & I_{k-1} \\ I'_{l+1} & \mathbf{0} \end{pmatrix}$, then*

$$\text{sat}(P, m, n) \geq (k-2) \cdot \max(m, n) + (l+1)(m+n) - O(1).$$

Notice that in Corollary 1.10 by setting $l = 0$ we get $P = J_k$.

Semisaturation result. The semisaturation problem turns out to be considerably simpler than the saturation problem. A dichotomy holds also in this case, but here we were able to characterize the patterns that have bounded semisaturation function.

Theorem 1.11. *Given a pattern P , $\text{ssat}(P, n) = O(1)$ if and only if all the following properties hold for P :*

1. *The first and last row of P both contain a 1 entry that is the only 1 entry in its column,*
2. *The first and last column of P both contain a 1 entry that is the only 1 entry in its row,*
3. *P contains a 1 entry that is the only 1 entry in its row and column,*

and $\text{ssat}(P, n) = \Theta(n)$ otherwise.

Observe that Theorem 1.11 implies among others that every permutation pattern (including I_k and J_k) has bounded semisaturation function.

2 Saturation

Proof of Theorem 1.4. Recall that $Q = \begin{pmatrix} & \bullet & & \\ & & \bullet & \\ \bullet & & & \\ & & & \bullet \end{pmatrix}$. It is enough to give a construction

of a saturating matrix M which we give in Figure 1a. The gray strips in the figure are arbitrarily wide and the 4 white corner areas contain at most a constant number of 1 entries. We obtain the actual construction by replacing some of the 0 entries in the corner white areas

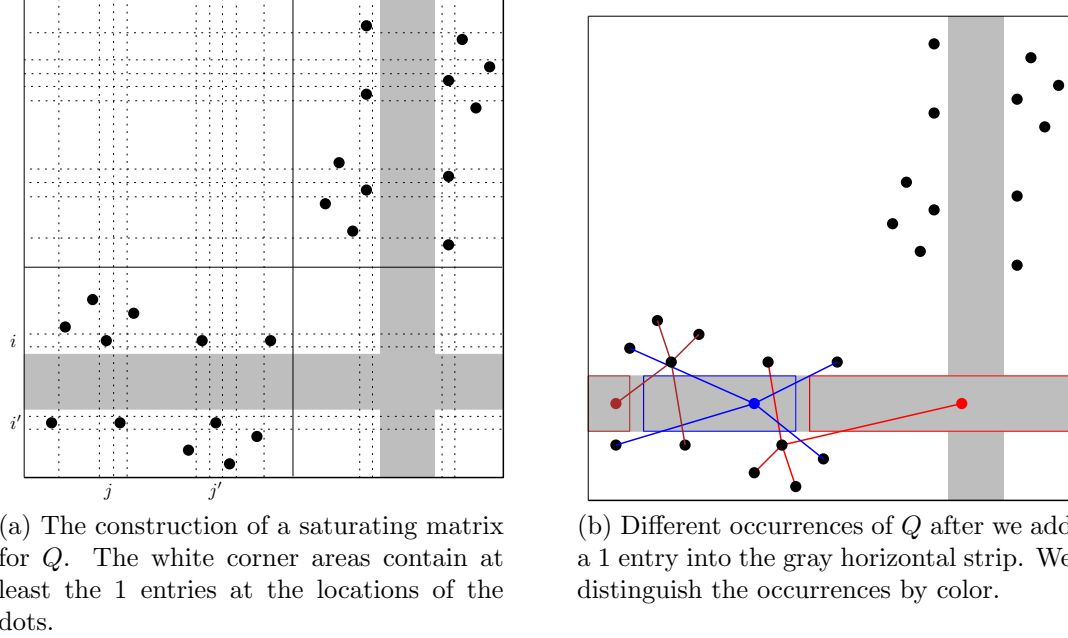


Figure 1: Saturating matrix M in the proof of Theorem 1.4.

by 1 entries so that the resulting matrix does not contain an occurrence of Q and turning any additional 0 entry in the white areas into a 1 entry would introduce an occurrence of Q .

It remains to check that the construction of the saturating matrix is correct. Figure 1b shows that the introduction of a 1 entry in the horizontal gray strip introduces an occurrence of Q . This takes care also of the vertical strip by symmetry. It remains to check that the matrix in Figure 1a does not contain an occurrence of Q which can be carried out by the following analysis.

Refer to Figure 1a. First, we observe that an occurrence Q' of Q in M must be contained in one of the two opposite “quadrants” (containing all the 1 entries) determined by the pair of thin full lines crossing inside of M in the figure. Due to symmetry it is enough to consider the case when Q' is contained in the bottom-left quadrant of M denoted by M' . We claim that there exists two 1 entries in the bottom-left quadrant that could possibly be the 1 entry in the 3rd row and the 3rd column of Q' . Namely, m_{ij} and $m_{i'j'}$ as depicted in the figure. Indeed, for every such 1 entry m_{lk} , there must exist in M' four 1 entries, one in every quadrant centered at m_{lk} , that is, four 1 entries $m_{l_0k_0}$, $m_{l_1k_1}$, $m_{l_2k_2}$, and $m_{l_3k_3}$, such that $l > l_0$ and $k > k_0$, $l < l_1$ and $k > k_1$, $l > l_2$ and $k < k_2$, and $l < l_3$ and $k < k_3$, respectively. However, neither of these two 1 entries can be contained in an occurrence of Q by a simple case analysis.

We refrain from optimizing the exact value of $\text{sat}(Q, n)$. The upper bound of 400 in the statement of the theorem is obtained by observing that the white areas form a 20×20 submatrix and in the construction 1 entries are only in the white areas. \square

Proof of Claim 1.5. Let P have at least two 1-entries in every row. Then if an $n \times n$ matrix M contains a row with no 1-entries then we can add a 1 entry in this row without introducing a copy of P . Thus, if M is saturating for P then it contains a 1-entry in every row. Thus, the weight of M is at least n and so $\text{sat}(P, n) \geq n$, finishing the proof. \square

Proof of Theorem 1.6. By symmetry we can assume that the pattern P has all 0 entries in its first row. Now assume that M is an $n \times n$ matrix does not contain P and is maximal to this property. We claim that the first row of M cannot contain a 0 entry. Indeed, assuming on the contrary that it contains, exchanging that 0 entry to a 1 entry cannot introduce a copy of P in M as this 1 entry could be only in the first row of P which contains all 0 entries. Thus, the weight of M is at least n and so $\text{sat}(P, n) \geq n$, finishing the proof. \square

Proof of Theorem 1.7. Let M denote a 0-1 matrix of size $n \times n$ saturating for P , that is it avoids P but turning any 0 entry of M into 1 results in a matrix M' that contains P . We prove that M cannot have an all-zero row or column which clearly proves the theorem. Recall that $P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

For the sake of contradiction suppose that $M = (m_{ij})$ contains an all-zero row (the proof of the non-existence of an all-zero column is analogous.) Let the l -th row of M be all-zero. By turning m_{lk} to 1, for an arbitrary k , $1 \leq k \leq n$, we obtain the matrix M_k containing P . Note that m_{l1} is contained in an occurrence of A in M_1 that is the submatrix of an occurrence of P while this is not true for m_{ln} .

It follows that there exists a maximum index $1 \leq k' < n$ such that $m_{lk'}$ is contained in an occurrence of A in $M_{k'}$ that is a submatrix of an occurrence P' of P in $M_{k'}$. Let B' denote the occurrence of B contained in P' . Then $m_{l(k'+1)}$ is contained in an occurrence of B in $M_{k'+1}$ that is a submatrix of an occurrence P'' of P in $M_{k'+1}$. Let A'' denote the occurrence of A contained in P'' .

Note that the occurrence A'' of A and B' and B are submatrices of an occurrence of P in M , a contradiction. \square

Proof of Theorem 1.3. For a pattern P with all 0's, trivially $\text{sat}(P, m, n) = 0$. Now let P be a $k \times l$ (i.e., has k rows and l columns) not all-0 pattern, having a 1 entry in the intersection of the k' th row and l' th column.

Now let M be the $m \times n$ matrix which has all 1 entries in its first $k' - 1$ and last $k - k'$ rows, first $l' - 1$ and last $l - l'$ columns. It is easy to see that M is saturating for P . Counting the 1 entries in M we get that $\text{sat}(P, m, n) \leq (k - 1)n + (l - 1)m - (k - 1)(l - 1)$, showing that $\text{sat}(P, n, n)$, $\text{sat}(P, m, n_0)$, $\text{sat}(P, m_0, n)$ can grow at most linearly.

Next we prove that these functions are either bounded or have linear growth.

Let $k' = \max(k, l)$. Assume that $\text{sat}(P, n_0, n_0) < \frac{n_0}{k'-1}$ for some $n_0 \geq k' - 1$, we want to show that in this case $\text{sat}(P, n, n) \leq \text{sat}(P, n_0, n_0)$ for $n \geq n_0$. Indeed, in this case there exists an $n_0 \times n_0$ size saturating matrix M_0 ($n_0 \geq k'$) with weight less than $\frac{n_0}{k'-1}$. Then there must

$$I_4 = \begin{pmatrix} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{pmatrix} \quad I'_4 = \begin{pmatrix} & & & \bullet \\ & & \bullet & \\ & \bullet & & \\ \bullet & & & \end{pmatrix} \quad J_5 = \begin{pmatrix} & \bullet & & & \\ & & \bullet & & \\ & & & \bullet & \\ & & & & \bullet \\ \bullet & & & & \end{pmatrix} \quad J'_5 = \begin{pmatrix} & & & \bullet & \\ & & \bullet & & \\ & \bullet & & & \\ \bullet & & & & \\ & \bullet & & & \end{pmatrix}$$

Figure 2: Examples for I_k , I'_k , J_k and J'_k .

be $k - 1$ consecutive empty rows and $l - 1$ consecutive empty columns in M . For any $n \geq n_0$ let M be the $n \times n$ size matrix which we get from M_0 by replacing these empty rows and columns with as many new empty rows and columns as necessary.

We claim that M is a saturating matrix as well. First, it cannot contain a copy of P as (using that P is non-empty) this copy uses at most $k - 1$ new empty rows and $l - 1$ empty columns. This implies that P can already be found in M_0 , a contradiction. On the other hand we claim that M is a maximal matrix avoiding P . Indeed, if we could replace a 0 entry with a 1 in M then we could find a corresponding 0 entry in M_0 with the same property, contradicting that M_0 is a saturating matrix³.

Similarly (by multiplying either only the empty rows or the empty columns) we can see that if $\text{sat}(P, m_0, n_0) < \frac{n_0}{k-1}$ for some $n_0 \geq k - 1$ and $m_0 \geq l - 1$, then $\text{sat}(P, m_0, n) \leq \text{sat}(P, m_0, n_0)$ (for $n \geq n_0$) and if $\text{sat}(P, m_0, n_0) < \frac{m_0}{l-1}$ for some $n_0 \geq k$ and $m_0 \geq l$, then $\text{sat}(P, m, n_0) \leq \text{sat}(P, m_0, n_0)$ (for $m \geq m_0$).

These together imply the theorem. \square

3 Saturation of J_k

See the examples on Figure 2 for the following definition.

Definition 3.1. I_k denotes the identity matrix of size $k \times k$. I'_k denotes the matrix we get from I_k by reflecting it vertically.

J_k is the matrix we get from I_k by cyclically shifting its rows so that its first row follows the last row. J'_k is the matrix we get from J_k by reflecting it vertically.

Definition 3.2. A staircase in a matrix M is a set S of positions of M such that the top-right and bottom-left positions of M are in S and for every position p in S the following two properties hold:

³We note that for maximality we do not need multiple consecutive empty rows and columns, one of each would be enough. Multiple empty rows and columns are only needed to show that M avoids P .

- exactly one of the neighboring positions above p and right to p is in S except when p is the top-right position,
- exactly one of the neighboring positions below p and left to p is in S except when p is the bottom-left position.

Note that a staircase in an $m \times n$ matrix M has exactly $m + n - 1$ positions.

Lemma 3.3. *Given a non- (1×1) pattern P in which the last row and column both contain exactly one 1 entry, which is in their intersection. Then in any matrix M saturating for P there is a staircase S in M such that all positions in S contain a 1 entry and all positions that are below S contain only 0 entries.*

Proof. Let $m, n \geq k$ and the $m \times n$ matrix M be saturating for P . Let S be the set of positions of 1 entries of M for which there is no 1 entry in M that has a larger row index and bigger column index. We claim that S , the set of these extremal 1 entries, is a staircase.

First observe that positions $(m, 1)$ and $(1, n)$ in M must be 1 entries and so these are in S .

Let p be a 1 entry of S different from $(m, 1)$. We need to show that exactly one of the neighboring positions p' left to p and p'' below p is in S . Both of them cannot be in S by definition of S . For the sake of contradiction, we assume that none of them is in S .

First, we consider the case when p is the lowest 1 entry in its column, in which case p is not in the first column. Then we use the saturation property for p' to get an occurrence of P , in which p' must be the bottom-right 1 entry (by the definition of S) and then replacing p' with p we get an occurrence of P in M (contradiction).

Second, we consider the case when p is not the lowest 1 entry in its column. Then there is a 1 entry q below p'' . We use the saturation property for p'' to get an occurrence of P in which p'' must be the bottom-right 1 entry (by the definition of S) and then replacing p'' with q we get an occurrence of P in M (contradiction).

Symmetrically it follows that if p be a 1 entry of S different from $(1, n)$ then exactly one of the neighboring positions p' right to p and p'' above p is in S . \square

By symmetry, whatever bounds we show about saturating J'_k , the same holds for saturating J_k . Let M be an $m \times n$ matrix saturating for J'_k . We need to show that for $w(M)$, the weight of M , we have $w(M) \geq n(k-1) + m - 1 - \frac{(k-2)(k-1)}{2}$ as then $w(M) \geq m(k-1) + n - 1 - \frac{(k-2)(k-1)}{2}$ will follow by symmetry, together implying the required lower bound.

Let S be the staircase in M , whose existence is guaranteed by applying Lemma 3.3 with $P = J'_k$. There are exactly $m + n - 1$ positions in S all of which are 1 entries, and the rest of the 1 entries of M are above S .

Definition 3.4. *Let p be a position in M above S . Let the matrix $M_p = M$ if p is a 1 entry, otherwise let M_p be a matrix that we get from M by turning p into a 1 entry. The position p*

in M has level l , $l(p) = l$ for short, if the following holds. The value of l is the largest natural number such that there exists an occurrence of I'_l in M_p which lies completely above S (that is, every 0 and 1 entry of the copy of I'_l is above S) and p is its bottom-left 1 entry. A 1 entry of M has/is on level l if its position has level l . Positions on S and below S have no level.

Lemma 3.5. *For the matrix M which is saturating for J'_k , the following hold:*

- (i) *The level of every 1 entry is at most $k - 2$.*
- (ii) *The level of every 0 entry is at most $k - 1$.*
- (iii) *For every row $i \leq m - 1$, the first position in this row is on level i if $i \leq k - 2$, and otherwise on level $k - 2$ or $k - 1$.*
- (iv) *For every row $i \leq m - 1$, the last position in this row which is above S is on level 1.*
- (v) *If two positions above S are next to each other in a row, p' left to p , then $l(p) \leq l(p') \leq l(p) + 1$.*

Proof. Notice that every position (m, j) and (j, n) , such that $1 \leq j \leq k - 1$, is a 1 entry, which then must be in S and also every position (i, j) , such that $i + j \leq k - 1$, is a 1 entry. Indeed, otherwise exchanging a 0 entry in one of these positions to a 1 entry should introduce a copy of J'_k , which is impossible. These imply (iii).

A 0 entry with level at least k or a 1 entry with level at least $k - 1$ would mean the existence of a copy of I'_{k-1} in M completely above S which together with a 1 entry on S would be a copy of J'_k , a contradiction. These imply (i) and (ii).

For any position p which is last in its row above S it is true that the position to the right of p is on S . Thus, a maximum size square matrix whose bottom-left position is p and is completely above S has size 1×1 . This implies (iv).

Finally, if a position p is on level $l(p)$ then there is a copy of $I_{l(p)}$ in M_p above S . This copy is in the intersection of the set of rows R and set of columns C . Now if in C we exchange the first column (note that p is in this column) with the column containing p' (the position left to p) to get C_1 , then in the intersection of C_1 and R there is a copy of $I_{l(p)}$ in $M_{p'}$ above S , showing that $l(p') \geq l(p)$.

Conversely, if p' is on level $l(p')$ then there is a copy of $I_{l(p')}$ in $M_{p'}$ above S . This copy is in the intersection of the set of rows R' and set of columns C' . Now if in C' we exchange the first column (note that p' is in this column) with the column containing p to get C_2 (if the column of p was already in C then we simply delete the column of p' to get C_2), then in the intersection of C_2 and an appropriate subset of R' of size $l(p') - 1$ there is a copy of $I_{l(p')-1}$ in M_p above S , showing that $l(p) \geq l(p') - 1$. This implies (v). \square

contradiction.

Let q' be the rightmost 1 entry of N' for which there exists a 1 entry q of N which has a smaller row-index and a larger column-index compared to q' (and let q be the leftmost such entry). If q and q' exist then let N_1 be the 1 entries of N in the columns of N left to the column of q and let N'_1 be the 1 entries of N' in the columns of N to the right of the column of p and to the left (including) the column of q' . If q and q' do not exist, then let $N_1 = N$ be all the 1 entries of N and let N'_1 be all the 1 entries of N' in the columns of N to the right of the column of p .

Let the 1 entry in N in the column of N preceding q be r (this must exist) and the 1 entry in N' in the column of N' after q' be r' (if it exists). The submatrix of M with opposite corners q' and q cannot contain r or r' by the choice of q and q' . Also, r and q must form an I'_2 and also r' and q' must form an I'_2 , which implies that r and r' must also form an I'_2 .

Both N_1 and N'_1 form occurrences of a vertically reflected identity matrix, say N_1 is an occurrence of I'_s , and N'_1 is an occurrence of $I'_{s'}$ for some s, s' .

First, assume $s > s'$, then by replacing in N' , the sub-matrix N'_1 by N_1 we get an occurrence of I'_{k-1} above S (using that r and r' must form an I'_2), a contradiction (as then M contains J'_k).

Second, if $s \leq s'$, then by replacing in N , the sub-matrix N_1 by N'_1 we get an occurrence of I'_l with the following property. If we introduce a 1 entry in position p then this 1 entry is extendable to an occurrence of I'_{l+1} above S having its bottom-left 1 entry in position p (using that q and q' form an I'_2), implying that the level of p is at least $l + 1$, a contradiction. \square

Proof of Theorem 1.8. There are $n + m - 1$ many 1 entries in S . Additionally, by Lemma 3.6 for every $1 \leq l \leq k - 2$ there are $n - l$ rows with a 1 entry on level l . Altogether this is at least

$$n + m - 1 + \sum_{l=1}^{k-2} (n - l) = n + m - 1 + n(k - 2) - \sum_{l=1}^{k-2} l = n(k - 1) + m - 1 - \frac{(k - 2)(k - 1)}{2}$$

1 entries in M , as required. \square

Let L_l denote the set of leftmost entries at level l in (the rows of) M . By changing “bottom-left” to “top-right” in Definition 3.4, we analogously define L'_l to be the set of top-most entries at level l in (the columns) of M . It holds that L_1 and L'_{k-2} (and by symmetry also L_{k-2} and L'_1) are disjoint (except for the top-left entry in the case $k = 3$) as otherwise for a 1 entry p in $L_1 \cap L_{k-2}$ there must be a 1 entry in M above p and a copy of I'_{k-2} whose top-left 1 entry is left of p such that these $(k - 1)$ 1 entries together with some 1 entry in S would form an occurrence of J'_k , a contradiction.

In particular, for $k = 3$ in an $m \times n$ matrix M avoiding J'_3 , by counting 1 entries using Lemma 3.6 in both L_1 and L'_1 (their only common 1 entry being the top-left 1 entry of M),

similarly as in the proof of Theorem 1.8 and adding the 1 entries in S we get the exact bound $2(m+n-2)$ for J'_3 and in turn also for J_3 , obtained also by Brualdi and Cao [1].

Proof of Theorem 1.9. Take any $m-1 \times n-1$ matrix M' saturating for P' and add a new column and row to it with all 1 entries to get the matrix M with $w(M) = \text{sat}(P', m-1, n-1) + m+n-1$. It is easy to see that M is saturating for P (notice that so far we did not need that P' is an extension of A , we just needed that P is the extension of P'). First, if M' is chosen to have minimal weight then we get that $\text{sat}(P, m, n) \leq w(M') + m+n-1 = \text{sat}(P', m-1, n-1) + m+n-1$. Second, if M' is chosen to have maximal weight then we get that $\text{ex}(P, m, n) \geq w(M') + m+n-1 = \text{ex}(P', m-1, n-1) + m+n-1$.

We are left to prove that $\text{ex}(P, m, n) \leq \text{ex}(P', m-1, n-1) + m+n-1$ and $\text{sat}(P, m, n) \geq \text{sat}(P', m-1, n-1) + m+n-1$. Let A , P and P' be matrices as defined in the theorem and M be a matrix saturating for P . Applying Lemma 3.3 to P we get a staircase S such that it has $(m+n+1)$ 1 entries and there exists no 1 entry that lies below and to the right from a 1 entry of S .

Now let the set of positions of M above S be M_1 and the set of positions below S be M_2 (notice that M_2 has only 0 entries). By deleting S and shifting every position (and the 0 entries in it) of M_2 by one to the right and by one upwards we get the set of positions M'_2 . Then M_1 together with M'_2 forms an $m-1 \times n-1$ matrix M' with $w(M') = w(M) - (m+n-1)$. We claim that M' is saturating for P' . Using this we can finish the proof. First, if M is chosen to have minimal weight then we get that $\text{sat}(P, m, n) = w(M) = w(M') + (m+n-1) \geq \text{sat}(P', m-1, n-1) + m+n-1$. Second, if M is chosen to have maximal weight then we get that $\text{ex}(P, m, n) = w(M) = w(M') + (m+n-1) \leq \text{ex}(P', m-1, n-1) + m+n-1$.

So we are left to prove that M' is saturating for P' . First we claim that M' avoids P' . Assume on the contrary that M' contains P' . As M'_2 is empty of 1 entries, the bottom-right 1 entry in this copy of P' must be in M_1 in some position p . Then by the definition of S there is a 1 entry in S in M which is to the right and below p . This together with the copy of P' forms a copy of P , contradicting that M was P -free.

Second we claim that turning any 0 entry of M' into a 1 entry introduces a copy of P' in M' . To see this take an arbitrary 0 entry in M' . First assume that it is in M_1 . Turning this into an 1 entry in M introduces a copy of P in M as M was saturating for P . Necessarily the bottom-right 1 entry of this copy is either on S or above S . In both cases after deleting S and shifting M_2 we still have a copy of P' in M' , as required. Finally, if this 0 entry is in M'_2 then again turning the respective 0 entry into a 1 entry in M_2 we get a copy of P in M . In this copy the 1 entry of P which is in its penultimate row and column must be either on S or above S . In both cases after deleting S and shifting M_2 we still have a copy of P' in M' , as required. \square

4 Semisaturation

Proof of Theorem 1.11. We call the 1 entries required by the properties *special*. First we show that if all the required properties hold for a $k \times l$ pattern P then indeed $ssat(P, n) = O(1)$. If $n \leq \max(2k - 2, 2l - 2)$ then M can simply be the all-1 $n \times n$ matrix. Otherwise, let M be the matrix that has a $(k - 1) \times (l - 1)$ matrix with all 1 entries in all four of its corners and has 0 entries everywhere else. We claim that M semisaturates P , that is, exchanging any 0 entry in M to a 1 entry introduces a new copy of P . This shows that if all properties hold for P then $ssat(P, n) \leq (2k - 2)(2l - 2) = O(1)$.

Let us first assume that the new 1 entry is in the first $k - 1$ rows, that is, the coordinates of it are (x, y) where $x \leq k - 1$ (and thus $l - 1 < y < m - l + 1$). In this case it is easy to see that there is a new copy of P in M in which this new 1 entry plays the role of the special 1 entry required by the properties in the first row of P . Similarly, if the new 1 entry is in the last $k - 1$ rows or first $l - 1$ columns or last $l - 1$ columns then a symmetric argument works (note that the properties are also symmetric). Second, assume that the new 1 entry has coordinates (x, y) where $k - 1 < x < n - k + 1$ and $l - 1 < y < m - l + 1$. Then again there is a new copy of P , in which the new 1 entry plays the role of the 1 entry required by the last property.

Now we show that if P does not have at least one of the properties required, then $P = \Theta(n)$. First, by Theorem 1.3 we know that $ssat(P, n) \leq sat(P, n) = O(n)$.⁴ Let M be a semisaturating matrix. First, assume that there is no special 1 entry in the first row of P . Then we claim that every column of M contains a 1 entry. Indeed, if there would be a column with no 1 entries then exchanging its first (top) 0 entry with a 1 entry could not introduce a new copy of P as in such a copy this new 1 entry would have to be special of this type. Thus in this case the weight of M is at least n . If there is no special 1 entry in the last row or first column or last column of P then a symmetric argument shows that the weight of M is at least n . Finally, if there is no special 1 entry which is the single 1 entry in its row and column then M cannot contain both an empty row and an empty column. Indeed, in this case adding a new 1 entry in the intersection of the empty row and empty column could not introduce a new copy of P as in such a copy this new 1 entry would have to be special of this type. Thus, the weight of M is at least n in this case as well. We conclude that if any of the properties fail then $ssat(P, n) \geq n$ and thus $ssat(P, n) = \Theta(n)$. \square

5 Discussion

We proved constant vs. linear dichotomy for the saturation function in the context of 0-1 matrices. On the one hand, we identified large classes of forbidden matrices with linear

⁴Note that the semisaturation function of any $k \times l$ pattern P with a single 1 entry attains the upper bound of Theorem 1.3, that is, $ssat(P, m, n) = sat(P, m, n) = (k - 1)n + (l - 1)m - (k - 1)(l - 1)$ for such a pattern.

saturation functions. On the other hand, up to left-to-right and bottom-to-top symmetry we discovered only one matrix, the permutation matrix Q , whose saturation function is in $\Theta(1)$. The only other matrix with the constant saturation function that we are aware of is the 1×1 identity matrix I_1 .

We suspect that there exist many more forbidden matrices with a constant saturation function, but we were unable to find any such matrix besides Q and I_1 . Here, a natural problem is to characterize forbidden permutation matrices with a linear/constant saturating function. To the best of our knowledge, this is open already if we restrict ourselves to 4×4 permutation matrices. Here, an interesting question is to decide whether Q becomes linear after the removal of the 3rd column and 3rd row from it.

Finally, it is not known if there exists an algorithm that always stops that decides for a given 0-1 matrix P whether $\text{sat}(P, n) = O(1)$. To this end, notice that similarly as in the case of the pattern Q , if a pattern P contains at most s empty rows and empty columns (e.g., in case of Q we have $s = 0$), then if a matrix M is saturating for P and contains $s + 1$ consecutive empty rows and columns, then any matrix M' that we get from M by further multiplying these empty rows and columns must also be saturating for P and thus P has a constant saturating function. Let us call such a matrix M a *witness* of P having a constant saturating function. Notice also that if P has a constant saturating function then it necessarily has a witness for this, and so a P has a constant saturating function if and only if it has a witness matrix for this.

Thus, to give a decision algorithm, it would be enough to prove that there exists a function $f(k, l)$ such that for any $k \times l$ matrix P if there exists some witness M for P then there is also an $f(k, l) \times f(k, l)$ matrix M' which is a witness for P . Indeed, then it would be enough to check all $f(k, l) \times f(k, l)$ 0-1 matrices, which is $2^{(f(k, l))^2}$ matrices. If any of them is a witness matrix, then P has a constant saturating function. If this is not the case then P has a linear saturating function.

References

- [1] Richard A Brualdi and Lei Cao. Pattern-avoiding $(0, 1)$ -matrices. *arXiv preprint arXiv:2005.00379*, 2020.
- [2] Gábor Damásdi, Balázs Keszegh, David Malec, Casey Tompkins, Zhiyu Wang, and Oscar Zamora. Saturation problems in the ramsey theory of graphs, posets and point sets, 2020. *arXiv:2004.06097*.
- [3] Andrzej Dudek, Oleg Pikhurko, and Andrew Thomason. On minimum saturated matrices. *Graphs and Combinatorics*, 29(5):1269–1286, 2013.

- [4] P. Erdős, A. Hajnal, and J. W. Moon. A problem in graph theory. *The American Mathematical Monthly*, 71(10):1107–1110, 1964.
- [5] Michael Ferrara, Bill Kay, Lucas Kramer, Ryan R Martin, Benjamin Reiniger, Heather C Smith, and Eric Sullivan. The saturation number of induced subposets of the boolean lattice. *Discrete Mathematics*, 340(10):2479–2487, 2017.
- [6] Nóra Frankl, Sergei Kiselev, Andrey Kupavskii, and Balázs Patkós. Vc-saturated set systems. *arXiv preprint arXiv:2005.12545*, 2020.
- [7] Zoltán Füredi and Younjin Kim. Cycle-saturated graphs with minimum number of edges. *Journal of Graph Theory*, 73(2):203–215, 2013.
- [8] Radoslav Fulek. Linear bound on extremal functions of some forbidden patterns in 0–1 matrices. *Discrete Mathematics*, 309(6):1736–1739, 2009.
- [9] Zoltán Füredi. The maximum number of unit distances in a convex n -gon. *J. Comb. Theory, Ser. A*, 55(2):316–320, 1990.
- [10] Zoltán Füredi and Péter Hajnal. Davenport-Schinzel theory of matrices. *Discrete Mathematics*, 103(3):233–251, 1992.
- [11] Jesse T Geneson. Extremal functions of forbidden double permutation matrices. *Journal of Combinatorial Theory, Series A*, 116(7):1235–1244, 2009.
- [12] Dániel Gerbner, Balázs Keszegh, Nathan Lemons, Cory Palmer, Dömötör Pálvölgyi, and Balázs Patkós. Saturating sperner families. *Graphs and Combinatorics*, 29(5):1355–1364, 2013.
- [13] Balázs Keszegh. On linear forbidden submatrices. *Journal of Combinatorial Theory, Series A*, 116(1):232–241, 2009.
- [14] Balázs Keszegh, Nathan Lemons, Ryan R. Martin, Dömötör Pálvölgyi, and Balázs Patkós. Induced and non-induced poset saturation problems, 2020. [arXiv:2003.04282](#).
- [15] Martin Klazar. The Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture. In *Formal power series and algebraic combinatorics*, pages 250–255. Springer, 2000.
- [16] Dániel Korándi, Gábor Tardos, István Tomon, and Craig Weidert. On the Turán number of ordered forests. *Journal of Combinatorial Theory, Series A*, 165:32–43, 2019.
- [17] L. Kászonyi and Zs. Tuza. Saturated graphs with minimal number of edges. *Journal of Graph Theory*, 10(2):203–210, 1986.

- [18] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the Stanley-Wilf conjecture. *Journal of Combinatorial Theory, Series A*, 107(1):153–160, 2004.
- [19] Joseph SB Mitchell. L1 shortest paths among polygonal obstacles in the plane. *Algorithmica*, 8(1-6):55–88, 1992.
- [20] János Pach and Gábor Tardos. Forbidden paths and cycles in ordered graphs and matrices. *Israel Journal of Mathematics*, 155(1):359–380, 2006.
- [21] Seth Pettie. Degrees of nonlinearity in forbidden 0–1 matrix problems. *Discrete Mathematics*, 311(21):2396–2410, 2011.
- [22] Gábor Tardos. On 0–1 matrices and small excluded submatrices. *Journal of Combinatorial Theory, Series A*, 111(2):266–288, 2005.