The action of a compact Lie group on nilpotent Lie algebras of type $\{n, 2\}$ *†‡

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Abstract

We classify finite-dimensional nilpotent Lie algebras with 2-dimensional central commutator ideals admitting a Lie group of automorphisms isomorphic to $\mathrm{SO}_2(\mathbb{R})$. This enables one to enlarge the class of nilpotent Lie algebras of type $\{n, 2\}$.

1. Introduction

The most simple (non-abelian) Lie algebras are the generalized Heisenberg Lie algebras, defined on a $(n+1)$-dimensional vector space $\mathfrak{h} = V \oplus \langle x \rangle$ by a non-degenerate alternating form F on the n-dimensional subspace V (n even), putting $[u, v] = F(u, v)x$, for any $u, v \in V$.

According to the literature beginning with Vergne [8], metabelian Lie algebras $\mathfrak{h} = V \oplus \langle x, y \rangle$ of dimension $(n+2)$ defined by a pair of alternating form F_1, F_2 on the n-dimensional vector space V, putting, for any $u, v \in V$, $[u, v] = F_1(u, v)x + F_2(u, v)y$, are called nilpotent Lie algebra of type $\{n, 2\}$, where the *type* $\{p_1, \ldots, p_c\}$ of a nilpotent Lie algebra $\mathfrak{g}$ with descending central series $\mathfrak{g}^{(i)} = [\mathfrak{g}, \mathfrak{g}^{(i-1)}]$ is defined by the integers $p_i = \dim \frac{\mathfrak{g}^{(i-1)}}{\mathfrak{g}^{(i)}}$.

Nilpotent Lie algebras of type $\{n, 2\}$ have been classified firstly by Gauger [5], applying the canonical reduction of the pair F_1, F_2. We mention that also nilpotent Lie algebras of type $\{n, 2, 1\}$ can be explicitly described (cf. [1]). According to results of Belitskii, Lipyanski, and Sergeichuk [3], this line of investigation cannot be carried further. A possible way of broadening these families of Lie algebras appears therefore that of considering their derivations.

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In this paper we want to study derivations of a nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$, whereas derivations of a nilpotent Lie algebra of type $\{n, 2, 1\}$ are being considered in another paper [2].

As a first question, we ask whether $\mathfrak{h}$ admits a compact Lie algebra of derivations. A non-commutative simple compact Lie algebra of derivations of a nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$ must induce the null map on the two–dimensional commutator ideal.

The smallest example, that is, the 5-dimensional Lie algebra of type $\{3, 2\}$ defined by

$$[u_1, u_2] = x, \quad [u_1, u_3] = y,$$

does not have compact non-commutative Lie algebras of derivations, since its derivations inducing the null map on $\mathfrak{h}'$ are defined with respect to the basis $\{u_1, u_2, u_3, x, y\}$ by the matrices

$$\left(\begin{array}{c|cc|cc} a & 0 & 0 & 0 & 0 \\ \hline b & -a & 0 & 0 & 0 \\ c & 0 & -a & 0 & 0 \\ \hline d_1 & d_2 & d_3 & 0 & 0 \\ d_4 & d_5 & d_6 & 0 & 0 \end{array}\right).$$

But in general the structure of a Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$ is not particularly rigid, and the following example shows that, as soon as $n = 4$, the algebra of derivations contains compact simple subalgebras.

Example 1. Let $\mathcal{B} = \{u_1, u_2, u_3, u_4, x, y\}$ be a basis of the 6-dimensional Lie algebra $\mathfrak{h}$ of type $\{4, 2\}$ defined by

$$[u_1, u_3] = x, \quad [u_1, u_4] = -y, \quad [u_2, u_3] = y, \quad [u_2, u_4] = x.$$

A direct computation shows that, with respect to the basis $\mathcal{B}$, the derivations of $\mathfrak{h}$ inducing the null map on $\mathfrak{h}'$ are represented by matrices of the form

$$\left(\begin{array}{cccc|cc} a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ -a_2 & a_1 & a_4 & -a_3 & 0 & 0 \\ -b_2 & c_2 & -a_1 & a_2 & 0 & 0 \\ c_2 & b_2 & -a_2 & -a_1 & 0 & 0 \\ \hline d_1 & d_2 & d_3 & d_4 & 0 & 0 \\ d_5 & d_6 & d_7 & d_8 & 0 & 0 \end{array}\right),$$

and, with $a_1 = 0$, $c_2 = -a_4$, $b_2 = a_3$, and all the entries d_i equal to zero, we get a subalgebra isomorphic to a real form of $\mathfrak{su}_2(\mathbb{C})$.

This, and the fact that any maximal compact subgroup of a connected real solvable Lie group is a torus, legitimate one, in our opinion, to study the action of a torus on nilpotent Lie algebras of type $\{n, 2\}$.

The simple structure of a nilpotent Lie algebra of type $\{n, 2\}$ admitting a one-dimensional compact group T of automorphisms is hidden by three obstacles that can be removed by a clear notation. The first is the representation of $2h \times 2k$ real matrices as $h \times k$ matrices with coefficients in the algebra of split-quaternions. The second is the reduction to canonical form of a pair of alternating forms. The third is the reduction to the T-*undecomposable* case. In Section 2 we summarize some known facts, fix the notation and find the relations (6), that are basic for the classification. Thereafter, we consider, in Section 3, the case where T induces the identity on the commutator ideal. It turns out that the classification is parametrized only by the dimension of $\mathfrak{h}$ and by a complex eigenvalue q (cf. Theorem 3). In Section 4, we consider the case where T operates effectively on the commutator ideal. In this case, the classification is more rich and we consider four cases. The classification is always reached by parameters that are linked to the dimension of the eigenspaces of T and by a certain arbitrarity in the reduction to the echelon form of the blocks of the matrices describing the Lie algebra $\mathfrak{h}$. But in one case (cf. Theorem 9), a class is possibly given by the real form of an arbitrary skew-symmetric complex matrix.

2. Notation

Split-quaternions and matrix notation. We denote:

i) by 0 any $n \times m$ (real or complex) zero matrix, by I_m the (real or complex) m-dimensional identity matrix, and by $\widetilde{I}_{n\times m}$ a $n \times m$ matrix having rank m, obtained by the identity matrix I_m by inserting $n - m$ zero rows (without specifying, however, which ones);

ii) by $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the real matrix corresponding to the imaginary unit;

iii) by A' the transpose of A, and by $A^\dagger$ the conjugate transpose of A;

iv) by $A \oplus B$ the diagonal block matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, and by $(\oplus A)$ the diagonal block matrix $A \oplus \cdots \oplus A$.

Throughout the paper, we represent the Clifford algebra of split-quaternions as the set

$$\mathbb{H}_- = \{z_1 + z_2\omega : z_i \in \mathbb{C}, \omega z = \bar{z}\omega, \omega^2 = 1\}.$$

We recall that, through the usual identification of the complex number $z = a + ib$ with the real matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and of the reflection $\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with the split-quaternion ω, one obtains an isomorphism of the algebra of real 2×2 matrices with the algebra $\mathbb{H}_-$ and, more generally, of the space of $\mathbb{R}^{2n\times 2m}$ matrices with the space of $\mathbb{H}_-^{n\times m}$ matrices.

In more details, with the identification of the split-quaternion matrix ωI_n with the $2n\times 2n$ reflection $\Omega_{2n}=\Omega\oplus\cdots\oplus\Omega$, any matrix $A\in\mathbb{R}^{2n\times 2m}$ can be written in a unique way as $A=A_1+A_2\Omega_{2m}$, where A_1 and A_2 are *real forms* of complex matrices $\widehat{A}_1=(z_{ij})$, $\widehat{A}_2=(u_{ij})\in\mathbb{C}^{n\times m}$ such that, for $\overline{\widehat{A}}_1=(\bar z_{ij})$ and $\overline{\widehat{A}}_2=(\bar u_{ij})$, one has $\omega I_n\widehat{A}_i=\overline{\widehat{A}}_i\,\omega I_m\ \ (i=1,2)$.

Canonical form of a pair of alternating forms. The problem of the simultaneous reduction to canonical form of a pair of symmetric or alternating forms is classic and, even if the alternating case has been settled only in 1976 by R. Scharlau [7], it goes back to a paper of Kronecker [6]. A summary is given in [4], here we point out that an unordered pair of alternating complex bilinear forms can be simultaneously reduced to the direct sum of the canonical forms $(L_t,R_t)^\nabla$ and $(I_t,J_t(q))^\nabla$, where

$$(A,B)^\nabla=\left(\left(\begin{array}{c|c}0&-A\\\hline A'&0\end{array}\right),\left(\begin{array}{c|c}0&-B\\\hline B'&0\end{array}\right)\right);$$

L_t and R_t are $t\times(t+1)$ matrices mapping the t-tuple $(x_1,\dots,x_t)$ to $(x_1,\dots,x_t,0)$ and $(0,x_1,\dots,x_t)$, respectively; I_t is the identity $t\times t$ matrix, and $J_t(q)$ is the $t\times t$ Jordan block of eigenvalue q. Over the field of real numbers, together with $(L_t,R_t)^\nabla$ and $(I_t,J_t(q))^\nabla$ $(q\in\mathbb{R})$, one has to consider also the real form of $(\widehat{I}_t,\widehat{J}_t(q))^\nabla$ $(q\in\mathbb{C})$.

Let $\{e_1,\dots,e_n,x,y\}$ be a basis of a nilpotent Lie algebra with a central commutator ideal $\mathfrak{h}'$, such that $\{x,y\}$ is a basis of $\mathfrak{h}'$. We define the pair of alternating matrices $\Big(T_1=(a_{ij}),T_2=(b_{ij})\Big)$ by

$$[e_i,e_j]=a_{ij}x+b_{ij}y.$$

Manifestly, such nilpotent Lie algebras can be thoroughly described by the canonical forms $(L_t,R_t)^\nabla$, $(I_t,J_t(q))^\nabla$ $(q\in\mathbb{R})$ and the real form of $(\widehat{I}_t,\widehat{J}_t(q))^\nabla$ $(q\in\mathbb{C})$. Notice that the module $(L_0,R_0)^\nabla=\big((0),(0)\big)$ is a component of the pair (T_1,T_2) if and only if the centre of $\mathfrak{h}$ contains the commutator ideal properly.

T-undecomposable nilpotent Lie algebra of type $\{n,2\}$. Let $\mathfrak{h}$ be a nilpotent Lie algebra of type $\{n,2\}$, that is, with a 2-dimensional commutator ideal $\mathfrak{h}'$ coinciding with the centre $\mathfrak{z}$. Let T be a group of automorphisms of $\mathfrak{h}$ isomorphic to $\mathrm{SO}_2(\mathbb{R})$ and $\mathfrak{t}$ the corresponding compact algebra of derivations of $\mathfrak{h}$, let $n=2m$ if n is even, and $n=2m+1$ if n is odd, with $n\ge 3$. By the complete reducibility of T, we find a basis $\{e_1,\dots,e_n,x,y\}$ of $\mathfrak{h}$, such that $\{x,y\}$ is a basis of $\mathfrak{h}'=\mathfrak{z}$ and such that $\mathfrak{t}$ operates on $\mathfrak{h}$ as the algebra of matrices

$$\begin{array}{c}\Big\{\partial(t):=\Big(\alpha_1t\cdot J\oplus\cdots\oplus\alpha_mt\cdot J\Big)\oplus\beta t\cdot J|\ t\in\mathbb{R}\Big\}\ \text{ for }\ n=2m,\\ \Big\{\partial(t):=\Big(0\oplus\alpha_1t\cdot J\oplus\cdots\oplus\alpha_mt\cdot J\Big)\oplus\beta t\cdot J|\ t\in\mathbb{R}\Big\}\ \text{ for } n=2m+1,\end{array}\qquad(1)$$

where $\beta t \cdot J$ is the 2×2 matrix operating on $\mathfrak{h}' = \langle x, y\rangle$. Notice that, up to rescaling the parameter t, we can assume that either $\beta = 0$ or $\beta = 1$. Moreover, up to interchanging the basis vector of each T-invariant plane in $\mathfrak{h}$, we can assume that α_i is non-negative, for all $i = 1, \dots, m$, and, up to interchanging the ordering of the planes in the basis, we can assume that $\alpha_i \leq \alpha_{i+1}$, for all $i = 1, \dots, m-1$.

If $\mathfrak{h}$ contains two proper ideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ which are invariant under T such that $[\mathfrak{i}_1, \mathfrak{i}_2] = 0$ and $\mathfrak{i}_1 \cap \mathfrak{i}_2 = \mathfrak{h}'$, we say that $\mathfrak{h}$ is T-*decomposable* into the direct sum of $\mathfrak{i}_1$ and $\mathfrak{i}_2$ with amalgamated centre, and we restrict our interest on T-undecomposable Lie algebras $\mathfrak{h}$ of type $\{n, 2\}$. Namely, if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two T-undecomposable nilpotent Lie algebras of type $\{n, 2\}$ such that the action of the group T on the centre $\mathfrak{z}_1$ of $\mathfrak{h}_1$ and on $\mathfrak{z}_2$ of $\mathfrak{h}_2$ coincides, then the direct sum of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ with amalgamated centre is a T-decomposable nilpotent Lie algebra of type $\{n, 2\}$, and any T-decomposable nilpotent Lie algebra of type $\{n, 2\}$ is obtained in this way.

As above, we define the pair of alternating matrices $\Big(T_1 = (a_{ij}), T_2 = (b_{ij})\Big)$ by putting

$$[e_i, e_j] = a_{ij} x + b_{ij} y.$$

Clearly, the Lie algebra $\mathfrak{h}$ is T-decomposable if and only if the matrices T_1 and T_2 can be put into the same diagonal block form and T leaves invariant the subspaces corresponding to the blocks.

Writing

$$\begin{aligned} &\Big\{\partial_0(t) := \Big(\alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J\Big) \mid t \in \mathbb{R}\Big\}, \quad \text{for } n \text{ even} \\ &\Big\{\partial_0(t) := \Big(0 \oplus \alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J\Big) \mid t \in \mathbb{R}\Big\}, \ \text{for } n \text{ odd,} \end{aligned}$$

since $\mathfrak{t}$ operates as an algebra of derivations of $\mathfrak{h}$, that is,

$$[e_i, e_j]^{\partial(t)} = [e_i^{\partial(t)}, e_j] + [e_i, e_j^{\partial(t)}],$$

for a generator of $\mathfrak{t}$, e. g. for $t = 1$, we get

$$\begin{aligned} \beta T_2 &= \partial_0(1)' T_1 + T_1 \partial_0(1) \\ -\beta T_1 &= \partial_0(1)' T_2 + T_2 \partial_0(1), \end{aligned} \tag{2}$$

hence we find

$$\begin{aligned} \beta^2 T_1 &= -\partial_0(1)'^2 T_1 - 2\partial_0(1)' T_1 \partial_0(1) - T_1 \partial_0(1)^2 \\ \beta^2 T_2 &= -\partial_0(1)'^2 T_2 - 2\partial_0(1)' T_2 \partial_0(1) - T_2 \partial_0(1)^2. \end{aligned} \tag{3}$$

We arrange the matrices T_1 and T_2 into 2×2 blocks A_{hk} and B_{hk} with $h, k = 1, \dots, m$ (in the case where $n = 2m+1$, we denote the 1×2 blocks of the first row with A_{0k} and B_{0k} and we put $\alpha_0 = 0$). Then (2) is equivalent to

$$\begin{aligned} \beta B_{hk} &= -\alpha_h J A_{hk} + \alpha_k A_{hk} J \\ -\beta A_{hk} &= -\alpha_h J B_{hk} + \alpha_k B_{hk} J \end{aligned} \tag{4}$$

and (3) is equivalent to

$$\begin{aligned} -\beta^2 \cdot A_{hk} &= -(\alpha_h^2 + \alpha_k^2) A_{hk} + 2\alpha_h \alpha_k J' A_{hk} J \\ -\beta^2 \cdot B_{hk} &= -(\alpha_h^2 + \alpha_k^2) B_{hk} + 2\alpha_h \alpha_k J' B_{hk} J. \end{aligned} \tag{5}$$

Notice that, since $\alpha_0 = 0$, the above equations still hold, with a slight abuse of notation, in the case where $h = 0$.

Considering T_1 and T_2 as split-quaternion matrices, we write $A_{hk} = z_1 + z_2\omega$ for suitable complex numbers z_1 and z_2. Then the equations (5) give

$$\begin{aligned} (\alpha_h - \alpha_k)^2 z_1 &= \beta^2 z_1, \\ (\alpha_h + \alpha_k)^2 z_2 &= \beta^2 z_2. \end{aligned} \tag{6}$$

Remark 2. Notice that, if h is such that, for any k with $\alpha_h \neq \alpha_k$, all the blocks A_{hk} and B_{hk} are zero, then $\mathfrak{h}$ is T-decomposable. By the way, from equations (4) it follows that, in the case where $\beta \neq 0$, A_{hk} is zero if and only if B_{hk} is zero.

3. The case where $\beta = 0$

This is the case where T is contained in a non-commutative compact group of automorphisms of $\mathfrak{h}$ (see Remark 6).

Theorem 3. *With the above notations, let $\mathfrak{h}$ be a T-undecomposable Lie algebra of type $\{n, 2\}$ and let $\beta = 0$. Then n is even and, up to a change of basis, the pair (T_1, T_2) is the real form of the pair of complex matrices $\left(L_{\frac{n+2}{4}}, R_{\frac{n+2}{4}}\right)^{\nabla}$, if $n \equiv 2 \bmod 4$, or the real form of the pair of complex matrices $\left(I_{\frac{n}{4}}, J_{\frac{n}{4}}(q)\right)^{\nabla}$, if $n \equiv 0 \bmod 4$. The group T operates, with respect to the chosen basis, as the group of automorphisms $\exp\big(\partial(t)\big)$, where*

$$\partial(t) = t \cdot \Big((\oplus \alpha J) \Big) \oplus 0.$$

Proof. From the equations (4) we deduce that, if $\alpha_h = 0 \neq \alpha_k$, then A_{hk} and B_{hk} are zero. As $\mathfrak{h}$ is T-undecomposable, it follows from Remark 2 that α_h is positive for any $h = 1, \dots, m$ and n is even. Write $A_{hk} = z_1 + z_2\omega$ for suitable complex numbers z_1 and z_2. Then the equations (5) give

$$\begin{aligned} (\alpha_h - \alpha_k)^2 z_1 &= 0, \\ (\alpha_h + \alpha_k)^2 z_2 &= 0. \end{aligned}$$

This latter forces $z_2 = 0$, that is, A_{hk} and B_{hk} are the real form of two complex numbers. Moreover, from the former we obtain that either A_{hk} and B_{hk} are zero, or $\alpha_h = \alpha_k$. As $\mathfrak{h}$ is T-undecomposable, we exclude the first case, hence we have that $\partial(t) = t \cdot \Big((\oplus \alpha J) \Big) \oplus 0$. Since T_1 and T_2 are the real form of complex $m \times m$ matrices $\widehat{T_1}$ and $\widehat{T_2}$ and T operates on them as the complex scalar matrix $\alpha i I_m$, up to a change of basis in the m-dimensional complex space, which leaves T invariant, we can assume that $(\widehat{T_1}, \widehat{T_2})$ is either $\Big(L_{\frac{n+2}{4}}, R_{\frac{n+2}{4}} \Big)^{\nabla}$ (if $n \equiv 2 \bmod 4$), or $\Big(I_{\frac{n}{4}}, J_{\frac{n}{4}}(q) \Big)^{\nabla}$ (if $n \equiv 0 \bmod 4$). These are T-undecomposable over the real numbers, since the only T-invariant real planes are $\Pi_1 = \langle e_1, e_2 \rangle, \ldots, \Pi_{\frac{n}{2}} = \langle e_{n-1}, e_n \rangle$. □

Remark 4. Up to rescaling the parameter t, we can assume $\alpha = 1$ in the above theorem, but we prefer to leave it, because, in the case where $\mathfrak{h}$ is T-decomposable, different values of α can occur.

Remark 5. For $n = 4$ and $q = i$, that is, $\big(I_{\frac{n}{4}}, J_{\frac{n}{4}}(q) \big) = \big((1), (i) \big)$, we obtain Example 1 in the Introduction.

Remark 6. Let $\mathfrak{k}$ be a simple compact algebra of derivations of the nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$, hence it induces on the 2-dimensional commutator subalgebra $\mathfrak{h}'$ the null map. Any element in $\mathfrak{k}$ generates a 1-dimensional compact subalgebra of derivations of $\mathfrak{h}$, thus $\mathfrak{h}$ has the structure given in Theorem 3, and its algebra of derivations can be directly computed.

4. The cases where $\beta \neq 0$

Up to rescaling the parameter t, if $\beta \neq 0$, then we can assume that $\beta = 1$. From now on, we need to distinguish the cases where the smallest coefficient α_h is zero or, respectively, smaller, equal or greater than $1/2$. The arguments are more or less the ones we give in the following theorem.

Theorem 7. *With the notations given in (1), if $\beta = 1$ and if the smallest coefficient α_h is zero, then, with respect to a suitable basis of $\mathfrak{h}$, the group T operates as the group of automorphisms* $\exp\big(\partial(t)\big)$, *where*

$$\partial(t) = t \cdot \Big((\oplus 0) \oplus (\oplus J) \oplus (\oplus 2J) \oplus \cdots \oplus (\oplus lJ) \Big) \oplus t \cdot J \tag{7}$$

with the diagonal blocks $(\oplus iJ)$ of dimension $d_i \times d_i$ (with d_i even for $i > 0$), and the T-undecomposable Lie algebra $\mathfrak{h}$ is described by the pair (T_1, T_2), where $T_2 = T_1(\oplus J)$ and

$$T_1 = \left(\begin{array}{c|c|c|c|c|c} 0 & W_0 & & & & \\ \hline -W_0' & 0 & W_1 & & & \\ \hline & -W_1' & 0 & W_2 & & \\ \hline & & -W_2' & \ddots & \ddots & \\ \hline & & & \ddots & 0 & W_l \\ \hline & & & & -W_l' & 0 \end{array} \right). \tag{8}$$

The blocks W_i have dimension $d_i \times d_{i+1}$ and:

i) the block W_0 is, in the most general case, in the echelon form

$$W_0 = \left(\begin{array}{c|c|ccc|c} 0 & \oplus L_1 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{2s} & 0 & \Omega_{2s} \\ 0 & 0 & 0 & I_{2t} & 0 \end{array} \right),$$

where $L_1 = (1,0)$ and Ω_{2s} is the real form of the split-quaternion matrix ωI_s, $s \geq 0$, $t \geq 0$, and the first zero columns, as well as the blocks L_1, are not necessarily being,

ii) for any $i > 0$, the block W_i is the real form of a complex matrix $\widehat{W_i}$ that can be reduced to the almost echelon *form $\widehat{W_i} = (0|\widetilde{I}_{r_i})$ (and, in particular, $\widehat{W_l} = \widetilde{I}_{r_l}$), where $\widetilde{I}_{r_i}$ is obtained by the complex identity matrix by possibly adding zero rows,*

and such that no two successive columns of T_1 of indices $2j-1, 2j$ are both zero.

Proof. Recall that we have chosen a basis such that $0 \leq \alpha_i \leq \alpha_{i+1}$ and that, by the equations (4), we get

$$A_{hk} = 0 \iff B_{hk} = 0.$$

If the coefficient α_h is zero, then the equations (5) become

$$\begin{aligned} -A_{hk} &= -\alpha_k^2 A_{hk} \\ -B_{hk} &= -\alpha_k^2 B_{hk}, \end{aligned}$$

and we see that, if $\alpha_k \neq 1$, then $A_{hk} = B_{hk} = 0$. As $\mathfrak{h}$ is T-undecomposable, by Remark 2 we find that the smallest non-zero coefficients must be equal to 1. For the same reason, the next possible coefficients must be equal to 2 and so on, that is, the derivations must be of the form given in (7).

Moreover, equations (6), which with $\beta = 1$ become

$$\begin{aligned} (\alpha_h - \alpha_k)^2 z_1 &= z_1, \\ (\alpha_h + \alpha_k)^2 z_2 &= z_2, \end{aligned}$$

show that $A_{hk} \neq 0$ only if $|\alpha_h - \alpha_k| = 1$, and that A_{hk} is the real form of a non-zero complex number, as soon as $0 < \alpha_h = \alpha_k - 1$. Thus T_1 must be of the form given in (8) and the blocks W_i are, for $i > 0$, the real form of complex matrices $\widehat{W_i}$. By the first of equations (4), which for $\beta = 1$ gives

$$B_{hk} = -\alpha_h \cdot JA_{hk} + \alpha_k \cdot A_{hk} J,$$

the block of T_2, corresponding to $\alpha_h = 0$ and $\alpha_k = 1$, is equal to $W_0(\oplus J)$, and the same holds for the blocks corresponding to $0 < \alpha_h = \alpha_k - 1$, because in these cases A_{hk} commutes with J, hence they are of the form $W_i(\oplus J)$ also for $i > 0$.

Consider a basis change block diagonal matrix of the form

$$X = (X_0 \oplus X_1 \oplus \cdots \oplus X_l) \oplus I_2,$$

where, for $i > 0$, the blocks X_i are $d_i \times d_i$ matrices that are the real form of complex matrices $\widehat{X}_i$, and notice that it leaves the derivations invariant, and changes the block W_i of T_1 into the block $X_i' W_i X_{i+1}$ (and the corresponding block of T_2, accordingly).

In order to reduce the blocks W_i into the form given in the claim, we now perform the following algorithm:

i) Starting from the last block W_l and working upward one by one, by left multiplication $X_i' W_i$ with a suitable matrix X_i, we can assume that the blocks W_i are reduced to *lower* echelon form, that is, such that, for $h < k$, the pivot of the h-th row is on the *right* of the pivot of the k-th row. Moreover we annihilate the entries also below any pivot (as well as above it). For $i = 0$, the matrices X_0 and W_0 are not necessarily the real forms of complex matrices. For $i > 0$, on the contrary, they are.

ii) Let $i > 0$. In order to reduce to zero all the row entries which are on the right of any pivot, we operate on the real form W_i of the complex matrix $\widehat{W_i}$ by adding to a given (complex) column a linear combination of the *previous* (complex) columns of W_i, that is, by the multiplication

$$\left(\begin{array}{c} W_i \\ \hline 0 \\ \hline -W_{i+1}' \end{array} \right) X_{i+1}$$

with a suitable matrix X_{i+1} which is the real form of an upper triangular complex matrix with any diagonal (complex) entry equal to I and this operation did not change the lower echelon form of W_{i+1}. Notice, in fact, that the transpose of a lower echelon matrix is still a lower echelon matrix. Thus we can assume that the columns of all blocks W_i are either zero or vectors from the canonical basis (taken in the reverse order).

iii) We have to distinguish the case where $i = 0$, because W_0 is not necessarily the real form of a complex matrix, but X_1 is such, hence the real matrix X_1 operates on *pairs* of columns, with indices $2j - 1, 2j$. As W_0 is a lower echelon matrix with zeros above and below any pivot, considering the pivot of a row and its position with respect to the pivot of the previous row, we have the following three cases:

$$\left(\begin{array}{c|cc|c} \cdots & 0 & 1 & \cdots \\ \cdots & 1 & 0 & \cdots \end{array} \right), \left(\begin{array}{c|cc|c} \cdots & 0 & 0 & \cdots \\ \cdots & 1 & a & \cdots \end{array} \right), \left(\begin{array}{c|cc|c} \cdots & 0 & 0 & \cdots \\ \cdots & 0 & 1 & \cdots \end{array} \right).$$

As any row is virtually the second row of a two-rows real form of a single complex row, in the last two cases, the second row can be reduced to a vector e_{2i+1} of the real canonical basis, by multiplying on the right with the real form X_1 of an upper triangular complex matrix $\widehat{X}_1$ which has the identity in any (complex) diagonal entry but the one corresponding to the pivot, which has to be

$$\frac{1}{1+a^2}\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{9}$$

respectively. In order to show that we can assume that also the entries below $(1,0)$ are zero, we consider, for instance, the minimal case of the matrix

$$\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & a \\ 1 & 0 & 0 & b \end{array}\right).$$

The following multiplications, first on the left

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -b & 1 & 0 \\ a & 0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & a \\ 1 & 0 & 0 & b \end{array}\right) = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & -b & a \\ 1 & 0 & a & b \end{array}\right)$$

with a real matrix, and second on the right

$$\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & -b & a \\ 1 & 0 & a & b \end{array}\right)\left(\begin{array}{cc|cc} 1 & 0 & -a & -b \\ 0 & 1 & b & -a \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

with the real form of a complex matrix, show therefore that the block $L_1 = (1,0)$ has only zero blocks on left and right, and above and below.

We are left with the first case

$$\left(\begin{array}{c|cc|c} \cdots & 0 & 1 & \cdots \\ \cdots & 1 & 0 & \cdots \end{array}\right),$$

which is the case where the two rows can be seen as a row with entries in the algebra $\mathbb{H}_-$ of split-quaternions, and with pivot ω. By multiplying on the right with the real form of an upper triangular complex matrix which has the identity in any (complex) diagonal entry, we reduce each non zero entry $z_1 + z_2\omega$ to z_1 and, by another multiplication, we can assume that, in the row, only the most leftward entry z_1 is non-zero. Finally, we reduce it to 1 by multiplying on the right with the real form of an upper triangular complex matrix which has z_1^{-1} in the suitable (complex) diagonal entry.

Thus, also in the case $i = 0$, we can assume that the columns of the block W_0 are either zero or vectors from the canonical basis. Moreover this operation did not change the lower echelon form of W_1, however some row has been multiplied by a complex scalar at point (9) and in the above reduction of z_1 to 1. These rows can be reduced again to vectors of the canonical basis, by multiplying on the right with a suitable complex diagonal matrix, and so on with the successive blocks W_i.

iv) Starting again from the last block and working upward, we change now the lower echelon blocks into upper echelon blocks by multiplying on the left with the suitable permutation matrix. We still indicate the blocks by W_k.

v) Starting now from the first block W_0, by multiplying on the left by a real permutation matrix Q'_0 and on the right by a complex permutation matrix Q_1, in the most general case we reduce it to the form

$$Q'_0 W_0 Q_1 = \left(\begin{array}{c|c|cc|c} 0 & \oplus L_1 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{2s} & 0 & \Omega_{2s} \\ 0 & 0 & 0 & I_{2t} & 0 \end{array}\right),$$

where Ω_{2s} is the real form of the split-quaternion matrix ωI_s, $s \geq 0$, $t \geq 0$, and the first zero columns, as well as the blocks L_1, are not necessarily being (notice that W_0 cannot have zero rows). The second block is now $Q'_1 W_1$ and its (complex) columns are vectors from the canonical basis, in the order permuted by the multiplication by Q'_1. The zero rows are not necessarily at the bottom now, and we cannot move them to the bottom without permuting the columns of W_0. On the contrary, by multiplying on the right by a complex permutation matrix Q_2, we can permute the columns and reduce W_1 to the *almost echelon* form $\widehat{W_1} = (0|\widetilde{I}_{r_1})$, where $\widetilde{I}_{r_1}$ is obtained by the complex identity matrix by adding zero rows. Going down, we cannot move the rows of $Q'_2 W_2$ without permuting the columns of W_1, but we can permute the columns and reduce $\widehat{W_2}$ to the form $\widehat{W_2} = (0|\widetilde{I}_{r_2})$, where $\widetilde{I}_{r_2}$ is obtained by the complex identity matrix by adding zero rows.

The claim follows after repeating the same argument till the last block $\widehat{W_l}$, which, in particular, will be reduced to the form $\widetilde{I}_{r_l}$, because it cannot have zero (complex) columns. □

Theorem 8. *With the notations given in (1), if $\beta = 1$ and if the smallest coefficient α_h is greater than $\frac{1}{2}$, then, with respect to a suitable basis of $\mathfrak{h}$, the group T operates as the group of automorphisms* $\exp\big(\partial(t)\big)$*, where*

$$\partial(t) = t \cdot \Big((\oplus\alpha J) \oplus (\oplus(\alpha+1)J) \oplus \cdots \oplus (\oplus(\alpha+l)J)\Big) \oplus t \cdot J,$$

with the diagonal blocks $(\oplus(\alpha+i-1)J)$ of dimension $d_i \times d_i$ (with d_i even), then the T-undecomposable Lie algebra $\mathfrak{h}$ is described by the pair (T_1, T_2), where $T_2 = T_1(\oplus J)$ and

$$T_1 = \left(\begin{array}{c|c|c|c|c|c} 0 & W_1 & & & & \\ \hline -W_1' & 0 & W_2 & & & \\ \hline & -W_2' & 0 & W_3 & & \\ \hline & & -W_3' & \ddots & \ddots & \\ \hline & & & \ddots & 0 & W_l \\ \hline & & & & -W_l' & 0 \end{array}\right).$$

The block W_i has dimension $d_i \times d_{i+1}$ and is the real form of a complex matrix $\widehat{W}_i$ that can be reduced to the almost echelon *form $\widehat{W}_i = (0|\widetilde{I}_{r_i})$ (and, in particular, $\widehat{W}_1 = (0|I_{r_1})$ and $\widehat{W}_l = \widetilde{I}_{r_l}$), where $\widetilde{I}_{r_i}$ is obtained by the complex identity matrix by adding zero rows and such that no two successive columns of T_1 of indices $2j-1, 2j$ are both zero.*

Proof. Let the smallest coefficient α_h be greater than $\frac{1}{2}$ and let $\alpha_k \neq \alpha_h$. By equations (6), if $1-\alpha_h \neq \alpha_k \neq 1+\alpha_h$, then $A_{hk} = B_{hk} = 0$. Since $\mathfrak{h}$ is T-undecomposable, we have that the closest coefficients are $\alpha_k = 1-\alpha_h$ or $\alpha_k = 1+\alpha_h$. But, since $\alpha_h > \frac{1}{2}$, we have that $1-\alpha_h < \alpha_h$, thus $\alpha_k = 1-\alpha_h$ would be a contradiction to the minimality of α_h. Therefore, in this case, the coefficients are α_h, $1+\alpha_h$, $2+\alpha_h$, and so on. With the same arguments as in Theorem 7, the claim follows. In this case, also the first block W_1 is the real form of a complex matrix $\widehat{W}_1$. □

The following case is, somehow, exceptional. In fact, in addition to the various choices of $\widetilde{I}_r$, here it is possible that the real form of an arbitrary skew-Hermitian matrix gives a class of T-undecomposable Lie algebra $\mathfrak{h}$.

Theorem 9. *With the notations given in (1), if $\beta = 1$ and if the smallest coefficient α_h is equal to $\frac{1}{2}$, then, with respect to a suitable basis of $\mathfrak{h}$, the group T operates as the group of automorphisms* $\exp\big(\partial(t)\big)$*, where*

$$\partial(t) = t \cdot \Big((\oplus\frac{1}{2}J) \oplus (\oplus\frac{3}{2}J) \oplus \cdots \oplus (\oplus\frac{2l+1}{2}J)\Big) \oplus t \cdot J.$$

If we denote by $d_i \times d_i$ (d_i even) the dimension of the block $(\oplus\frac{2i-1}{2}J)$, then the T-undecomposable Lie algebra $\mathfrak{h}$ is described by the pair (T_1, T_2), with $T_2 = T_1(\oplus J)$ and

$$T_1 = \left(\begin{array}{c|c|c|c|c|c} \Omega_{d_1}H & W_1 & & & & \\ \hline -W_1' & 0 & W_2 & & & \\ \hline & -W_2' & 0 & W_3 & & \\ \hline & & -W_3' & \ddots & \ddots & \\ \hline & & & \ddots & 0 & W_l \\ \hline & & & & -W_l' & 0 \end{array}\right) \qquad (10)$$

where Ω_{d_1} is the real form of the split-quaternion matrix $\omega I_{\frac{d_1}{2}}$ and:

i) H and W_1 are the real form of the complex matrices

$$\widehat{H} = \left(\begin{array}{c|cc} \widehat{H}_0 & 0 & 0 \\ \hline 0 & 0 & I_r \\ 0 & -I_r & 0 \end{array}\right), \quad \widehat{W}_1 = \left(\begin{array}{c|cc} 0 & 0 & I_s \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

respectively, with $s \geq 0$, $r \geq 0$, $d_1 = 2(s+2r)$, and $\widehat{H}_0$ is a $s \times s$ skew-symmetric complex matrix,

ii) for any $i > 1$, the block W_i of dimension $d_i \times d_{i+1}$ is the real form of a complex matrix $\widehat{W}_i$ that can be reduced to the almost echelon *form $\widehat{W}_i = (0|\widetilde{I}_{r_i})$ (and, in particular, $\widehat{W}_l = \widetilde{I}_{r_l}$), where $\widetilde{I}_{r_k}$ is obtained by the complex identity matrix by adding zero rows,*

and such that no two successive columns of T_1 of indices $2j-1, 2j$ are both zero.

Proof. If $\alpha_h = \frac{1}{2}$, then the assumption $\alpha_k = 1 - \alpha_h$ leads to a contradiction to $\alpha_k \neq \alpha_h$. Thus, also in this case, the distinct coefficients are α_h, $1 + \alpha_h$, $2 + \alpha_h$, and so on. By equations (6), we see, however, that, if $\alpha_k = \alpha_h$, then $A_{hk} = z_2\omega$, hence the $d_1 \times d_1$ block corresponding to the values of α_k equal to $\frac{1}{2}$ is the real form of a split-quaternion matrix $\omega\widehat{H}$. Notice that, whereas the real form of $\widehat{H}$ is skew-symmetric if and only if $\widehat{H}$ is skew-Hermitian, the real form of $\omega\widehat{H}$ is skew-symmetric if and only if $\widehat{H}$ is skew-symmetric. This forces T_1 and T_2 to be of the form given in (10).

With the same argument of Theorem 7, we reduce W_i to the almost echelon form $(0|\widehat{I}_{r_i})$, and in particular $W_l = \widehat{I}_{r_l}$ and, since we can operate on the first row by multiplication on the left with a further matrix, we reduce W_1 to the real form of

$$\widehat{W}_1 = \left(\begin{array}{c|c} 0 & I_s \\ \hline 0 & 0 \end{array}\right).$$

A basis change matrix of the form $X_1 \oplus I_{d_2} \oplus \cdots \oplus I_{d_l}$ transforms the blocks W_1 and $\Omega_{d_1}H$ into $X_1'W_1$ and $X_1'\Omega_{d_1}HX_1$, respectively.
In order to leave invariant the echelon form of W_1, we have to take $\widehat{X}_1 = \left(\begin{array}{c|c} I_s & 0 \\ \hline C & D \end{array}\right)$.

Notice now that $X_1'\Omega_{d_1}HX_1$ is the real form of

$$\widehat{X}_1^\dagger \omega I_{\frac{d_1}{2}} \widehat{H}\widehat{X}_1 = \omega I_{\frac{d_1}{2}} \widehat{X}_1'\widehat{H}\widehat{X}_1.$$

If we write $\widehat{H} = \left(\begin{array}{c|c} H_0 & H_1 \\ \hline -H_1' & H_2 \end{array}\right)$, with H_0 of dimension $s \times s$, we see that the congruence $\widehat{X}_1'\widehat{H}\widehat{X}_1$ changes H_2 into $D'H_2D$ and H_1 into $(H_1 + C'H_2)D$.

Thus we can reduce, firstly, H_2 to the canonical form $\left(\begin{array}{c|c} 0 & I_r \\ \hline -I_r & 0 \end{array}\right)$, because H_2 has to be non-degenerate, or T_1 would have a zero (complex) row. Finally, since H_2 is non-degenerate, we reduce H_1 to zero, taking $C' = -H_1 H_2^{-1}$. $\square$

Remark 10. In the case where $\partial(t) = (\oplus\frac{1}{2}t \cdot J) \oplus t \cdot J$, we find that $(T_1, T_2) = (\oplus I_2, \oplus J)^{\nabla}$.

In the following last theorem we will change the ordering of the coefficients α_h defining the derivations $\partial(t)$.

Theorem 11. *With the notations given in (1), if $\beta = 1$ and if the smallest coefficient α_h is smaller than $\frac{1}{2}$, then, with respect to a suitable basis of $\mathfrak{h}$, the group T operates, in the most general case, as the group of automorphisms $\exp\big(\partial(t)\big)$, where $\partial(t) = \partial_1(t) \oplus \partial_2(t) \oplus t \cdot J$ with*

$$\partial_1(t) = t \cdot \Big((\oplus(l_1 - \alpha)J) \oplus \cdots \oplus (\oplus(2-\alpha)J) \oplus (\oplus(1-\alpha)J) \Big)$$

$$\partial_2(t) = t \cdot \Big((\oplus \alpha J) \oplus (\oplus(\alpha+1)J) \oplus \cdots \oplus (\oplus(\alpha + l_2)J) \Big)$$

and the T-undecomposable Lie algebra $\mathfrak{h}$ is described by the pair (T_1, T_2), with $T_2 = T_1(\oplus J)$ and

$$T_1 = \left(\begin{array}{c|c|c|c|c|c|c|c}
0 & V_{l_1-1} & & & & & & \\ \hline
-V'_{l_1-1} & \ddots & \ddots & & & & & \\ \hline
 & \ddots & 0 & V_1 & & & & \\ \hline
 & & -V'_1 & 0 & \Omega_{d_1} S & & & \\ \hline
 & & & -S'\Omega_{d_1} & 0 & W_1 & & \\ \hline
 & & & & -W'_1 & 0 & \ddots & \\ \hline
 & & & & & \ddots & \ddots & W_{l_2} \\ \hline
 & & & & & & -W'_{l_2} & 0
\end{array}\right) \tag{11}$$

where the blocks V_i, W_i and S are the real form of complex matrices $\widehat{V_i}$, $\widehat{W_i}$ and $\widehat{S}$ that can be reduced to the almost echelon *form $\widehat{V_i} = (0|\widetilde{I}_{r_i})$, $\widehat{W_i} = (0|\widetilde{I}_{s_i})$ and $\widehat{S} = (0|\widetilde{I}_s)$ (and, in particular, $\widehat{V}_{l_1-1} = (0|I_{r_{l_1-1}})$ and $\widehat{W}_{l_2} = \widetilde{I}_{r_{l_2}}$), where $\widetilde{I}_{r_k}$ is obtained by the complex identity matrix I_l by adding zero rows, Ω_{d_1} is the real form of the split-quaternion matrix $\omega I_{\frac{d_1}{2}}$, and such that no two successive columns of T_1 of indices $2j-1, 2j$ are both zero.*

Proof. At last, let the smallest coefficient α_h be smaller than $\frac{1}{2}$, thus the closest possible coefficients are $\alpha_k = 1 - \alpha_h$ and $\alpha_k = 1 + \alpha_h$. The latter gives, as above, the coefficients α_h, $1 + \alpha_h$, $2 + \alpha_h$, and so on. The former gives moreover the coefficients

$1-\alpha_h$, $2-\alpha_h$, and so on, and no other, since $(1-\alpha_h)-1$ is negative and $1-(1-\alpha_h)$ is again α_h. Thus, in this case, the coefficients are

$$l_1-\alpha,\ldots,2-\alpha,1-\alpha,\alpha,1+\alpha,2+\alpha,\ldots,l_2+\alpha.$$

Notice that no coefficient of the form $a-\alpha_h$ can be equal to $b+\alpha_h$, because $\alpha_h<\frac{1}{2}$. It follows that:

i) if $\alpha_h=a+\alpha$ and $\alpha_k=b+\alpha$, then $\alpha_h+\alpha_k$ is not an integer and, by equations (6), we see that A_{hk} is non-zero only if $|b-a|=1$ and that A_{hk} is the real form of a complex number,

ii) if $\alpha_h=a+\alpha$ and $\alpha_k=b-\alpha$, then $|\alpha_h-\alpha_k|$ is not an integer and, by equations (6), we see that A_{hk} is non-zero only if $a+b=1$, that is $\alpha_h=\alpha$ and $\alpha_k=1-\alpha$, and that A_{hk} is the real form of a split-quaternion ωz_2,

iii) if $\alpha_h=a-\alpha$ and $\alpha_k=b-\alpha$, then $\alpha_h+\alpha_k$ is not an integer and, by equations (6), we see that again A_{hk} is non-zero only if $|b-a|=1$ and that A_{hk} is the real form of a complex number.

Thus, T_1 is of the form given in (11). With the same arguments as in Theorem 7, we can reduce the blocks V_i, W_i and S to the form given in the claim. □

Remark 12. In Theorem 11 it can happen that $\partial(t)=\Big(\partial_1(t)\oplus(\oplus\alpha t\cdot J)\Big)\oplus t\cdot J$ and

$$T_1=\left(\begin{array}{c|c|c|c|c} 0 & V_{l_1-1} & & & \\ \hline -V'_{l_1-1} & \ddots & \ddots & & \\ \hline & \ddots & 0 & V_1 & \\ \hline & & -V'_1 & 0 & \Omega_{d_1}S \\ \hline & & & -S'\Omega_{d_1} & 0 \end{array}\right)$$

or that $\partial(t)=\partial_2(t)\oplus t\cdot J$ and

$$T_1=\left(\begin{array}{c|c|c|c} 0 & W_1 & & \\ \hline -W'_1 & 0 & \ddots & \\ \hline & \ddots & \ddots & W_{l_2} \\ \hline & & -W'_{l_2} & 0 \end{array}\right).$$

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