FROM THE INGHAM–JESSEN PROPERTY TO MIXED-MEAN INEQUALITIES

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ABSTRACT. For every symmetric mean $\mathcal{M}: \bigcup_{n=1}^{\infty} I^n \to I$ (where I an interval) and a nonzero function $W: \{1, \ldots, n\} \to \mathbb{N} \cup \{0\}$, define an *n*-variable mean by

$$\mathcal{M}_W(x) := \mathcal{M}\Big(\underbrace{x_1, \dots, x_1}_{W(1)\text{-times}}, \dots, \underbrace{x_n, \dots, x_n}_{W(n)\text{-times}}\Big) \text{ for } x = (x_1, \dots, x_n) \in I^n.$$

Given two symmetric means $\mathcal{M}, \mathcal{N}: \bigcup_{n=1}^{\infty} I^n \to I$ satisfying the so-called Ingham–Jessen inequality and some nonzero functions $F_1, \ldots, F_k, G_1, \ldots, G_l: \{1, \ldots, n\} \to \mathbb{N} \cup \{0\}$, we establish sufficient conditions for inequalities of the form

 $\mathcal{N}\big(\mathcal{M}_{F_1}(x),\ldots,\mathcal{M}_{F_k}(x)\big) \le \mathcal{M}\big(\mathcal{N}_{G_1}(x),\ldots,\mathcal{N}_{G_l}(x)\big) \qquad (x \in I^n).$

Our results provide a unified approach to the celebrated inequalities obtained by Kedlaya in 1994 and by Leng–Si–Zhu in 2004 and offer also new families of mixed-mean inequalities.

1. INTRODUCTION

In this paper a function $\mathcal{M} \colon \bigcup_{n=1}^{\infty} I^n \to I$ will be called a *mean* if, for all $n \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in I^n$

$$\min(x_1,\ldots,x_n) \le \mathcal{M}(x_1,\ldots,x_n) \le \max(x_1,\ldots,x_n).$$

One of the most extensively studied branches of the theory of means is the extension and generalization of the important and classical inequalities involving means. Beyond the comparison theory of means, the so-called mixed-mean type inequalities are in the focus of many recent research papers. These type of inequalities involve two or more means which are substituted into each other.

For example Kedlaya [8], giving an affirmative answer to Holland's conjecture [7], proved that, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n > 0$, the following inequality holds:

$$\frac{x_1 + \sqrt{x_1 x_2} + \dots + \sqrt[n]{x_1 x_2 \dots x_n}}{n} \le \sqrt[n]{x_1 \cdot \frac{x_1 + x_2}{2} \dots \frac{x_1 + x_2 + \dots + x_n}{n}},$$

Another result of this type was established by Leng-Si-Zhu [11] in 2004

$$\frac{n - \sqrt[n]{x_2 x_3 \cdots x_n} + \sqrt[n-1]{x_1 x_3 \cdots x_n} + \dots + \sqrt[n-1]{x_1 x_2 \cdots x_{n-1}}}{n}$$

$$\leq \sqrt[n]{\frac{x_2 + x_3 + \dots + x_n}{n-1} \cdot \frac{x_1 + x_3 + \dots + x_n}{n-1} \cdots \frac{x_1 + x_2 + \dots + x_{n-1}}{n-1}}.$$

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More general inequalities than this that involve power means (instead the arithmetic and geometric means) were obtained by Carlson–Meany–Nelson [3] in 1971. For further examples of inequalities of this type we refer the reader to the seminal monograph [2]. The typical tool to prove such inequalities is the Hölder inequality. In our paper we aim to derive the above, as well as some more general inequalities with the help of the so-called Ingham–Jessen inequality. For the setting of the arithmetic and geometric means this states that, for all $n, m \in \mathbb{N}$ and $x_{ij} > 0$ $(i \in \mathbb{N}_n, j \in \mathbb{N}_m)$,

$$\frac{\sqrt[m]{x_{11}x_{12}\cdots x_{1m}} + \sqrt[m]{x_{21}x_{22}\cdots x_{2m}} + \dots + \sqrt[m]{x_{n1}x_{n2}\cdots x_{nm}}}{n}$$

$$\leq \sqrt[m]{\frac{x_{11} + x_{21} + \dots + x_{n1}}{n}} \cdot \frac{x_{12} + x_{22} + \dots + x_{n2}}{n} \cdots \frac{x_{1m} + x_{2m} + \dots + x_{nm}}{n}$$

Here and also in the sequel, we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and, for every $k \in \mathbb{N}$, we denote $\mathbb{N}_k := \{1, \ldots, k\} := [1, k] \cap \mathbb{N}$. More generally, we say that a pair $(\mathcal{M}, \mathcal{N})$ of means on I forms an Ingham-Jessen pair (briefly is an I-J pair) if, for $x_{ij} \in I$ $(i \in \mathbb{N}_n, j \in \mathbb{N}_m)$, the following inequality is satisfied:

(1.1)
$$\mathcal{N}\Big(\mathcal{M}(x_{11}, x_{12}, \dots, x_{1m}), \mathcal{M}(x_{21}, x_{22}, \dots, x_{2m}), \dots, \mathcal{M}(x_{n1}, x_{n2}, \dots, x_{nm})\Big) \\ \leq \mathcal{M}\Big(\mathcal{N}(x_{11}, x_{21}, \dots, x_{n1}), \mathcal{N}(x_{12}, x_{22}, \dots, x_{n2}), \dots, \mathcal{N}(x_{1m}, x_{2m}, \dots, x_{nm})\Big).$$

The Ingham–Jessen pairs were characterized for several families of means. In particular, a pair of power means $(\mathcal{P}_p, \mathcal{P}_q)$ is an Ingham–Jessen pair if and only if $p \leq q$ (see [6]). In a more general setting, a pair of Gini means $(\mathcal{G}_{p,q}, \mathcal{G}_{r,s})$ forms an Ingham–Jessen pair if and only if pqrs = 0 and $\min(p,q) \leq \min(r,s) \leq \max(p,q) \leq \max(r,s)$ (see [13, Theorem 5]). Extending these results, Páles also characterized the Ingham–Jessen property in the class of quasideviation means [14].

In order to formulate our main statements, we need to introduce some further notations. Given a mean \mathcal{M} on $I, n \in \mathbb{N}$, and a non-identically zero function $F \colon \mathbb{N}_n \to \mathbb{N}_0$, we define $\mathcal{M}_F \colon I^n \to I$ by

$$\mathcal{M}_F(x_1,\ldots,x_n) := \mathcal{M}\Big(\underbrace{x_1,\ldots,x_1}_{F(1)\text{-times}},\ldots,\underbrace{x_n,\ldots,x_n}_{F(n)\text{-times}}\Big) \quad \text{for} \quad (x_1,\ldots,x_n) \in I^n$$

For $k, l, n \in \mathbb{N}$ and sequences $(F_i)_{i=1}^k$ and $(G_j)_{j=1}^l$ of non-identically zero functions mapping \mathbb{N}_n into \mathbb{N}_0 , and means \mathcal{M}, \mathcal{N} on I, we will investigate the validity of the inequality

(1.2)
$$\mathcal{N}\big(\mathcal{M}_{F_1}(x),\ldots,\mathcal{M}_{F_k}(x)\big) \leq \mathcal{M}\big(\mathcal{N}_{G_1}(x),\ldots,\mathcal{N}_{G_l}(x)\big) \quad \text{for } x \in I^n.$$

This inequality generalizes the comparison inequality of means \mathcal{M} and \mathcal{N} . Indeed, for k = l = 1 and $F_1 = G_1 \equiv 1$, (1.2) becomes $\mathcal{M} \leq \mathcal{N}$; while for k = l = n and $F_i = G_i = \mathbf{1}_{\{i\}}$ for $i \in \mathbb{N}_n$, (1.2) reduces to $\mathcal{N} \leq \mathcal{M}$. Note that the Kedlaya inequality is a particular case of (1.2), too. In fact, it is enough to take \mathcal{M} and \mathcal{N} to be the geometric and arithmetic mean on $(0, +\infty)$, respectively, k = l = n, and $F_i = G_i = \mathbf{1}_{\{1,\ldots,i\}}$ for $i \in \mathbb{N}_n$.

In fact, the Ingham–Jessen inequality (1.1) is a particular case of (1.2). Indeed, for $k, l \in \mathbb{N}, n = kl$,

(1.3)
$$F_i = \mathbf{1}_{\{(i-1)k+1, (i-1)k+2, \dots, ik\}} \quad \text{for } i \in \mathbb{N}_l$$

and

(1.4)
$$G_j = \mathbf{1}_{\{j,k+j,\dots,(l-1)k+j\}} \quad \text{for } j \in \mathbb{N}_k$$

inequality (1.2) takes the form

$$\mathcal{N}\Big(\mathcal{M}(x_1, x_2, \dots, x_k), \mathcal{M}(x_{k+1}, x_{k+2}, \dots, x_{2k}), \dots, \mathcal{M}(x_{(l-1)k+1}, x_{(l-1)k+2}, \dots, x_{lk})\Big) \\ \leq \mathcal{M}\Big(\mathcal{N}(x_1, x_{k+1}, \dots, x_{(l-1)k+1}), \mathcal{N}(x_2, x_{k+2}, \dots, x_{(l-1)k+2}), \dots, \mathcal{N}(x_k, x_{2k}, \dots, x_{lk})\Big).$$

The above inequality, with an obvious substitution, becomes equivalent to (1.1).

The notion of quasiarithmetic means was introduced in 1930s by Aumann, Knopp [9] and Jessen independently and then characterized by Kolmogorov, Nagumo and de Finetti [5,10, 12]. For a continuous strictly monotone function $f: I \to \mathbb{R}$, we define the quasiarithmetic mean $\mathcal{A}^{[f]}$ on I by

$$\mathcal{A}^{[f]}(x_1,\ldots,x_n) := f^{-1}\Big(\frac{f(x_1)+\cdots+f(x_n)}{n}\Big).$$

Quasiarithmetic means naturally generalize power means. Indeed, whenever $I = \mathbb{R}_+$ and $f = \pi_p$, where $\pi_p(x) := x^p$ if $p \neq 0$ and $\pi_0(x) := \ln x$, then the mean $\mathcal{A}^{[f]}$ coincides with the *p*-th power mean (this is what was noticed by Knopp [9]). These means share most of the properties of power means. In particular, it is easy to verify that they are symmetric and repetition invariant. Let us recall here that a mean \mathcal{M} on the interval I is called *symmetric* provided the *n*-variable restriction $M|_{I^n}$ is symmetric for every $n \in \mathbb{N}$; and it is said to be *repetition invariant* if, for all $n, m \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in I^n$, the following identity is satisfied

$$\mathfrak{M}(\underbrace{x_1,\ldots,x_1}_{m\text{-times}},\ldots,\underbrace{x_n,\ldots,x_n}_{m\text{-times}})=\mathfrak{M}(x_1,\ldots,x_n).$$

(cf. [15]). Quasiarithmetic means share even more properties of power means (cf. [1], [10]).

Generalizing the above idea, for a given mean \mathcal{M} on I and a homeomorphism $f: J \to I$, we define a mean $\mathcal{M}^{[f]}$ on J by

$$\mathfrak{M}^{[f]}(x_1,\ldots,x_n) := f^{-1}\big(\mathfrak{M}(f(x_1),\ldots,f(x_n))\big).$$

The mean $\mathcal{M}^{[f]}$ will be called the *f*-conjugate of \mathcal{M} . In this terminology, quasiarithmetic means are just the conjugates of the arithmetic mean.

It can be observed that if f is continuous and strictly increasing and $(\mathcal{M}, \mathcal{N})$ is an Ingham– Jessen pair, then so is $(\mathcal{M}^{[f]}, \mathcal{N}^{[f]})$. Additionally, the Ingham–Jessen property of the pairs $(\mathcal{M}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{M})$ is equivalent to the convexity and to the concavity of \mathcal{M} , respectively. Using these facts, $(\mathcal{M}, \mathcal{A}^{[f]})$ is an Ingham–Jessen pair if and only if $\mathcal{M}^{[f^{-1}]}$ is convex. Similarly, the Ingham–Jessen property of $(\mathcal{A}^{[f]}, \mathcal{M})$ is equivalent to concavity of $\mathcal{M}^{[f^{-1}]}$. Finally, if $g: I \to \mathbb{R}$ is continuous and strictly monotone, then $(\mathcal{A}^{[f]}, \mathcal{A}^{[g]})$ is an Ingham–Jessen pair if and only if $(\mathcal{A}^{[g]})^{[f^{-1}]} = \mathcal{A}^{[g \circ f^{-1}]}$ is concave. Fortunately, the concavity of quasiarithmetic means has been characterized recently in [4] and in [17]. Namely, for a strictly monotone \mathcal{C}^2 function $f: I \to \mathbb{R}$, the quasiarithmetic mean $\mathcal{A}^{[f]}$ is concave if and only if either f'' is nowhere vanishing and f'/f'' is convex and negative, or $f'' \equiv 0$.

2. Conjugated families of probability distributions

Let \mathbb{F} be a subfield of \mathbb{R} . For $n \in \mathbb{N}$, let $\Pi_n(\mathbb{F})$ denote the set of all \mathbb{F} -valued probability distributions (\mathbb{F} -p.d. for short) over the set \mathbb{N}_n , i.e.,

$$\Pi_n(\mathbb{F}) := \{ (\pi_1, \dots, \pi_n) \mid \pi_1, \dots, \pi_n \in [0, 1] \cap \mathbb{F}, \, \pi_1 + \dots + \pi_n = 1 \}.$$

The sequences $(P_1, \ldots, P_k) \in \Pi_n(\mathbb{R})^k$ and $(Q_1, \ldots, Q_m) \in \Pi_n(\mathbb{R})^m$ are said to be \mathbb{F} -conjugated provided that there exists a matrix $(R_{i,j}) \in \Pi_n(\mathbb{F})^{k \times m}$ such that

(2.1)
$$P_i = \frac{1}{m} \sum_{j=1}^m R_{i,j} \quad \text{for all } i \in \mathbb{N}_k \quad \text{and} \quad Q_j = \frac{1}{k} \sum_{i=1}^k R_{i,j} \quad \text{for all } j \in \mathbb{N}_m.$$

In such a case, $(R_{i,j})$ is called a *transition matrix* between (P_1, \ldots, P_k) and (Q_1, \ldots, Q_m) . A sequence which is \mathbb{F} -conjugated to itself is called \mathbb{F} -selfconjugated. Obviously, every element of $\Pi_n(\mathbb{F})$ is \mathbb{F} -selfconjugated.

Remark. Note that a necessary condition for two sequences (P_1, \ldots, P_k) and (Q_1, \ldots, Q_m) of \mathbb{R} -p.d. to be \mathbb{F} -conjugated is

$$\frac{1}{k}\sum_{i=1}^{k} P_i = \frac{1}{m}\sum_{j=1}^{m} Q_j.$$

In the following examples we present some pairs of Q-conjugated sequences of Q-p.d.

Example 1. For $k, l \in \mathbb{N}$, the sequences $(\frac{1}{k}F_i)_{i=1}^l$ and $(\frac{1}{l}G_j)_{j=1}^k$, where F_i and G_j are given by (1.3) and (1.4) are \mathbb{Q} -conjugated with the transition matrix $(\mathbf{1}_{\{(i-1)k+j\}})_{(i,j)\in\mathbb{N}_k\times\mathbb{N}_l}$.

Example 2 (Leng–Si–Zhu [11]). For $k, n \in \mathbb{N}$ with $\frac{n}{2} < k \leq n$, the sequence consisting of the normalized characteristic functions of all k-elements subsets of \mathbb{N}_n is Q-selfconjugated. In the sequel, we will denote this sequence by C_n^k (ordered lexicographically).

Further nontrivial examples of the conjugated pairs will be discussed in the last section of this paper.

The following lemma will be useful for our considerations mainly for the case when $\mathbb{F} = \mathbb{Q}$.

Lemma 2.1. If two sequences of \mathbb{F} -p.d.-s are \mathbb{R} -conjugated then they are also \mathbb{F} -conjugated.

Proof. Let $k, m, n \in \mathbb{N}$, and $(P_1, \ldots, P_k) \in \Pi_n(\mathbb{F})^k$ and $(Q_1, \ldots, Q_m) \in \Pi_n(\mathbb{F})^m$ that are \mathbb{R} -conjugated. Then there exists a matrix $(R_{i,j})_{(i,j)\in\mathbb{N}_k\times\mathbb{N}_m}$ of real-valued probability distributions such that (2.1) is valid. Define $r_{i,j,t} := R_{i,j}(t)$ for $(i, j, t) \in \mathbb{N}_k \times \mathbb{N}_m \times \mathbb{N}_n$.

Now consider the following system of linear equations and inequalities:

(2.2)

$$\begin{aligned}
x_{i,j,t} &\geq 0 & \text{for all } (i,j,t) \in \mathbb{N}_k \times \mathbb{N}_m \times \mathbb{N}_n; \\
\sum_{t \in \mathbb{N}_n} x_{i,j,t} &= 1 & \text{for all } (i,j) \in \mathbb{N}_k \times \mathbb{N}_m; \\
P_i(t) &= \frac{1}{m} \sum_{j=1}^m x_{i,j,t} & \text{for all } (i,t) \in \mathbb{N}_k \times \mathbb{N}_n; \\
Q_j(t) &= \frac{1}{k} \sum_{i=1}^k x_{i,j,t} & \text{for all } (j,t) \in \mathbb{N}_m \times \mathbb{N}_n.
\end{aligned}$$

By the conjugacy, $(x_{i,j,t}) := (r_{i,j,t})$ is a solution for the above system. On the other hand, the solution set S of this system forms a compact convex polyhedron in the space \mathbb{R}^{kmn} . Therefore, S has an extreme point x^0 due to the Minkowski(-Krejn-Milmann) Theorem. Then there exist kmn supporting hyperplanes to S whose intersection is exactly x^0 . Since the coefficients of these hyperplanes belong to \mathbb{F} , therefore the point x^0 is a unique solution of a system of linear equations with coefficients belonging to \mathbb{F} . Then Cramer's Rule implies that all the coordinates of x^0 are in \mathbb{F} . Now setting the matrix $R_{i,j}^0(t) := x_{i,j,t}^0$ for $(i, j, t) \in \mathbb{N}_k \times \mathbb{N}_m \times \mathbb{N}_n$, we get that R^0 is a transition matrix between the sequences $(P_i)_{i=1}^k$ and $(Q_j)_{i=1}^m$ whose entries are \mathbb{F} -p.d.

Now we are going to formulate our main result. To this end, if $W \colon \mathbb{N}_n \to \mathbb{N}_0$ is not identically zero, then its associated \mathbb{Q} -valued probability distribution \mathcal{D}_W over \mathbb{N}_n is defined as the *n*-tuple

$$\mathcal{D}_W := \left(\frac{W(1)}{W(1) + \dots + W(n)}, \dots, \frac{W(n)}{W(1) + \dots + W(n)}\right).$$

Our main result is contained in the subsequent theorem.

Theorem 2.2. Assume that $(\mathcal{M}, \mathcal{N})$ is an Ingham–Jessen pair of symmetric and repetition invariant means on I. Let $k, m, n \in \mathbb{N}$, $F_i, G_j: \mathbb{N}_n \to \mathbb{N}_0$ for $i \in \mathbb{N}_k$, $j \in \mathbb{N}_m$. If the sequences $(\mathcal{D}_{F_i})_{i=1}^k$, $(\mathcal{D}_{G_j})_{j=1}^m$ are \mathbb{R} -conjugated, then the inequality (1.2) is valid for every n-tuple $x \in I^n$.

We formulate now an auxiliary result which will play a key role in the proof of this theorem. We need to introduce some further definitions. Let R be a subring of \mathbb{R} . We say that $D \subseteq \mathbb{R}$ is an *R*-interval if D is of the form [a, b), where $a, b \in R$. The Cartesian product of two *R*-intervals will be called an *R*-rectangle. A set being a finite union of *R*-intervals or a finite union of *R*-rectangles is called *R*-simple. For an *R*-simple set $H \subseteq \mathbb{R}$, we denote its Lebesgue measure by |H|. If D and E are *R*-intervals, $\theta \in [0, 1]$, then $H \subset D \times E$ is called a θ -proportional subset of $D \times E$ if

(a) for all $x \in D$, $|\{y \in E : (x, y) \in H\}| = \theta \cdot |E|$,

(b) for all
$$y \in E$$
, $|\{x \in D : (x, y) \in H\}| = \theta \cdot |D|$.

The following lemma generalizes Lemma 2.6 in [16] to the case $n \ge 2$.

Lemma 2.3. Let R be a subring of \mathbb{R} such that $\mathbb{Q}R \subseteq R$. Then, for every R-rectangle $D \times E$, $n \in \mathbb{N}$ and $\theta_1, \ldots, \theta_n \in \mathbb{Q} \cap [0, 1]$ with $\theta_1 + \cdots + \theta_n \leq 1$, there exist R-simple disjoint subsets $H_1, \ldots, H_n \subseteq D \times E$ such that, for every $k \in \mathbb{N}_n$, H_k is a θ_k -proportional subset of $D \times E$. Moreover, if $\theta_1 + \cdots + \theta_n = 1$, then $(H_k)_{k=1}^n$ is a partition of $D \times E$.

Proof. Let $D \times E$ be an *R*-rectangle. Then there exists an affine bijection $\varphi \colon [0,1)^2 \to D \times E$. Indeed, if D = [a, b) and E = [c, d), then such an affine bijection can be given by

$$\varphi(t,s) = ((1-t)a + tb, (1-s)c + sd)$$
 for $(t,s) \in [0,1)^2$.

Let $\theta_k \in \mathbb{Q} \cap (0,1]$ for $k \in \mathbb{N}_n$ with $\theta_1 + \cdots + \theta_n \leq 1$. Assume that, for every $k \in \mathbb{N}_n$, $\theta_k = p_k/q$, where $q \in \mathbb{N}$ and $p_k \in \mathbb{N}_q \cup \{0\}$. Put $P_0 := 0$, $P_k := p_1 + \cdots + p_k$ for $k \in \mathbb{N}_n$ and

$$A_{i,j} := \left[\frac{i-1}{q}, \frac{i}{q}\right] \times \left[\frac{j-1}{q}, \frac{j}{q}\right] \quad \text{for} \quad i, j \in \mathbb{N}_q.$$

Furthermore, let

$$S_k := \bigcup_{i=1}^q \bigcup_{j=i+P_{k-1}}^{i+P_k-1} A_{i,j \pmod{q}+1} \quad \text{for} \quad k \in \mathbb{N}_n.$$

Note that $(S_k)_{k \in \mathbb{N}_n}$ is a family of pairwise disjoint \mathbb{Q} -simple subsets of $[0,1)^2$ and, for every $k \in \mathbb{N}_n$, S_k is θ_k -proportional. Therefore, for every $k \in \mathbb{N}_n$, $H_k := \varphi(S_k)$ is a θ_k -proportional subset of $D \times E$. Moreover, as φ is a bijection and S_k -s are pairwise disjoint, so are H_k -s. Due to the assumption $\mathbb{Q}R \subseteq R$, it also follows that each H_k is R-simple.

If $\theta_1 + \cdots + \theta_n = 1$, then $P_n = p_1 + \cdots + p_n = q$. Hence the family $(S_k)_{k \in \mathbb{N}_n}$ is a partition of $[0, 1)^2$, and so the family $(H_k)_{k \in \mathbb{N}_n}$ is a partition of $D \times E$.

Now we are in the position to proceed with the proof of theorem.

Proof of Theorem 2.2. Let $x = (x_1, \ldots, x_n) \in I^n$ and assume that $\left(\mathcal{D}_{F_i}\right)_{i=1}^k$ and $\left(\mathcal{D}_{G_j}\right)_{j=1}^m$ are \mathbb{R} -conjugated. In view of Lemma 2.1, we have that they are also \mathbb{Q} -conjugated, that is, there exists a transition matrix $R = (R_{i,j})_{(i,j)\in\mathbb{N}_k\times\mathbb{N}_m}$ with \mathbb{Q} -p.d. entries. Applying Lemma 2.3, we conclude that, for every $(i, j) \in \mathbb{N}_k \times \mathbb{N}_m$, there exists a family $(H_{i,j}^s)_{s=1}^n$ of pairwise disjoint \mathbb{Q} -simple sets such that, for every $(i, j, s) \in \mathbb{N}_k \times \mathbb{N}_m \times \mathbb{N}_n$, the set $H_{i,j}^s$ is an $R_{i,j}(s)$ -proportional in $[i-1,i) \times [j-1,j]$. Moreover, this family is a partition of $[i-1,i) \times [j-1,j]$. Furthermore, since the family is finite, there exists $\ell \in \mathbb{N}$ such that the family $(\ell H_{i,j}^s)_{(i,j,s)\in\mathbb{N}_k\times\mathbb{N}_m\times\mathbb{N}_n}$ is a partition of $[0,\ell k) \times [0,\ell n)$ consisting of \mathbb{Z} -simple sets. Thus, the family $(H^s)_{s\in\mathbb{N}_n}$ defined by $H^s := \bigcup_{(i,j)\in\mathbb{N}_k\times\mathbb{N}_m} \ell H_{i,j}^s$ for $s \in \mathbb{N}_n$ is also a partition of $[0,\ell k) \times [0,\ell n)$ consisting of \mathbb{Z} -simple sets.

Define now a matrix $A = (a_{i,j})_{(i,j) \in \mathbb{N}_{\ell k} \times \mathbb{N}_{\ell m}}$ in the following way:

$$a_{i,j} = s$$
 whenever $(i-1, j-1) \in H^s$, $s \in \mathbb{N}_n$.

For every $b \in \mathbb{N}_{\ell k}$, $c \in \mathbb{N}_{\ell m}$ and $s \in \mathbb{N}_n$, we denote by $B_b(s)$ and $C_c(s)$ the number of appearance of s in the b-th row and the c-th column of A, respectively.

Fix $b \in \mathbb{N}_{\ell k}$. Then there exists a unique $j_0 \in \mathbb{N}_k$ such that $\frac{b-1}{\ell} \in [j_0 - 1, j_0)$. Using the $p_{i,j_0}(s)$ -proportionality of H^s_{i,j_0} for $(i, s) \in \mathbb{N}_k \times \mathbb{N}_n$, for every $s \in \mathbb{N}_n$, we obtain

$$B_{b}(s) = \left| H^{s} \cap \left([0, \ell k) \times \{b\} \right) \right| = \left| \bigcup_{(i,j) \in \mathbb{N}_{k} \times \mathbb{N}_{m}} \ell H^{s}_{i,j} \cap \left([0, \ell k) \times \{b\} \right) \right|$$
$$= \ell \sum_{(i,j) \in \mathbb{N}_{k} \times \mathbb{N}_{m}} \left| H^{s}_{i,j} \cap \left([0, k) \times \{b/\ell\} \right) \right| = \ell \sum_{i \in \mathbb{N}_{k}} \left| H^{s}_{i,j_{0}} \cap \left([0, k) \times \{b/\ell\} \right) \right|$$
$$= \ell \sum_{i \in \mathbb{N}_{k}} \left| H^{s}_{i,j_{0}} \cap \left([0, k) \times [j_{0} - 1, j_{0}) \right) \right| = \ell \sum_{i \in \mathbb{N}_{k}} p_{i,j_{0}}(s).$$

Thus, in view of (2.1), we get

(2.3)
$$B_b(s) = \frac{\ell k}{\sum_{u \in \mathbb{N}_n} G_{j_0}(u)} G_{j_0}(s) \quad \text{for } s \in \mathbb{N}_n.$$

Similarly, for every $c \in \mathbb{N}_{m\ell}$, we obtain

(2.4)
$$C_c(s) = \frac{\ell m}{\sum_{u \in \mathbb{N}_n} F_{i_0}(u)} F_{i_0}(s) \quad \text{for } s \in \mathbb{N}_n,$$

where i_0 is a unique natural number such that $\frac{c-1}{\ell} \in [i_0 - 1, i_0)$.

After these preparations, we show that the Ingham–Jessen inequality for the pair $(\mathcal{M}, \mathcal{N})$ applied to the matrix $X = (x_{a_{i,j}})_{(i,j) \in \mathbb{N}_k \times \mathbb{N}_m}$ yields inequality (1.2). To this end, fix $b \in \mathbb{N}_{\ell k}$. Consider j_0 as previously and put

$$\lambda_{j_0} := \frac{\ell k}{\sum_{u \in \mathbb{N}_n} G_{j_0}(u)}$$

Then, according to the symmetry and repetition invariance of \mathcal{N} , in view of (2.3), we get

$$\mathcal{N}(x_{b,1},\ldots,x_{b,m\ell}) = \mathcal{N}(\underbrace{x_1,\ldots,x_1}_{B_b(1)\text{-times}},\ldots,\underbrace{x_n,\ldots,x_n}_{B_b(n)\text{-times}}) = \mathcal{N}(\underbrace{x_1,\ldots,x_1}_{\lambda_{j_0}G_{j_0}(1)\text{-times}},\ldots,\underbrace{x_n,\ldots,x_n}_{\lambda_{j_0}G_{j_0}(n)\text{-times}})$$
$$= \mathcal{N}(\underbrace{x_1,\ldots,x_1}_{G_{j_0}(1)\text{-times}},\ldots,\underbrace{x_n,\ldots,x_n}_{G_{j_0}(n)\text{-times}}) = \mathcal{N}_{G_{j_0}}(x_1,\ldots,x_n).$$

Hence, making use of repetition invariance of \mathcal{M} , we conclude that

$$\mathcal{M}\big(\mathcal{N}(x_{1,1},\ldots,x_{1,m\ell}),\ldots,\mathcal{N}(x_{k\ell,1},\ldots,x_{k\ell,m\ell})\big)=\mathcal{M}\big(\mathcal{N}_{G_1}(x_1,\ldots,x_n),\ldots,\mathcal{N}_{G_k}(x_1,\ldots,x_n)\big).$$

Similarly

$$\mathcal{N}\big(\mathcal{M}(x_{1,1},\ldots,x_{k\ell,1}),\ldots,\mathcal{M}(x_{1,m\ell},\ldots,x_{k\ell,m\ell})\big)=\mathcal{N}\big(\mathcal{M}_{F_1}(x_1,\ldots,x_n),\ldots,\mathcal{M}_{F_m}(x_1,\ldots,x_n)\big).$$

Since $(\mathcal{M}, \mathcal{N})$ is an Ingham–Jessen pair, the inequality (1.2) follows.

3. Applications

According to Theorem 2.2 each Ingham–Jessen pair of means jointly with a pair of conjugated sequences of function implies validity of a suitable inequality. A few examples of Ingham–Jessen pairs were already mentioned in the introduction, now we will focus our attention on establishing conjugated sequences.

3.1. Conjugacy related to Kedlaya's inequality.

Theorem 3.1. For every $n \in \mathbb{N}$, the sequence $(\frac{1}{i}\mathbf{1}_{\mathbb{N}_i})_{i=1}^n$ is \mathbb{Q} -selfconjugated.

Proof. Following Kedlaya's idea, define a family $(r_{i,j})_{(i,j)\in\mathbb{N}_n^2}$ of measures on \mathbb{N}_n by

$$r_{i,j}(k) = \frac{(n-i)!(n-j)!(i-1)!(j-1)!}{(n-1)!(k-1)!(n-i-j+k)!(i-k)!(j-k)!} \quad \text{for } k \in \mathbb{N}_n$$

We adopt a convention $m! = \infty$ for negative integers m. It has been proved in [8] that this family has the following properties:

(i)
$$r_{i,j}(k) \ge 0$$
 for $i, j, k \in \mathbb{N}_n$;
(ii) $\sum_{k=1}^n r_{i,j}(k) = 1$ for $i, j \in \mathbb{N}_n$;
(iii) $r_{i,j}(k) = r_{j,i}(k)$ for $i, j, k \in \mathbb{N}_n$;
(iv) $\sum_{i=1}^n r_{i,j}(k) = \begin{cases} n/j & \text{for } j, k \in \mathbb{N}_n \text{ with } k \le j, \\ 0 & \text{for } j, k \in \mathbb{N}_n \text{ with } k > j. \end{cases}$

It follows from (i) and (ii) that $(r_{i,j})_{(i,j)\in\mathbb{N}_n^2}$ is a family of probability distributions on \mathbb{N}_n . Furthermore, applying (iii) and (iv), one can verify that the matrix $R = (r_{i,j})_{i,j\in\mathbb{N}_n}$ validates the \mathbb{Q} -selfconfugacy of the family $(\frac{1}{i}\mathbf{1}_{\mathbb{N}_i})_{i=1}^n$.

Using that the geometric mean-arithmetic mean, i.e., the pair $(\mathcal{P}_0, \mathcal{P}_1)$ is an Ingham– Jessen pair and then applying Theorem 2.2 and the result above, we can derive Kedlaya's inequality.

3.2. Conjugacy related to combinations. Let, for every $n \in \mathbb{N}$ and $k \in \mathbb{N}_n$, C_n^k denote the lexicographically ordered sequence consisting of the characteristic functions of all k-element subsets of \mathbb{N}_n (cf. Example 2).

Theorem 3.2. For every $k, l, n \in \mathbb{N}$ with $\max(k, l) \leq n \leq k + l - 1$ the families C_n^k and C_n^l are conjugated.

Proof. Fix $k, l, n \in \mathbb{N}$ with $\max(k, l) \le n \le k + l - 1$. Set

$$C_n^k =: \left(F_1, \dots, F_{\binom{n}{k}}\right)$$
 and $C_n^l =: \left(G_1, \dots, G_{\binom{n}{l}}\right).$

We need to show that there exists a transition matrix $R = (r_{ij}) \in \Pi_n(\mathbb{R})^{\binom{n}{k} \times \binom{n}{l}}$ of probability distributions such that

(3.1)
$$\mathcal{D}_F(i) = \frac{1}{\binom{n}{l}} \sum_{j=1}^{\binom{n}{l}} R_{i,j} \text{ for } i \in \mathbb{N}_{\binom{n}{k}} \text{ and } \mathcal{D}_G(j) = \frac{1}{\binom{n}{k}} \sum_{i=1}^{\binom{n}{k}} R_{i,j} \text{ for } j \in \mathbb{N}_{\binom{n}{l}}.$$

Note that, by the inequality $n \leq k+l-1$, we have that $F_i \cdot G_j$ is not identically zero for all i, j. Therefore, for every $(i, j) \in \mathbb{N}_{\binom{n}{k}} \times \mathbb{N}_{\binom{n}{l}}$, we can define $r_{i,j}$ as the uniform probability distribution on the support of $F_i \cdot G_j$. Then, for $i \in \mathbb{N}_{\binom{n}{k}}$ and $s \in \text{supp } F_i$, we have

$$\sum_{j=1}^{\binom{n}{l}} r_{i,j}(s) = \sum_{j: s \in \text{supp } G_j} r_{i,j}(s) = \sum_{j: s \in \text{supp } G_j} \frac{1}{|\operatorname{supp } F_i \cap \operatorname{supp } G_j|}$$
$$= \sum_{m=1}^n \frac{|\{j: s \in \text{supp } G_j, |\operatorname{supp } F_i \cap \operatorname{supp } G_j| = m\}|}{m}$$
$$= \sum_{m=1}^n \frac{1}{m} \binom{k-1}{m-1} \binom{n-k}{l-m} = \sum_{m=1}^n \frac{1}{k} \binom{k}{m} \binom{n-k}{l-m} = \frac{1}{k} \binom{n}{l}$$

Hence, for every $i \in \mathbb{N}_{\binom{n}{k}}$, $\binom{n}{l}^{-1} \sum_{j=1}^{\binom{n}{l}} r_{i,j}$ is a uniform probability distribution on supp F_i . On the other hand, for every $i \in \mathbb{N}_{\binom{n}{k}}$, the function $\frac{F_i}{\sum_{s \in \mathbb{N}_n} F_i(s)}$ is also a uniform distribution on supp F_i , which implies the first equality in (3.1). The second one can be reached in similar way. This proves the Q-conjugacy of C_n^k and C_n^l .

Since the pair $(\mathcal{P}_0, \mathcal{P}_1)$ is an Ingham–Jessen pair, from Theorem 2.2 and the result above, we can obtain the inequality due to Leng–Si–Zhu [11].

3.3. Conjugacy of cyclic blocks. For the purpose of this section, for a given $k, n \in \mathbb{N}$ with $k \leq n$, denote

$$O_n^k := \frac{1}{k} \big(\mathbf{1}_{\{1,\dots,k\}}, \mathbf{1}_{\{2,\dots,k+1\}}, \dots, \mathbf{1}_{\{n-k+1,\dots,n\}}, \mathbf{1}_{\{n-k+2,\dots,n,1\}}, \dots, \mathbf{1}_{\{n,1,\dots,k-1\}} \big).$$

We conclude this paper by providing a necessary and sufficient condition for \mathbb{Q} -conjugacy of two families O_n^k . For the sake of convenience, define the canonical projection $\mathfrak{I}_n \colon \mathbb{Z} \to \mathbb{N}_n$ by $\mathfrak{I}_n(k) \equiv k \pmod{n}$ for $k \in \mathbb{Z}$.

Theorem 3.3. Assume that $n, k, l \in \mathbb{N}$ are such that $k \leq l \leq n \leq k + l - 1$. Then the sequences O_n^k and O_n^l are \mathbb{Q} -conjugated if and only if there exists a matrix $(A_{i,j})_{(i,j)\in\mathbb{N}_k\times\mathbb{N}_n}$ having nonnegative real entries such that

(i)
$$\sum_{i=1}^{k} A_{i,j} = 1$$
 for $j \in \mathbb{N}_n$,
(ii) $\sum_{j=1}^{n} A_{i,j} = n/k$ for $i \in \mathbb{N}_k$,
(iii) $\sum_{i=1}^{k} A_{i,\mathfrak{I}_n(i+j-1)}$ is equal to n/l for $j \in \mathbb{N}_l$ and equals zero for $j \in \mathbb{N}_n \setminus \mathbb{N}_l$.

Before we start the proof, let us apply this result in a simple example.

Example 3. Each of the sequences O_7^3 and O_7^4 are \mathbb{Q} -conjugated to O_7^5 . Indeed, one can easily check that the matrices

$$\frac{1}{15} \cdot \begin{bmatrix} 15 & 10 & 6 & 3 & 1 & 0 & 0 \\ 0 & 5 & 8 & 9 & 8 & 5 & 0 \\ 0 & 0 & 1 & 3 & 6 & 10 & 15 \end{bmatrix} \quad \text{and} \quad \frac{1}{20} \cdot \begin{bmatrix} 10 & 11 & 4 & 6 & 4 & 0 & 0 \\ 0 & 9 & 11 & 10 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 & 10 & 11 & 9 \\ 10 & 0 & 0 & 4 & 6 & 4 & 11 \end{bmatrix}$$

satisfy the conditions (i)–(iii) in the proposition above. Therefore, in view of Theorem 2.2 and the Ingham–Jessen property of the geometric-arithmetic means, the Q-conjugacy of O_7^3 and O_7^5 implies the following inequality:

$$\frac{\sqrt[3]{x_1x_2x_3} + \sqrt[3]{x_2x_3x_4} + \dots + \sqrt[3]{x_7x_1x_2}}{7}}{5} \le \sqrt[7]{\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}} \cdot \frac{x_2 + x_3 + x_4 + x_5 + x_6}{5} \cdots \frac{x_7 + x_1 + x_2 + x_3 + x_4}{5}}{5},$$

Analogously, the conjugancy of O_7^4 and O_7^5 yields

$$\frac{\sqrt[4]{x_1x_2x_3x_4} + \sqrt[4]{x_2x_3x_4x_5} + \dots + \sqrt[4]{x_7x_1x_2x_3}}{7}}{5} \le \sqrt[7]{\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} \cdot \frac{x_2 + x_3 + x_4 + x_5 + x_6}{5} \cdots \frac{x_7 + x_1 + x_2 + x_3 + x_4}{5}}{5}}.$$

Proof of Theorem 3.3. Assume that the sequences O_n^k and O_n^l are \mathbb{Q} -conjugated. Let R be a transition matrix of probability distributions between them.

For the sake of convenience, let $T: \mathbb{N}_n \to \mathbb{N}_n$ be given by $T(k) := \mathcal{I}_n(k+1)$. Furthermore, for every matrix $(\mu_{i,j})_{i,j\in\mathbb{N}_n}$ with entries being measures on \mathbb{N}_n , define

$$F((\mu_{i,j})_{i,j\in\mathbb{N}_n}) := \left(\mu_{T(i),T(j)} \circ T\right)_{i,j\in\mathbb{N}_n}.$$

Then F is linear, F^n is the identity and, for every $\alpha \in \mathbb{N}_n$, $F^{\alpha}(R)$ is a transition matrix for the considered pair of sequences. Consequently,

$$Q := (q_{i,j})_{i,j \in \mathbb{N}_n} := \frac{1}{n} \sum_{\alpha=0}^{n-1} F^{\alpha}(R)$$

is a fixed point of F. Moreover, as every convex combination of transition matrices is again a transition matrix, Q is a transition matrix for O_n^k and O_n^l . Set

$$A_{i,j} := q_{1,\mathfrak{I}_n(j-l+1)}(i) \quad \text{for} \quad (i,j) \in \mathbb{N}_k \times \mathbb{N}_n.$$

We are going to prove that the matrix $A = (A_{i,j})$ satisfies the conditions (i)–(iii). Since $q_{1,\mathcal{I}_n(j-l+1)}$ -s are probability distributions on \mathbb{N}_k , we get (i). Furthermore, for $i \in \mathbb{N}_k$, we have

$$\frac{1}{n}\sum_{j=1}^{n}A_{i,j} = \frac{1}{n}\sum_{j=1}^{n}q_{1,\mathfrak{I}_n(j-l+1)}(i) = \frac{1}{n}\sum_{j=1}^{n}q_{1,j}(i) = [O_n^k]_1(i) = \frac{1}{k}\mathbf{1}_{\{1,\dots,k\}}(i) = \frac{1}{k},$$

which validates (ii). Finally, as Q is a fixed point of F^{n+1-j} , for every $i \in \mathbb{N}_n$, we obtain

$$\sum_{j=1}^{k} A_{j,\mathfrak{I}_{n}(i+j-1)} = \sum_{j=1}^{k} q_{1,\mathfrak{I}_{n}(i+j-l)}(j) = \sum_{j=1}^{k} q_{T^{n+1-j}(1),T^{n+1-j}\circ\mathfrak{I}_{n}(i+j-l)}(T^{n+1-j}(j))$$
$$= \sum_{j=1}^{k} q_{\mathfrak{I}_{n}(2-j),\mathfrak{I}_{n}(i-l+1)}(1) = q_{1,\mathfrak{I}_{n}(i-l+1)}(1) + \sum_{j=n+2-k}^{n} q_{j,\mathfrak{I}_{n}(i-l+1)}(1) + \sum_{j=n+2-k}^{$$

However, since $1 \notin \text{supp}([O_n^k]_j)$ for $j \in \{2, \ldots, n+1-k\}$ and Q is a transition matrix between the sequences O_n^k and O_n^l , we have $q_{j,\mathfrak{I}_n(i-l+1)}(1) = 0$ for $j \in \{2, \ldots, n+1-k\}$. Thus, using again that Q is a transition matrix, we get

$$\sum_{j=1}^{k} A_{j, \mathcal{I}_{n}(i+j-1)} = q_{1, \mathcal{I}_{n}(i-l+1)}(1) + \sum_{j=n+2-k}^{n} q_{j, \mathcal{I}_{n}(i-l+1)}(1)$$
$$= \sum_{j=1}^{n} q_{j, \mathcal{I}_{n}(i-l+1)}(1) = n[O_{n}^{l}]_{\mathcal{I}_{n}(i-l+1)}(1) = \begin{cases} \frac{n}{l} & \text{for } i \in \mathbb{N}_{l}, \\ 0 & \text{for } i \in \mathbb{N}_{n} \setminus \mathbb{N}_{l}. \end{cases}$$

Therefore (iii) holds, and so the proof of the necessity of (i)–(iii) is completed.

To prove the converse implication, fix a matrix $A = (A_{i,j})_{(i,j) \in \mathbb{N}_k \times \mathbb{N}_n}$ with the properties (i)–(iii). Extend the matrix A by putting

(3.2)
$$A_{i,j} := 0 \quad \text{for } (i,j) \in (\mathbb{N}_n \setminus \mathbb{N}_k) \times \mathbb{N}_n.$$

Let $R = (r_{i,j})_{i,j \in \{1,...,n\}}$ be a matrix with entries being measures defined as follows

$$r_{i,j}(s) := A_{\mathfrak{I}_n(s-i+1),\mathfrak{I}_n(j-i+1)} \qquad s \in \mathbb{N}_n.$$

It follows from (i) that, for every $(i, j) \in \mathbb{N}_n \times \mathbb{N}_n$, $r_{i,j}$ is a probability distribution on \mathbb{N}_n . Furthermore, for every $j, s \in \mathbb{N}_n$, we have

$$\sum_{i=1}^{n} r_{i,j}(s) = \sum_{i=1}^{n} A_{\mathcal{I}_n(s-i+1),\mathcal{I}_n(j-i+1)} = \sum_{i=1}^{n} A_{\mathcal{I}_n(s-i+1),\mathcal{I}_n((j-s+1)+(s-i+1)-1)} = \sum_{i=1}^{n} A_{i,\mathcal{I}_n((j-s+1)+i-1)}$$

and so, in view of (3.2) and (iii), we obtain

$$\frac{1}{n}\sum_{i=1}^{n}r_{i,j}(s) = \begin{cases} 1/l & \text{whenever } \mathfrak{I}_n(s-j+1) \in \mathbb{N}_l, \\ 0 & \text{otherwise.} \end{cases} = [O_n^l]_j(s).$$

Hence the second equality in (2.1) is valid with $Q := O_n^l$. In order to get the first one with $P := O_n^k$, it is enough to note that, for every $i, s \in \mathbb{N}_n$, it holds

$$\sum_{j=1}^{n} r_{i,j}(s) = \sum_{j=1}^{n} A_{\mathcal{I}_n(s-i+1), \mathcal{I}_n(j-i+1)} = \sum_{j=1}^{n} A_{\mathcal{I}_n(s-i+1), j},$$

whence, by (ii), we get

$$\frac{1}{n}\sum_{j=1}^{n}r_{i,j}(s) = \begin{cases} 1/k & \text{whenever } \mathcal{I}_n(s-i+1) \in \mathbb{N}_k, \\ 0 & \text{otherwise.} \end{cases} = [O_n^k]_i(s).$$

Therefore, the matrix R provides transition between the mentioned families.

Remark. Note that in the case where $n \ge k+l$ neither the considered families are conjugated nor the matrix with the properties (i)-(iii) exists. In a light of Theorems 2.2 and 3.3 the following question arises naturally. Does for every $k, l, n \in \mathbb{N}$ with $k \le l \le k + n - 1$ there exist a matrix $(A_{i,j})_{(i,j)\in\mathbb{N}_k\times\mathbb{N}_n}$ with the properties (i)–(iii). We mention that for $n \le 17$, we have verified the existence of such a matrix by manual calculations.

However, in the particular case when l = n - k + 1, we can prove the existence of the matrix A and therefore the transition matrix between O_n^k and O_n^l .

Theorem 3.4. Let $m, n \in \mathbb{N}$ with $m \leq n$. Then the sequences O_n^m and O_n^{n-m+1} are \mathbb{Q} -conjugated.

Proof. Let $k := \min(m, n - m + 1)$ and $l := \max(m, n - m + 1)$. Then, obviously, we have that $k \le l \le n = k + l - 1$. It is now sufficient to show that O_n^k and O_n^l are \mathbb{Q} -conjugated, i.e. construct a matrix $A = (A_{i,j})_{(i,j) \in \mathbb{N}_k \times \mathbb{N}_n}$ which satisfies conditions (i), (ii), and (iii) of Theorem 3.3.

For the costruction of the matrix A, we have to recall the notion of Pochhammer symbols and their basic properties. For $x \in \mathbb{R}$ and $\alpha \in \mathbb{N} \cup \{0\}$ set

$$x^{\underline{\alpha}} := \prod_{j=0}^{\alpha-1} (x-j)$$
 and $x^{\overline{\alpha}} := \prod_{j=0}^{\alpha-1} (x+j).$

They are binded by the following easy identities $x^{\underline{\alpha}} = (x - \alpha + 1)^{\overline{\alpha}}$ and $x^{\overline{\alpha}} = (x + \alpha - 1)^{\underline{\alpha}}$. If $x \in \mathbb{N}$ we also have $x^{\underline{\alpha}} = \frac{x!}{(x-\alpha)!}$ ($\alpha \in \{0, \ldots, x\}$) and $x^{\overline{\alpha}} = \frac{(x+\alpha-1)!}{(x-1)!}$ ($\alpha \in \mathbb{N}$). Furthermore, the following variants of the binomial theorem are well-known:

(3.3)
$$(x+y)^{\underline{\alpha}} = \sum_{s=0}^{\alpha} {\alpha \choose s} x^{\underline{s}} y^{\underline{\alpha-s}} \quad \text{and} \quad (x+y)^{\overline{\alpha}} = \sum_{s=0}^{\alpha} {\alpha \choose s} x^{\overline{s}} y^{\overline{\alpha-s}}.$$

Now, we are in the position to define the matrix A: Let

$$A_{i,j} := \binom{k-1}{i-1} \frac{(j-1)^{\underline{i-1}}(n-j)^{\underline{k-i}}}{(n-1)^{\underline{k-1}}}.$$

Trivially, all the entries of A are nonnegative. To show that condition (i) of Theorem 3.3, we use the first formula from (3.3). Then, for $j \in \mathbb{N}_n$, we have

$$\sum_{i=1}^{k} A_{i,j} = \sum_{i=1}^{k} \binom{k-1}{i-1} \frac{(j-1)^{\underline{i-1}}(n-j)^{\underline{k-i}}}{(n-1)^{\underline{k-1}}} = \frac{(n-1)^{\underline{k-1}}}{(n-1)^{\underline{k-1}}} = 1.$$

For $i \in \mathbb{N}_k$, we have

$$\sum_{j=1}^{n} A_{i,j} = \sum_{j=1}^{n} \binom{k-1}{i-1} \frac{(j-1)^{\underline{i-1}}(n-j)^{\underline{k-i}}}{(n-1)^{\underline{k-1}}} = \sum_{j=i}^{n-k+i} \binom{k-1}{i-1} \frac{(j-1)^{\underline{i-1}}(n-j)^{\underline{k-i}}}{(n-1)^{\underline{k-1}}}$$
$$= \sum_{\alpha=0}^{n-k} \binom{k-1}{i-1} \frac{(\alpha+i-1)^{\underline{i-1}}(n-\alpha-i)^{\underline{k-i}}}{(n-1)^{\underline{k-1}}}$$
$$= \sum_{\alpha=0}^{n-k} \frac{(k-1)!}{(k-i)!(i-1)!} \cdot \frac{(n-k)!(\alpha+i-1)!(n-i-\alpha)!}{(n-1)!\alpha!(n-k-\alpha)!}$$
$$= \frac{(k-1)!}{(n-1)!} \sum_{\alpha=0}^{n-k} \frac{(n-k)!}{\alpha!(n-k-\alpha)!} \cdot \frac{(\alpha+i-1)!(n-i-\alpha)!}{(i-1)!(k-i)!}$$
$$= \frac{(k-1)!}{(n-1)!} \sum_{\alpha=0}^{n-k} \binom{n-k}{\alpha} i^{\overline{\alpha}}(k-i+1)^{\overline{n-k-\alpha}}$$
$$= \frac{(k-1)!}{(n-1)!} (k+1)^{\overline{n-k}} = \frac{k!(k+1)^{\overline{n-k}}}{k(n-1)!} = \frac{n!}{k}.$$

Finally, observe that if $i \in \mathbb{N}_k$ and $j \in \mathbb{N}_{n-k+1}$, then, we have $i + j - 1 \leq n$, and hence $\mathcal{I}_n(i+j-1) = i+j-1$. For an arbitrary $j \in \mathbb{N}_{n-k+1}$ we calculate

$$\sum_{i=1}^{k} A_{i,\mathcal{I}_{n}(i+j-1)} = \sum_{i=0}^{k-1} A_{i+1,i+j} = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{(i+j-1)^{\underline{i}(n-i-j)\underline{k-i-1}}}{(n-1)\underline{k-1}}$$
$$= \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{j^{\overline{i}}(n-j-k+2)^{\overline{k-i-1}}}{(n-k+1)^{\overline{k-1}}} = \frac{(n-k+2)^{\overline{k-1}}}{(n-k+1)^{\overline{k-1}}} = \frac{n}{n-k+1}.$$

To conclude the proof observe that, as all entries of A are nonnegative and

$$\sum_{j=n-k+2}^{n} \sum_{i=1}^{k} A_{i,\mathcal{I}_n(i+j-1)} = \sum_{j=1}^{n} \sum_{i=1}^{k} A_{i,j} - \sum_{j=1}^{n-k+1} \sum_{i=1}^{k} A_{i,\mathcal{I}_n(i+j-1)} = n - \frac{(n-k+1)n}{n-k+1} = 0,$$

we obtain $A_{i,\mathcal{I}_n(i+j-1)} = 0$ for all $i \in \mathbb{N}_k$ and $j \in \mathbb{N}_n \setminus \mathbb{N}_{n-k+1}$. In fact it is a particular case of a more general rule which emerges directly from (iii) restricted to $j \in \mathbb{N}_l$ and (i). \Box

This theorem in the classical arithmetic-geometric setting yields the inequality

$$\frac{n - m + \sqrt{x_1 x_2 \cdots x_{n-m+1}} + \frac{n - m + \sqrt{x_2 x_3 \cdots x_{n-m+1}} + \dots + \frac{n - m + \sqrt{x_n x_1 \cdots x_{n-m}}}{n}}{n} \\
\leq \sqrt{\frac{x_1 + \dots + x_m}{m} \cdot \frac{x_2 + \dots + x_{m+1}}{m} \cdots \frac{x_n + x_1 + \dots + x_{m-1}}{m}},$$

which is valid for $n \ge 2$, $m \le n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$. Remarkably, in the boundary case m = n it reduces to the comparability of arithmetic and geometric mean. For n = m - 1 it is just the Leng-Si-Zhu inequality.

Obviously arithmetic and geometic means in the inequality above can be replaced by power means or (even more generally) by any Ingham-Jessen pair of means.

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