CANONICAL GRAPH CONTRACTIONS OF LINEAR RELATIONS ON HILBERT SPACES

ZSIGMOND TARCSAY AND ZOLTÁN SEBESTYÉN

Dedicated to Henk de Snoo on the occasion of his 75th birthday

ABSTRACT. Given a closed linear relation T between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the corresponding first and second coordinate projections P_T and Q_T are both linear contractions from T to $\mathcal{H}$, and to $\mathcal{K}$, respectively. In this paper we investigate the features of these graph contractions. We show among others that $P_T P_T^* = (I + T^*T)^{-1}$, and that $Q_T Q_T^* = I - (I + TT^*)^{-1}$. The ranges $\operatorname{ran} P_T^*$ and $\operatorname{ran} Q_T^*$ are proved to be closely related to the so called 'regular part' of T. The connection of the graph projections to Stone's decomposition of a closed linear relation is also discussed.

1. INTRODUCTION

When dealing with (unbounded) operators, it is sometimes beneficial to identify them with their graph, that is, to treat them as linear subspaces of the corresponding product space. This approach is especially useful if the operator in question is non-closable, that is, when the closure of its graph is not the graph of a 'single-valued' operator anymore. Similarly, the adjoint of a linear transformation can be interpreted as an operator only if it is densely defined.

The theory of linear relations (or 'multi-valued' linear operators in other words) between Hilbert spaces goes back at least to the fundamental paper by R. Arens [1]. By definition, a linear relation T between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is just a vector subspace of the product Hilbert space $\mathcal{H} \times \mathcal{K}$. In this way, the only (but significant) difference between operators and relations is that $\{0, k\} \in T$ does not necessarily imply $k = 0$. However, this generality greatly simplifies the handling of operations such as taking closure, adjoint, or inverse.

A linear relation T consists of certain ordered pairs $\{x, y\}$ of $\mathcal{H} \times \mathcal{K}$, so one may consider the first and second coordinate projections of T into $\mathcal{H}$ and $\mathcal{K}$, respectively:

$$P_T\{x, y\} := x, \qquad Q_T\{x, y\} := y, \qquad \{x, y\} \in \mathcal{H}.$$

Note that both P_T and Q_T are continuous (with norm bound 1) if we endow T with the inner product coming form that of $\mathcal{H} \times \mathcal{K}$. We shall therefore call P_T and Q_T the *canonical contractions* of T. Assume in addition that T is a closed relation,

then the domain of P_T and Q_T becomes a Hilbert space and thus we may take the adjoint operators $P_T^* : \mathcal{H} \to T$ and $Q_T^* : \mathcal{K} \to T$, and also the product operators $P_T P_T^*, P_T Q_T^*, Q_T P_T^*$ and $Q_T Q_T^*$ are well defined contractions.
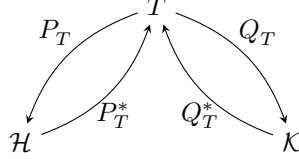

FIGURE 1. The canonical graph contractions and their adjoint.

The present paper is devoted to the study of these canonical contractions and their connection with the closed linear relation T. First we are going to show that the range $\operatorname{ran} P_T^* \subseteq T$ is always a regular relation (that is, the graph of a closable operator). Namely, its closure is identical with the regular part of T. (Recall that the regular part T_{s} of T is defined as $T_{\mathrm{s}} := (I - P_m)T$ where P_m is the orthogonal projection of $\mathcal{K}$ onto $\operatorname{mul} T$, see [6].) It will also turn out that $\operatorname{ran} P_T^*$ is the graph of the restriction of T_{s} to $\operatorname{dom} T^* T$.

In [15] the authors established necessary and sufficient conditions for a pair S, T of operators in order that they satisfy

(1.1) $S^* = T$ and $T^* = S$,

cf. also [10, 13, 14, 16] and [11] for the relation case. Below we provide some further characterizations for (1.1) by means of the corresponding graph contractions P_T, Q_T and P_S, Q_S. As an application we offer a new proof of the self-adjointness of the relations $T^* T^{**}$ and $T^{**} T^*$ by proving that $(I + T^* T^{**})^{-1} = P_T P_T^*$ and $Q_T Q_T^* = I - (I + T^{**} T^*)^{-1}$. Finally, we show how Stone's decomposition [17] of a closed linear relation T can be obtain by applying the results of the paper.

2. LINEAR RELATIONS

Throughout the paper, $\mathcal{H}$ and $\mathcal{K}$ will denote real or complex Hilbert spaces. A linear relation T between $\mathcal{H}$ and $\mathcal{K}$ is nothing but a linear subspace of the product Hilbert space $\mathcal{H} \times \mathcal{K}$. We shall call the relation T closed if it is a closed subspace of $\mathcal{H} \times \mathcal{K}$. Accordingly, the closure $\overline{T}$ of T is always a closed linear relation, and being so, it becomes a Hilbert space with respect to the induced inner product

$$(\{x, y\} \mid \{u, v\})_T := (x \mid u)_{\mathcal{H}} + (y \mid v)_{\mathcal{K}}, \qquad \{x, y\}, \{u, v\} \in \overline{T}.$$

If we refer to $\overline{T}$ as the above Hilbert space, we shall denote it by $\mathcal{G}(T)$.

Recall that every linear operator $T : \mathcal{H} \to \mathcal{K}$ when identified with its graph is a linear relation:

$$T \equiv \{\{x, Tx\} \, : \, x \in \operatorname{dom} T\}.$$

Nevertheless, the closure (of the graph) of a linear operator is no longer an operator in general, namely, it may be that $\{0, k\} \in \overline{T}$ for some non-zero k. Accordingly, we call T closable if its closure $\overline{T}$ is itself an operator.

The domain, range, kernel and multivalued part of a linear relation T are defined to be the following linear subspaces, respectively:

$$\operatorname{dom} T := \{x \in \mathcal{H} \, : \, \{x, y\} \in T\}, \qquad \operatorname{ran} T := \{y \in \mathcal{K} \, : \, \{x, y\} \in T\},$$

CANONICAL GRAPH CONTRACTIONS   3

$$\ker T := \big\{x \in \mathcal{H} \ : \ \{x, 0\} \in T\big\}, \qquad \operatorname{mul} T := \big\{y \in \mathcal{K} \ : \ \{0, y\} \in T\big\}.$$

It is immediate that $\ker T$ and $\operatorname{mul} T$ are both closed subspaces whenever T itself is closed. It goes also without saying that T is (the graph of) and operator if and only if $\operatorname{mul} T = \{0\}$, and that T is (the graph of) a closable operator if and only of $\operatorname{mul} \overline{T} = \{0\}$.

The inverse of a linear relation T is defined as

$$T^{-1} := \big\{\{y, x\} \ : \ \{x, y\} \in T\big\}.$$

If S and T are both linear relations then their product TS is given by

$$TS := \big\{\{x, z\} \ : \ \{x, y\} \in S \quad \text{and} \quad \{y, z\} \in T \quad \text{for some } y\big\}.$$

The operatorlike sum of S and T is

$$S + T := \big\{\{x, y + z\} \ : \ \{x, y\} \in S, \{x, z\} \in T\big\},$$

just like in the case of operators, while the componentwise (or Minkowski) sum is

$$S \mathbin{\widehat{+}} T := \big\{\{x + v, y + z\} + \ : \ \{x, y\} \in S, \{v, z\} \in T\big\}.$$

The adjoint of a linear relation T is defined by

$$T^* := W(T)^{\perp},$$

where $W : \mathcal{H} \times \mathcal{K} \to \mathcal{K} \times \mathcal{H}$ is the 'flip' operator

(2.1) $$W\{h, k\} := \{k, -h\}, \qquad \{h, k\} \in \mathcal{H} \times \mathcal{K}.$$

It is immediate that T^* is a closed linear relation between $\mathcal{K}$ and $\mathcal{H}$ and that $T^{**} = \overline{T}$. Note that the following orthogonal decomposition of $\mathcal{K} \times \mathcal{H}$ holds also true:

(2.2) $$T^* \mathbin{\widehat{\oplus}} W(T^{**}) = \mathcal{K} \times \mathcal{H}.$$

Another equivalent definition of T^* might be given in terms of the inner product, namely,

$$\{k, h\} \in T^* \quad \Longleftrightarrow \quad (y \,|\, k)_{\mathcal{K}} = (x \,|\, h)_{\mathcal{H}}, \qquad \forall \{x, y\} \in T.$$

Recall also the following identities:

$$\ker T^* = (\operatorname{ran} T)^{\perp}, \qquad \operatorname{mul} T^* = (\operatorname{dom} T)^{\perp}.$$

For a given linear relation T, let us denote by P_m the orthogonal projection of $\mathcal{K}$ onto $\operatorname{mul} \overline{T}$. The *regular part* of T is defined as the linear relation

(2.3) $$T_{\mathrm{s}} := \big\{\{x, (I - P_m)y\} \ : \ \{x, y\} \in T\big\}.$$

It can be shown that T_{s} is (the graph of) a closable operator. In contrast, the *singular part* following linear relation

(2.4) $$T_{\mathrm{sing}} := \big\{\{x, P_m y\} \ : \ \{x, y\} \in T\big\}$$

is a so called singular relation which means that $\overline{T_{\mathrm{sing}}}$ is the product of two closed subspaces. By means of the regular and singular parts, the linear relation T allows the following canonical sum decomposition

$$T = T_{\mathrm{s}} + T_{\mathrm{sing}},$$

see [6, Theorem 4.1]. Note also immediately that the regular and singular parts may be written as

$$T_{\mathrm{s}} = (I - P_m)T, \qquad T_{\mathrm{sing}} = P_m T.$$

We shall also use the fact that "regular part" and "closure" operations commute in the sense that

$$(2.5) \qquad (T_{\mathrm{s}})^{**} = (T^{**})_{\mathrm{s}},$$

see [6, Proposition 4.5]. An important consequence of this result is that the regular part of a closed linear relation is closed itself, and also that $T_{\mathrm{s}} \subseteq T$, provided that T is closed.

The interested reader is referred to the books [2, 12] and papers [1, 6, 7] where, in addition to the proofs of the above statements, more information about linear relations can be found.

3. CANONICAL GRAPH CONTRACTIONS OF A LINEAR RELATION

Let T be a linear relation between the real or complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. The canonical graph contractions $P_T : \overline{T} \to \mathcal{H}$ and $Q_T : \overline{T} \to \mathcal{K}$ of T are defined as the mappings

$$P_T\{x, y\} := x, \qquad Q_T\{x, y\} := y, \qquad \{x, y\} \in \overline{T}.$$

Note that both of those mappings are linear contractions if we consider them as operators from the Hilbert space $\mathcal{G}(T)$ into $\mathcal{H}$ and $\mathcal{K}$, respectively:

$$P_T \in \mathscr{B}(\mathcal{G}(T); \mathcal{H}), \|P_T\| \leq 1 \qquad \text{and} \qquad Q_T \in \mathscr{B}(\mathcal{G}(T); \mathcal{K}), \|Q_T\| \leq 1.$$

Therefore, their adjoint operators $P_T^* \in \mathscr{B}(\mathcal{H}; \mathcal{G}(T))$ and $Q_T^* \in \mathscr{B}(\mathcal{K}; \mathcal{G}(T))$ are themselves linear contractions, and their ranges $\operatorname{ran} P_T^*$ and $\operatorname{ran} Q_T^*$ are linear relations.

Below we are going to examine the properties of the contractions P_T and Q_T and their connection with T in detail. First let us establish a few elementary facts.

Proposition 3.1. *Let T be a linear relation between $\mathcal{H}$ and $\mathcal{K}$. Then*
(a) $P_T = P_{\overline{T}}$ *and* $Q_T = Q_{\overline{T}}$,
(b) T *is (the graph of) a closable operator if and only if* $\ker P_T$ *is trivial.*

Proof. The proof of (a) is straightforward from the definition of P_T. Statement (b) is obtained by noticing that $\ker P_T = \{0\} \times \operatorname{mul} \overline{T}$. $\square$

In view of part (a) of the preceding proposition, there is no loss of generality in assuming that the linear relation T is closed. In light of this, with a few exceptions, we will do so.

We start out by analysing the first coordinate projection P_T.

Lemma 3.2. *Let T be a linear relation between $\mathcal{H}$ and $\mathcal{K}$, then for every $h \in \mathcal{H}$ we have $Q_T P_T^* h \in \operatorname{dom} T^*$, that is,*

$$(3.1) \qquad \operatorname{ran} Q_T P_T^* \subseteq \operatorname{dom} T^*.$$

Proof. Consider $h \in \mathcal{H}$ and let $P_T^* h := \{z, w\} \in \operatorname{ran} P_T^*$, then we have

$$(x \mid z)_{\mathcal{H}} + (y \mid w)_{\mathcal{K}} = \left(\{x, y\} \mid P_T^* h\right)_T = (x \mid h)_{\mathcal{H}},$$

for every $\{x, y\} \in T$. Hence we get

$$(y \mid w)_{\mathcal{K}} = (x \mid h - z)_{\mathcal{H}},$$

which implies that $\{w, h - z\} \in T^*$ and therefore $w = Q_T P_T^* h \in \operatorname{dom} T^*$. $\square$

Proposition 3.3. *Let T be a linear relation between $\mathcal{H}$ and $\mathcal{K}$, then*

$$\operatorname{dom} T^*T = P_T(T \cap \operatorname{ran} P_T^*).$$

Proof. Assume first that $\{x, y\} \in T \cap \operatorname{ran} P^*$, then $k \in \operatorname{ran} Q_T P_T^* \subseteq \operatorname{dom} T^*$ by Lemma 3.2, and therefore there exists $z \in \mathcal{H}$ such that $\{y, z\} \in T^*$. This means that $\{x, z\} \in T^*T$ and therefore $x \in \operatorname{dom} T^*T$.

Suppose on the converse that $x \in \operatorname{dom} T^*T$, then $\{x, y\} \in T$ and $\{y, z\} \in T^*$ for some y and z. It suffices to show that $\{x, y\} \in \operatorname{ran} P^*$. Let therefore $\{u, v\} \in T$, then we have

$$\bigl(\{x, y\} \,\big|\, \{u, v\}\bigr)_T = (x \,|\, u)_{\mathcal{H}} + (y \,|\, v)_{\mathcal{K}} = (x \,|\, u)_{\mathcal{H}} + (z \,|\, u)_{\mathcal{H}}$$
$$= \bigl(x + z \,\big|\, P_T\{u, v\}\bigr)_{\mathcal{H}} = \bigl(P_T^*(x + z) \,\big|\, \{u, v\}\bigr)_T,$$

whence it follows that $\{x, y\} = P_T^*(x + z) \in \operatorname{ran} P_T^*$. $\qquad\square$

From the above lemma we get the following two straightforward corollaries:

Corollary 3.4. *If T is (the graph of) an operator, then*

$$T|_{\operatorname{dom} T^*T} = T \cap \operatorname{ran} P_T^*.$$

If T is closed in addition, then

$$T|_{\operatorname{dom} T^*T} = \operatorname{ran} P_T^*.$$

Corollary 3.5. *If T is a closed linear relation then*

$$\operatorname{dom} T^*T = \operatorname{dom}(\operatorname{ran} P_T^*).$$

Next we deal with the linear relation $\operatorname{ran} P_T^* \subseteq \overline{T}$. As it will turn out from the ensuing result, it is closely related to the regular part of T:

Theorem 3.6. *Let T be a linear relation between $\mathcal{H}$ and $\mathcal{K}$, then*

$$\overline{\operatorname{ran} P_T^*} = \overline{T}_{\mathrm{s}}.$$

In particular, $\operatorname{ran} P_T^*$ is always (the graph of) a closable operator.

Proof. First of all note that if $\{x, (I - P_m)y\} \in \overline{T}$ for some $\{x, y\} \in \mathcal{H} \times \mathcal{K}$, then necessarily $\{x, y\} \in \overline{T}$. Indeed,

$$\{x, y\} = \{x, (I - P_m)y\} + \{0, P_m y\} \in \overline{T} \,\widehat{+}\, (\{0\} \times \operatorname{mul}(\overline{T})) \subseteq \overline{T} + \overline{T} = \overline{T}.$$

Consequently,

$$\bigl\{\{x, y\} \in \overline{T} \,:\, y \in \operatorname{ran} P_m^\perp\bigr\} = \bigl\{\{x, (I - P_m)y\} \,:\, \{x, y\} \in \mathcal{H} \times \mathcal{K}, \{x, (I - P_m)y\} \in \overline{T}\bigr\}$$
$$= \bigl\{\{x, (I - P_m)y\} \,:\, \{x, y\} \in \overline{T}, \{x, (I - P_m)y\} \in \overline{T}\bigr\}.$$

Since we have $\ker P_T = \{0\} \times \operatorname{mul} \overline{T}$, it follows that

$$\overline{\operatorname{ran} P_T^*} = \bigl(\{0\} \times \operatorname{mul} \overline{T}\bigr)^\perp$$
$$= \bigl\{\{x, y\} \in \overline{T} \,:\, y \in (\operatorname{mul} \overline{T})^\perp\bigr\}$$
$$= \bigl\{\{x, y\} \in \overline{T} \,:\, y \in \operatorname{ran}(I - P_m)\bigr\}$$
$$= \bigl\{\{x, (I - P_m)y\} \,:\, y \in \mathcal{K}, \{x, (I - P_m)y\} \in \overline{T}\bigr\}$$
$$= \bigl\{\{x, (I - P_m)y\} \,:\, \{x, y\} \in \overline{T}\bigr\}$$
$$= (\overline{T})_{\mathrm{s}} = \overline{T}_{\mathrm{s}},$$

as it is claimed. $\qquad\square$

We continue by describing the kernel and range spaces of the contractions P_T and P_T^*:

Theorem 3.7. *For every linear relation T between $\mathcal{H}$ and $\mathcal{K}$ we have*

    (a) $\ker P_T = \{0\} \times \operatorname{mul} \overline{T}$,
    (b) $\operatorname{ran} P_T = \operatorname{dom} \overline{T}$,
    (c) $\ker P_T^* = \operatorname{mul} T^*$,
    (d) $\operatorname{ran} P_T^* = (\overline{T_{\mathrm{s}}})|_{\operatorname{dom} T^* \overline{T}}$.

Proof. Statements (a)-(c) are all straightforward, only point (d) needs some explanation. Note that we have identity $P_T = P_{\overline{T}}$ for every linear relation. On the other hand, $\overline{T_{\mathrm{s}}} = (\overline{T})_{\mathrm{s}}$ according to (2.5). Therefore, without loss of generality we may assume that T is closed, in which case (d) reduces to

$$(3.2) \qquad\qquad \operatorname{ran} P_T^* = (T_{\mathrm{s}})|_{\operatorname{dom} T^*T}.$$

By Theorem 3.6 we have $\operatorname{ran} P_T^* \subseteq T_{\mathrm{s}}$ and by Corollary 3.5, $\operatorname{dom}(\operatorname{ran} P_T^*) = \operatorname{dom} T^*T$ whenever T is closed. Since T_{s} is the graph of an operator, we obtain (3.2). $\qquad\square$

4. LINEAR RELATIONS ADJOINT TO EACH OTHER

Let T and S be linear relations between $\mathcal{H}$ and $\mathcal{K}$, respectively, $\mathcal{K}$ and $\mathcal{H}$. We say that T and S are adjoint to each other (or that T, S form an adjoint pair), if they satisfy

$$(4.1) \qquad\qquad T \subset S^* \qquad \text{and} \qquad S \subset T^*,$$

or equivalently, if

$$(4.2) \qquad\qquad (y \,|\, v)_{\mathcal{K}} = (x \,|\, u)_{\mathcal{H}}, \qquad \forall\{x, y\} \in T, \quad \forall\{v, u\} \in S.$$

An important and natural question is under what conditions are the equations $T = S^*$ and $S = T^*$. Below we provide some necessary and sufficient conditions on the pair S, T by means of the corresponding graph contractions P_T, Q_T and P_S, Q_S in order that they satisfy the weaker identities $T^{**} = S^*$ and $S^{**} = T^*$

Theorem 4.1. *Let S, T be linear relations between $\mathcal{H}$ and $\mathcal{K}$, respectively, $\mathcal{K}$ and $\mathcal{H}$, that are adjoint to each other in the sense of* (4.1)*. Then the following statements are equivalent:*

    (i) $S^* = T^{**}$ *and* $T^* = S^{**}$,
   (ii) (a) $P_T P_T^* + Q_S Q_S^* = I_{\mathcal{H}}$,
       (b) $P_S P_S^* + Q_T Q_T^* = I_{\mathcal{K}}$,
       (c) $Q_T P_T^* = P_S Q_S^*$.

Proof. Before proving the corresponding equivalences, let us introduce the following operator matrix

$$(4.3) \qquad\qquad U_{T,S} := \begin{bmatrix} P_T & -Q_S \\ Q_T & P_S \end{bmatrix} : \mathcal{G}(T) \times \mathcal{G}(S) \to \mathcal{H} \times \mathcal{K},$$

which acts between $\overline{T} \times \overline{S}$ and $\mathcal{H} \times \mathcal{K}$ by the correspondence

$$U_{T,S} \begin{bmatrix} \{x, y\} \\ \{v, u\} \end{bmatrix} := \{x - u, y + v\}, \qquad \{x, y\} \in T, \{v, u\} \in S.$$

As T and S are adjoint to each other, one concludes that $U_{T,S}$ is an isometry: for let $\{x, y\} \in T$ and $\{v, u\} \in S$, then by (4.2)

$$\left\| U_{T,S} \begin{bmatrix} \{x, y\} \\ \{v, u\} \end{bmatrix} \right\|^2 = \|x - u\|^2 + \|y + v\|^2$$

$$= \|x\|^2 + \|y\|^2 + \|v\|^2 + \|u\|^2 + 2\operatorname{Re}[(y \,|\, v)_{\mathcal{K}} - (x \,|\, u)_{\mathcal{H}}]$$

$$= \left\| \begin{bmatrix} \{x, y\} \\ \{v, u\} \end{bmatrix} \right\|^2 .$$

Assume now (i). Since the corresponding canonical contractions of T and $\overline{T}$ (resp., of S and $\overline{S}$) are identical, we may assume without loss of generality that both S and T are closed. Denote by W the 'flip' operator (2.1). If we have $S^* = T$ and $T^* = S$, then the orthogonal decomposition

$$\mathcal{H} \times \mathcal{K} = T \widehat{\oplus} W(S)$$

implies that for every pair $\{h, k\} \in \mathcal{H} \times \mathcal{K}$ there exists $\{x, y\} \in T$ and $\{v, u\} \in S$ such that

$$\{h, k\} = \{x, y\} + \{-u, v\} = U_{T,S} \begin{bmatrix} \{x, y\} \\ \{v, u\} \end{bmatrix},$$

which means that $U_{S,T}$ is surjective, and hence a unitary operator. As a consequence we get $U_{T,S} U_{T,S}^* = I_{\mathcal{H} \times \mathcal{K}}$, that is,

$$\begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & I_{\mathcal{K}} \end{bmatrix} = \begin{bmatrix} P_T P_T^* + Q_S Q_S^* & P_T Q_T^* - Q_S P_S^* \\ Q_T P_T^* - P_S Q_S^* & P_S P_S^* + Q_T Q_T^* \end{bmatrix},$$

that clearly implies (ii).

For the converse, assume (ii) and also that S, T are closed. Since S and T are adjoint to each other, it suffices to show that $T^* \subset S$ and $S^* \subset T$. Consider a pair $\{w, z\} \in T^*$. By (ii) (a)-(c), we infer that $U_{T,S}$ is unitary and therefore we can find $\{x, y\} \in T$ and $\{v, u\} \in S$ such that

$$\{-z, w\} = U_{T,S} \begin{bmatrix} \{x, y\} \\ \{v, u\} \end{bmatrix} = \{x - u, y + v\}.$$

Here,

$$\big(\{x, y\} \,|\, \{-u, v\}\big) = 0 = \big(\{x, y\} \,|\, \{-z, w\}\big),$$

and hence,

$$\|\{x, y\}\|^2 = \big(\{x, y\} \,|\, \{x - u, y + v\}\big) = \big(\{x, y\} \,|\, \{-z, w\}\big).$$

This entails that $\{x, y\} = \{0, 0\}$, and hence that $\{w, z\} = \{v, u\} \in S$. An analogous argument shows that $S^* \subset T$. □

Remark 4.2. Some characterizations of those linear operators S, T which satisfy identities $S^* = T$ and $T^* = S$ where given in [15] by means of the operator matrix

$$M_{T,S} := \begin{bmatrix} I_{\mathcal{H}} & -S \\ T & I_{\mathcal{K}} \end{bmatrix},$$

cf. also [10, 14]. The general case of linear relations was discussed in [11] in the same spirit. For an exact interpretation of matrices with linear relation entries the reader is referred to [8].

5. Products of graph contractions

Let T be linear relation between the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and consider its canonical graph contractions $P_T : \mathcal{G}(T) \to \mathcal{H}$ and $Q_T : \mathcal{G}(T) \to \mathcal{K}$. Then the following four operators $P_T P_T^*, P_T Q_T^*, Q_T P_T^*$ and $Q_T Q_T^*$ are all well defined linear contractions between the appropriate Hilbert spaces. In this section we clarify their role and connection with the relations T and T^*.

Lemma 5.1. *Let T be a closed linear relation between $\mathcal{H}$ and $\mathcal{K}$, then*

(a) $\operatorname{mul}(TT^*) = \operatorname{mul} T$,
(b) $T^*T = (T_{\mathrm{s}})^* T_{\mathrm{s}}$,
(c) $(TT^*)_{\mathrm{s}} = T_{\mathrm{s}}(T^*)_{\mathrm{s}}$.

Proof. (a) Let $k \in \operatorname{mul} T$, then $\{0,0\} \in T^*$ and $\{0,k\} \in T$ implies that $k \in \operatorname{mul} TT^*$. Assume on the converse that $k \in \operatorname{mul} TT^*$, then there exists u such that $\{0,u\} \in T^*$ and $\{u,k\} \in T^*$. Since we have $u \in \operatorname{dom} T^* \cap \operatorname{mul} T = \{0\}$, it follows that $\{0,k\} \in T$ and therefore that $k \in \operatorname{mul} T$.

(b) First we show inclusion $T^*T \subset (T_{\mathrm{s}})^* T_{\mathrm{s}}$. Take $\{x,z\} \in T^*T$, then there exists y such that $\{x,y\} \in T$ and $\{y,z\} \in T^*$. In particular we have $y \in \operatorname{dom} T^* \subseteq \operatorname{mul} T^\perp$, thus $\{x,y\} = \{x, (I - P_m)y\} \in T_{\mathrm{s}}$. On the other hand, we have inclusion $T_{\mathrm{s}} \subset T$ by closedness, so $T^* \subset (T_{\mathrm{s}})^*$. Consequently, $\{x,z\} \in (T_{\mathrm{s}})^* T_{\mathrm{s}}$, indeed. Let now $\{x,z\} \in (T_{\mathrm{s}})^* T_{\mathrm{s}}$, then there exists y such that $\{x,y\} \in T_{\mathrm{s}}$ and $\{y,z\} \in (T_{\mathrm{s}})^*$. Here we have $\{x,y\} \in T$ as $T_{\mathrm{s}} \subseteq T$. Furhtermore, $(T_{\mathrm{s}})^*$ can be written as

$$(T_{\mathrm{s}})^* = T^* \,\widehat{\oplus}\, (\operatorname{mul} T \times \{0\}),$$

where $\widehat{\oplus}$ denotes Minkowski direct sum. This yields us $\{k,h\} \in T^*$ and $w \in \operatorname{mul} T$ such that $\{y,z\} = \{k,h\} + \{w,0\}$. Since $y, k \in \operatorname{mul} T^\perp$, we get $w = 0$ and $y = k$, consequently $\{y,z\} \in T^*$ and $\{x,z\} \in T^*T$.

(c) By (a) we have $\operatorname{mul} T = \operatorname{mul}(TT^*)$, hence

$$(TT^*)_{\mathrm{s}} = (I - P_m)TT^* = T_{\mathrm{s}}T^* \supset T_{\mathrm{s}}(T^*)_{\mathrm{s}},$$

because $(T^*)_{\mathrm{s}} \subset T^*$. To see the converse inclusion take $\{v, (I - P_m)w\} \in (TT^*)_{\mathrm{s}}$, and let $\{v,u\} \in T^*$ and $\{u,w\} \in T$ for some u, then $\{u, (I - P_m)w\} \in T_{\mathrm{s}}$ and from $u \in \operatorname{dom} T$ we get that $u \in (\operatorname{mul} \overline{T})^\perp$, hence $\{v,u\} \in (T^*)_{\mathrm{s}}$. Thus $\{v, (I - P_m)w\} \in T_{\mathrm{s}}(T^*)_{\mathrm{s}}$. $\qquad\square$

In the next theorem we are going to deal with the contractions $P_T P_T^*, Q_T Q_T^*, P_T Q_T^*$ and $Q_T P_T^*$.

Theorem 5.2. *Let T be a closed linear relation between $\mathcal{H}$ and $\mathcal{K}$. Then*

(a) $P_T P_T^* = (I + T^*T)^{-1}$,
(b) $Q_T P_T^* = T_{\mathrm{s}}(I + T^*T)^{-1}$,
(c) $P_T Q_T^* = (T^*)_{\mathrm{s}}(I + TT^*)^{-1}$,
(d) $Q_T Q_T^* = I - (I + TT^*)^{-1} = P_m + (TT^*)_{\mathrm{s}}(I + TT^*)^{-1}$.

Proof. (a) Let us introduce the linear operator

$$P_T^\dagger : \operatorname{dom} T \to \mathcal{H} \times \mathcal{K}, \qquad P_T^\dagger u := \{u, T_{\mathrm{s}}u\}.$$

Observe that $P_T^\dagger u \in T_{\mathrm{s}} \subseteq T$ for every $u \in \operatorname{dom} T$, and that

(5.1) $$P_T^\dagger P_T \{u, T_{\mathrm{s}}u\} = \{u, T_{\mathrm{s}}u\}.$$

Since $\operatorname{ran} P_T^* \subset T_{\mathrm{s}}$ by Theorem 3.6, from (5.1) it follows that $P_T^\dagger P_T P_T^* = P_T^*$.

Let now $x \in \operatorname{dom} T$ and $h \in \mathcal{H}$, then

$$
\begin{aligned}
(x \mid h)_{\mathcal{H}} &= (P_T P_T^{\dagger} x \mid h)_{\mathcal{H}} = (P_T^{\dagger} x \mid P_T^* h)_T = (P_T^{\dagger} x \mid P_T^{\dagger} P_T P_T^* h)_T \\
&= \big(\{x, T_{\mathrm{s}} x\} \mid \{P_T P_T^* h, T_{\mathrm{s}} P_T P_T^* h\} \big)_T \\
&= (x \mid P_T P_T^* h)_{\mathcal{H}} + (T_{\mathrm{s}} x \mid T_{\mathrm{s}} P_T P_T^* h)_{\mathcal{K}},
\end{aligned}
$$

consequently,

$$
(T_{\mathrm{s}} x \mid T_{\mathrm{s}} P_T P_T^* h)_{\mathcal{K}} = (x \mid h - P_T P_T^* h)_{\mathcal{H}}.
$$

This implies that

$$
P_T P_T^* h \in \operatorname{dom}(T_{\mathrm{s}})^* T_{\mathrm{s}} \qquad \text{and} \qquad h = (I + (T_{\mathrm{s}})^* T_{\mathrm{s}}) P_T P_T^* h,
$$

that is, $P_T P_T^* = (I + (T^*)_{\mathrm{s}} T_{\mathrm{s}})^{-1}$. Since we have identity $T^* T = (T^*)_{\mathrm{s}} T_{\mathrm{s}}$ by Lemma 5.1, the proof of part (a) is complete.

(b) Take any vector $h \in \mathcal{H}$. From (a) and equality $P_T^{\dagger} P_T P_T^* = P_T^*$ we conclude that

$$
\begin{aligned}
Q_T P_T^* h = Q_T P_T^{\dagger} P_T P_T^* h = Q_T P_T^{\dagger} (I + T^* T)^{-1} h \\
= Q_T \{ (I + T^* T)^{-1} h, T_{\mathrm{s}} (I + T^* T)^{-1} h \} \\
= T_{\mathrm{s}} (I + T^* T)^{-1} h,
\end{aligned}
$$

whence we get identity (b).

(c) Replacing T by T^* in (b), we obtain that

$$
Q_{T^*} P_{T^*}^* = (T^*)_{\mathrm{s}} (I + T T^*)^{-1}.
$$

On the other hand, it follows from Theorem 3.6 (ii) (c) that $P_T Q_T^* = Q_{T^*} P_{T^*}^*$, hence the desired identity follows.

(d) First we note that

$$
(I - P_m) Q_T = T_{\mathrm{s}} P_T,
$$

because for $\{x, y\} \in T$,

$$
(I - P_m) Q_T \{x, y\} = (I - P_m) y = T_{\mathrm{s}} x = T_{\mathrm{s}} P_T \{x, y\}.
$$

From this and (c) we get that

$$
\begin{aligned}
(I - P_m) Q_T Q_T^* = T_{\mathrm{s}} P_T Q_T^* = T_{\mathrm{s}} (T^*)_{\mathrm{s}} (I + T T^*)^{-1} \\
= (T T^*)_{\mathrm{s}} (I + T T^*)^{-1}.
\end{aligned}
$$

On the other hand, we have

$$
Q_T Q_T^* = I - P_{T^*} P_{T^*}^*,
$$

by Theorem 4.1 (ii) (b). Since $\operatorname{ran} P_{T^*} = \operatorname{dom} T^* \subset (\operatorname{mul} T)^{\perp}$, we get

$$
P_m Q_T Q_T^* = P_m - P_m P_{T^*} P_{T^*}^* = P_m.
$$

From the above identities we get

$$
Q_T Q_T^* = P_m Q_T Q_T^* + (I - P_m) Q_T Q_T^* = P_m + (T T^*)_{\mathrm{s}} (I + T T^*)^{-1},
$$

which completes the proof. $\qquad \square$

Remark 5.3. We notice that $P_T^{\dagger}$ appearing in the proof of the preceding theorem is identical with the Moore-Penrose inverse of P_T, cf. [3] or [4]. We also remark that the proof might be slightly simplified when T is a closed operator. Namely, in that case we have $T = T_{\mathrm{s}}$ and $P_T^{\dagger} = P_T^{-1}$.

Corollary 5.4. *Let T be a densely defined closed linear operator between two Hilbert spaces. Then*

    (a) $P_T P_T^* = (I + T^* T)^{-1}$,
    (b) $Q_T P_T^* = T(I + T^* T)^{-1}$,
    (c) $P_T Q_T^* = T^* (I + T T^*)^{-1}$,
    (d) $Q_T Q_T^* = T T^* (I + T T^*)^{-1}$.

Proof. The proof is straightforward from Theorem 5.2 by noticing that $P_m = 0$ whenever T is a densely defined closed operator.   □

In the next proposition we describe the kernel and range spaces of the contractions Q_T and Q_T^*.

Proposition 5.5. *For every linear relation T between $\mathcal{H}$ and $\mathcal{K}$ we have*

    (a) $\ker Q_T = (\mathcal{H} \times \{0\}) \cap \overline{T}$,
    (b) $\operatorname{ran} Q_T = \operatorname{ran} \overline{T}$,
    (c) $\ker Q_T^* = (\operatorname{ran} T)^{\perp}$,
    (d) $\operatorname{ran} Q_T^* = (\overline{T_{\mathrm{s}}})|_{\operatorname{ran} T^* \cap \operatorname{dom} \overline{T}} \,\widehat{\oplus}\, (\{0\} \times \operatorname{mul} \overline{T})$.

Proof. Throughout the proof we shall assume, for sake of simplicity, that T is closed. Statement (a) follows immediatly from

$$\ker Q_T = \big\{\{x, y\} \in T \,:\, y = 0\big\} = \big\{\{x, 0\} \,:\, \{x, 0\} \in T\big\}.$$

Statements (b) and (c) are straightforward form the very definition of Q_T. Finally, let us prove assertion (d). According to Theorem 5.2 (c) and (d) we have

$$\begin{aligned}
\operatorname{ran} Q_T^* &= \big\{\{P_T Q_T^* k, Q_T Q_T^* k\} \,:\, k \in \mathcal{K}\big\} \\
&= \big\{\{(T^*)_{\mathrm{s}}(I + T T^*)^{-1} k, T_{\mathrm{s}}(T^*)_{\mathrm{s}}(I + T T^*)^{-1} k + P_m k \,:\, k \in \mathcal{K}\big\} \\
&= \big\{\{(T^*)_{\mathrm{s}}(I + T T^*)^{-1} k, (T T^*)_{\mathrm{s}}(I + T T^*)^{-1} k\} \,:\, k \in \mathcal{K}\big\} \widehat{+} (\{0\} \times \operatorname{mul} T),
\end{aligned}$$

where in the last equality we used that

$$\ker(I + T T^*)^{-1} = \operatorname{mul} T T^* = \operatorname{mul} T = \operatorname{ran} P_m.$$

We have on the other hand

$$\begin{aligned}
\big\{\{(T^*)_{\mathrm{s}}(I + T T^*)^{-1} k, (T T^*)_{\mathrm{s}}(I + T T^*)^{-1} k\} \,:\, k \in \mathcal{K}\big\} &= \\
= \big\{\{(T^*)_{\mathrm{s}} z, T_{\mathrm{s}}(T^*)_{\mathrm{s}} z\} \,:\, z \in \operatorname{dom} T_{\mathrm{s}}(T^*)_{\mathrm{s}}\big\} & \\
= T_{\mathrm{s}}|_{\operatorname{ran}(T^*)_{\mathrm{s}} \cap \operatorname{dom} T_{\mathrm{s}}}. &
\end{aligned}$$

Finally we note that $\operatorname{dom} T_{\mathrm{s}} = \operatorname{dom} T$ and that $\operatorname{mul} T^* = \overline{\operatorname{dom} T}$ and therefore

$$\operatorname{ran}(T^*)_{\mathrm{s}} \cap \operatorname{dom} T_{\mathrm{s}} = \operatorname{ran} T^* \cap \operatorname{dom} T.$$

This together with the above observations yields identity (d).   □

Corollary 5.6. *Let T be a closed linear operator between $\mathcal{H}$ and $\mathcal{K}$, then*

$$\operatorname{ran} Q_T^* = T|_{\operatorname{ran} T^* \cap \operatorname{dom} T}.$$

Corollary 5.7. *Let T be a closed linear relation between $\mathcal{H}$ and $\mathcal{K}$, then*

$$T_{\mathrm{s}} = T_{\mathrm{s}}|_{\operatorname{dom} T^* T} \,\widehat{+}\, T_{\mathrm{s}}|_{\operatorname{ran} T^* \cap \operatorname{dom} T}.$$

If T is a closed operator, then

$$T = T|_{\operatorname{dom} T^* T} \,\widehat{+}\, T|_{\operatorname{ran} T^* \cap \operatorname{dom} T}.$$

Proof. Recall that the operator matrix U_{T,T^*} defined by (4.3) with $S := T^*$ is an isometry hence, in particular one has $P_T^* P_T + Q_T^* Q_T = I_{\mathcal{G}(T)}$. As a consequence, we have by [5, Theorem 2.2] that

$$T = \operatorname{ran}(P_T^* P_T + Q_T^* Q_T)^{1/2} = \operatorname{ran} P_T^* \mathbin{\widehat{+}} \operatorname{ran} Q_T^*$$
$$= T_{\mathrm{s}}|_{\operatorname{dom} T^* T} \mathbin{\widehat{+}} T_{\mathrm{s}}|_{\operatorname{ran} T^* \cap \operatorname{dom} T} \mathbin{\widehat{+}} (\{0\} \times \operatorname{mul} T).$$

Now the desired identity follows since $T_{\mathrm{s}} = T \mathbin{\widehat{\ominus}} (\{0\} \times \operatorname{mul} T)$. $\qquad\square$

We conclude the paper with an application of the results. Let T be a closed linear relation between the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and denote by E_T the orthogonal projection of $\mathcal{H} \times \mathcal{K}$ onto T. Then E_T admits a matrix representation as an operator in $\mathcal{H} \times \mathcal{K}$:

$$E_T = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

where the components E_{ij} are bounded operators between the appropriate Hilbert spaces. Recall that E_T is was called the characteristic projection of T by Stone, who proved that the entries E_{ij} may be expressed in terms of T and T^*, provided that T is a densely defined and closed operator (see [17, Theorem 4], cf. also [9, Theorem 3]):

$$E_T = \begin{bmatrix} (T^* T + I)^{-1} & T^*(TT^* + I)^{-1} \\ T(T^* T + I)^{-1} & TT^*(TT^* + I)^{-1} \end{bmatrix}.$$

In [6, Lemma 6.4], the above result of Stone was extended to closed linear relations. In the ensuing theorem we are going we restate this general result as a straightforward consequence of Theorem 5.2:

Theorem 5.8. *Let T be a closed linear relation between the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Then the characteristic projection E_T of T can be written as*

$$E_T = \begin{bmatrix} P_T P_T^* & P_T Q_T^* \\ Q_T P_T^* & Q_T Q_T^* \end{bmatrix} = \begin{bmatrix} (T^* T + I)^{-1} & (T^*)_{\mathrm{s}}(TT^* + I)^{-1} \\ T_{\mathrm{s}}(T^* T + I)^{-1} & I - (TT^* + I)^{-1} \end{bmatrix}$$

Proof. Consider the canonical embedding operator $V_T : \mathcal{G}(T) \to \mathcal{H} \times \mathcal{K}$, given by

$$V_T := \begin{bmatrix} P_T \\ Q_T \end{bmatrix} \{x, y\} := \{x, y\}, \qquad \{x, y\} \in T.$$

Clearly, V_T is a linear isometry with range T and therefore $V_T V_T^*$ is identical with E_T, i.e.,

$$E_T = \begin{bmatrix} P_T \\ Q_T \end{bmatrix} \begin{bmatrix} P_T^* & Q_T^* \end{bmatrix} = \begin{bmatrix} P_T P_T^* & P_T Q_T^* \\ Q_T P_T^* & Q_T Q_T^* \end{bmatrix}.$$

Theorem 5.2 completes now the proof. $\qquad\square$

References

[1] R. Arens, Operational calculus of linear relations, *Pacific J. Math.* **11** (1961), 9–23.

[2] J. Behrndt, S. Hassi, H.S.V. de Snoo, *Boundary value problems, Weyl functions, and differential operators*, Springer Nature, 2020.

[3] A. Ben-Israel and T. N. E. Greville, *Generalized inverses: theory and applications.* Vol. 15. Springer Science & Business Media, 2003.

[4] F. J. Beutler, Operator theory of the pseudo-inverse II. Unbounded operators with arbitrary range, *J. Math. Anal. Appl.*, 10, 471-493 (1965).

[5] P. A. Fillmore and J. P. Williams, On operator ranges, *Advances in Math.*, 7 (1971), 254–81.

[6] S. Hassi, Z. Sebestyén, H. S. V. de Snoo, and F. H. Szafraniec, A canonical decomposition for linear operators and linear relations, *Acta Math. Hungar.*, **115** (2007), 281–307.

[7] S. Hassi, H.S.V. de Snoo, and F. H. Szafraniec, Componentwise and canonical decompositions of linear relations, *Dissertationes Mathematicae*, 465 (2009), 59pp.

[8] S. Hassi, J.-Ph. Labrousse, H.S.V. de Snoo, Operational calculus for rows, columns, and blocks of linear relations, *Advances in Operator Theory*, 2020, online first; DOI: 10.1007/s43036-020-00085-3

[9] P. E. T. Jorgensen, E. Pearse, Feng Tian, Unbounded operators in Hilbert space, duality rules, characteristic projections, and their applications, *Anal.Math.Phys.* 8 (2018) 351382.

[10] D. Popovici, Z. Sebestyén, Operators which are adjoint to each other, *Acta Sci. Math. (Szeged)* 80, 175194 (2014).

[11] A. Sandovici, On the Adjoint of Linear Relations in Hilbert Spaces, *Mediterr. J. Math.* (2020) 17:68; DOI: 10.1007/s00009-020-1503-y

[12] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*, Vol. 265. Springer Science & Business Media, 2012.

[13] Z. Sebestyn and Zs. Tarcsay, Characterizations of selfadjoint operators, *Studia Sci. Math. Hungar.*, 50(4) (2013), 423-435.

[14] Z. Sebestyn and Zs. Tarcsay, Adjoint of sums and products of operators in Hilbert spaces, *Acta Sci. Math. (Szeged)*, 82 (2016), 175-191.

[15] Z. Sebestyn and Zs. Tarcsay, On the adjoint of Hilbert space operators, *Linear and Multilinear Algebra*, 67 (2019), 625-645.

[16] Z. Sebestyn and Zs. Tarcsay, Range-kernel characterizations of operators which are adjoint of each other, *Advances in Operator Theory*, online first, 2020, DOI: 10.1007/s43036-020-00068-4

[17] M. H. Stone, On unbounded operators in Hilbert spaces, *J. Indian Math. Soc.* 15, 155–192 (1952).

ZS. TARCSAY, DEPARTMENT OF APPLIED ANALYSIS AND COMPUTATIONAL MATHEMATICS, EÖTVÖS LORÁND UNIVERSITY, PÁZMÁNY PÉTER SÉTÁNY 1/C., BUDAPEST H-1117, HUNGARY
E-mail address: `tarcsay@cs.elte.hu`

Z. SEBESTYÉN, DEPARTMENT OF APPLIED ANALYSIS AND COMPUTATIONAL MATHEMATICS, EÖTVÖS LORÁND UNIVERSITY, PÁZMÁNY PÉTER SÉTÁNY 1/C., BUDAPEST H-1117, HUNGARY
E-mail address: `sebesty@cs.elte.hu`