

On line colorings of finite projective spaces*

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Abstract

In this paper, we prove lower and upper bounds on the achromatic and the pseudoachromatic indices of the n -dimensional finite projective space of order q .

1 Introduction

The results given in this paper are related to the well-known combinatorial problem called the Erdős-Faber-Lovász Conjecture (for short EFL Conjecture), see [12].

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Let \mathbf{S} be a finite linear space. A *coloring* of \mathbf{S} with k colors is an assignment of the lines of \mathbf{S} to a set of colors $[k] := \{1, \dots, k\}$. A coloring of \mathbf{S} is called *proper* if any two intersecting lines have different colors. The *chromatic index* $\chi'(\mathbf{S})$ of \mathbf{S} is the smallest k such that there exists a proper coloring of \mathbf{S} with k colors. Erdős, Faber and Lovász conjectured ([12, 13]) that the chromatic index of any finite linear space \mathbf{S} cannot exceed the number of its points, so if \mathbf{S} has v points then

$$\chi'(\mathbf{S}) \leq v.$$

In [8] the EFL Conjecture was proved for one of the most studied linear spaces, namely for the n -dimensional finite projective space of order q , $\text{PG}(n, q)$. In this case it is known that

$$\chi'(\text{PG}(n, q)) \leq \frac{q^{n+1} - 1}{q - 1}.$$

Three of this article's authors proved the EFL Conjecture for some linear spaces ([4, 5]). Moreover, in [1, 2, 3] two of them have considered different types of colorations that expand the notion of the chromatic index for graphs: the achromatic and the pseudoachromatic indices. Related problems were intensively studied by several authors, see [9, 14, 15, 17]. Furthermore, in [11] Colbourn and Colbourn investigated these parameters for block designs (see also [18]).

A coloring of \mathbf{S} is called *complete* if each pair of colors appears on at least one point of \mathbf{S} . It is not hard to see that any proper coloring of \mathbf{S} with $\chi'(\mathbf{S})$ colors is a complete coloring. The *achromatic index* $\alpha'(\mathbf{S})$ of \mathbf{S} is the largest k such that there exists a proper and complete coloring of \mathbf{S} with k colors. The *pseudoachromatic index* $\psi'(\mathbf{S})$ of \mathbf{S} is the largest k such that there exists a complete coloring (not necessarily proper) of \mathbf{S} with k colors. Clearly we have that

$$\chi'(\mathbf{S}) \leq \alpha'(\mathbf{S}) \leq \psi'(\mathbf{S}). \tag{1}$$

If Π_q is an arbitrary (not necessarily desarguesian) finite projective plane of order q , then

$$\chi'(\Pi_q) = \alpha'(\Pi_q) = \psi'(\Pi_q) = q^2 + q + 1,$$

because any two lines of Π_q have a point in common. The situation is much more complicated in higher dimensional projective spaces, the exact values of the chromatic indices are not known for $n \geq 3$. The aim of this paper is to study the achromatic and pseudoachromatic indices of finite projective spaces. Our main results are summarized in the following theorem.

Theorem 1.1. *Let $v = \frac{q^{n+1}-1}{q-1}$ denote the number of points of $\text{PG}(n, q)$.*

1. *If $n = 3 \cdot 2^i - 1$ ($i = 1, 2, \dots$) then*

$$c_n \frac{1}{q} v^{\frac{4}{3} + \frac{1}{3n}} < \alpha'(\text{PG}(n, q)), \quad (2)$$

where $\frac{1}{2^5} \leq c_n < \frac{1}{2^3}$ is a constant that depends only on n .

2. *If $n \geq 2$ is an arbitrary integer then*

$$\psi'(\text{PG}(n, q)) < \frac{1}{q} v^{\frac{3}{2}}.$$

In Section 2, we collect some known properties of projective spaces, spreads and packings, and we prove a lemma about the existence of a particular spread. In Section 3, we prove the main theorems about the achromatic and pseudoachromatic indices. Finally, in Section 4 (an Appendix is attached, where) we consider the smallest projective space, $\text{PG}(3, 2)$, and determine the exact value of its pseudoachromatic index without using a computer.

2 On projective spaces

It is well-known that, for any $n > 2$, the n -dimensional finite projective space of order q exists if and only if q is a prime power and it is unique up to isomorphism. Let V_{n+1} be an $(n + 1)$ -dimensional vector space over the Galois field $\text{GF}(q)$ with q elements. The n -dimensional finite projective space, denoted by $\text{PG}(n, q)$, is the geometry whose k -dimensional subspaces for $k = 0, 1, \dots, n$ are the $(k + 1)$ -dimensional subspaces of V_{n+1} . For the detailed description of these spaces we refer the reader to [16].

The basic combinatorial properties of $\text{PG}(n, q)$ can be described by the q -nomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$. This number is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n - 1)(q^n - q) \dots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \dots (q^k - q^{k-1})},$$

and it equals to the number of k -dimensional subspaces in an n -dimensional vector space over $\text{GF}(q)$. The proof of the following proposition is straightforward.

Proposition 2.1. *The following holds in $\text{PG}(n, q)$:*

- *the number of k -dimensional subspaces is $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$, in particular, the number of points equals to $\frac{q^{n+1}-1}{q-1}$ and the number of lines equals to $\frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}$;*
- *the number of k -dimensional subspaces through a given m -dimensional ($m \leq k$) subspace is $\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q$.*

A t -spread \mathcal{S}^t of $\text{PG}(n, q)$ is a set of t -dimensional subspaces (for short t -subspaces) of $\text{PG}(n, q)$ that partitions $\text{PG}(n, q)$. That is, each point of $\text{PG}(n, q)$ lies in exactly one element of \mathcal{S}^t . Hence any two elements of \mathcal{S}^t are disjoint. A 1-spread is also called *line spread* and it is denoted by \mathcal{S} . It is well-known that a t -spread of $\text{PG}(n, q)$ exists if and only if $(t+1)|(n+1)$, hence line spreads exist in projective spaces of odd dimension.

A t -packing \mathcal{P}^t of $\text{PG}(n, q)$ is a partition of the t -spaces of $\text{PG}(n, q)$ into t -spreads. A 1-packing is also called *line packing* or *parallelism* and it is denoted by \mathcal{P} . The next result is an obvious corollary of Proposition 2.1.

Proposition 2.2.

- *A t -spread in $\text{PG}(n, q)$ consists of $\frac{q^{n+1}-1}{q^{t+1}-1}$ t -subspaces.*
- *A t -packing in $\text{PG}(n, q)$ consists of $\begin{bmatrix} n \\ t \end{bmatrix}_q$ t -spreads.*

A necessary and sufficient condition for the existence of a t -packing of $\text{PG}(n, q)$ is not known in general. The following theorems give specific constructions in some particular cases.

Theorem 2.3 (Beutelspacher [7]). *If $n = 2^i - 1$ with $i = 1, 2, \dots$, then for every prime power q the finite projective space $\text{PG}(n, q)$ admits a line packing.*

Theorem 2.4 (Baker [6]). *For all integers $m > 0$ the finite projective space $\text{PG}(2m + 1, 2)$ admits a line packing.*

A *regulus* of $\text{PG}(3, q)$ is a set \mathcal{R} of $q + 1$ mutually skew lines such that any line of $\text{PG}(3, q)$ intersecting three distinct elements of \mathcal{R} intersects all elements of \mathcal{R} . It is known [10] that any three pairwise skew lines ℓ_1, ℓ_2, ℓ_3 of $\text{PG}(3, q)$ are contained in exactly one regulus $\mathcal{R} = \mathcal{R}(\ell_1, \ell_2, \ell_3)$ of $\text{PG}(3, q)$. A line spread \mathcal{S} of $\text{PG}(3, q)$ is called *regular*, if for any three distinct lines of \mathcal{S} the whole regulus $\mathcal{R} = \mathcal{R}(\ell_1, \ell_2, \ell_3)$ is contained in \mathcal{S} .

Theorem 2.5 (Beutelspacher [7]). *For any regular spread \mathcal{S} of $\text{PG}(3, q)$ there is a packing \mathcal{P} of $\text{PG}(3, q)$ which contains \mathcal{S} as one of its spreads.*

There is an important class of spreads. The notion of geometric spread was introduced by Segre [19] in the following way. Let $\langle X, Y \rangle$ be the subspace of $\text{PG}(n, q)$ generated by X and Y , where X and Y are two different elements of a t -spread \mathcal{S}^t of $\text{PG}(n, q)$. As X and Y are disjoint, from the dimension formula we get that $\langle X, Y \rangle$ is a $(2t + 1)$ -subspace. We say that \mathcal{S}^t *induces a spread* $\mathcal{S}_{\langle X, Y \rangle}^t$ in $\langle X, Y \rangle$, if any element Z of \mathcal{S}^t having at least one point in $\langle X, Y \rangle$ is totally contained in $\langle X, Y \rangle$. The t -spread \mathcal{S}^t is called *geometric* if \mathcal{S}^t induces a spread $\mathcal{S}_{\langle X, Y \rangle}^t$ in $\langle X, Y \rangle$ for any two distinct elements X and Y of \mathcal{S}^t .

It is not difficult to check (see [7], Section 4) that a t -spread \mathcal{S}^t of $\text{PG}(n, q)$ is geometric if and only if the following holds. If the elements X of \mathcal{S}^t are called *large points*, and for disjoint elements X, Y of \mathcal{S}^t the subspaces $\langle X, Y \rangle$ are called *large lines*, then the large points and large lines form a projective space. This space, $\Pi_{\mathcal{S}^t}$, has dimension $s = \frac{n+1}{t+1} - 1$ and order q^{t+1} , it is isomorphic to $\text{PG}\left(\frac{n+1}{t+1} - 1, q^{t+1}\right)$.

The following two results are due to Segre [19].

Theorem 2.6. *The finite projective space $\text{PG}(n, q)$ admits a geometric t -spread if and only if there exists a positive integer s such that $n + 1 = (t + 1)(s + 1)$ holds.*

Lemma 2.7. *If $\text{PG}(n, q)$ admits a geometric line spread \mathcal{S} then $\mathcal{S}_{\langle X, Y \rangle}$ is a regular line spread of the 3-dimensional subspace $\langle X, Y \rangle$ of $\text{PG}(n, q)$ for any $X, Y \in \mathcal{S}$ ($X \neq Y$).*

Combining the cited results of Beutelspacher and Segre, we prove a lemma that plays a crucial role in the proof of the lower bound in Theorem 1.1.

If $n = 3 \cdot 2^i - 1$ ($i = 1, 2, \dots$) then $n + 1 = (2^i - 1 + 1)(2 + 1)$, hence the projective space $\text{PG}(n, q)$ admits a geometric t -spread \mathcal{S}^t with $t = 2^i - 1$. The large points and large lines form a projective plane $\Pi_{\mathcal{S}^t}$ of order q^{t+1} . Consider the lines of $\Pi_{\mathcal{S}^t}$ and denote the corresponding $(2^{i+1} - 1)$ -subspaces of $\text{PG}(n, q)$ by \mathfrak{L}_j ($j = 1, \dots, q^{2t+2} + q^{t+1} + 1$). The t -spread \mathcal{S}^t is geometric, therefore for all j the elements X of \mathcal{S}^t with $X \cap \mathfrak{L}_j \neq \emptyset$ form a t -spread of \mathfrak{L}_j which will be denoted by \mathcal{S}_j^t . The spread \mathcal{S}_j^t induces a special line packing of \mathfrak{L}_j .

Lemma 2.8. *Let $\text{PG}(n, q)$ be the finite projective space of dimension $n = 3 \cdot 2^i - 1$ ($i = 1, 2, \dots$). Then there exists a geometric t -spread \mathcal{S}^t with $t = 2^i - 1$ having the property that any finite projective subspace \mathfrak{L}_j admits a line packing \mathcal{P}_j such that the set of lines contained in the elements of \mathcal{S}_j^t is the union of elements of some line spreads of \mathcal{P}_j .*

Proof. Since $n + 1 = (1 + 1)((3 \cdot 2^{i-1} - 1) + 1)$, it follows from Theorem 2.6 that $\text{PG}(3 \cdot 2^i - 1, q)$ admits a geometric line spread \mathcal{S} . The elements of \mathcal{S} and the 3-subspaces $\langle X, Y | X, Y \in \mathcal{S}, X \neq Y \rangle$ can be considered, respectively, as points and lines of a $(3 \cdot 2^{i-1} - 1)$ -dimensional space $\Pi_{\mathcal{S}}$ of order q^2 . Denote the 3-subspaces of $\text{PG}(3 \cdot 2^i - 1, q)$ corresponding to the lines of $\Pi_{\mathcal{S}}$ by \mathfrak{U}_j , where $j = 1, \dots, \binom{3 \cdot 2^{i-1} - 1}{2}_{q^2}$. Since \mathcal{S} is a geometric spread, as a consequence of Lemma 2.7, we have that the elements X of \mathcal{S} with $X \cap \mathfrak{U}_j \neq \emptyset$ form a regular line spread of \mathfrak{U}_j which will be denoted by $\mathcal{S}_{\mathfrak{U}_j}$. Moreover, by Theorem 2.5, we conclude that the 3-space \mathfrak{U}_j admits a packing \mathcal{P}_j such that $\mathcal{S}_{\mathfrak{U}_j} \in \mathcal{P}_j$. For $k = 1, 2, \dots, q^2 + q$ let $\mathcal{S}_{j,k}$ be the other spreads of \mathcal{P}_j , hence

$$\mathcal{P}_j = \{\mathcal{S}_{\mathfrak{U}_j}, \mathcal{S}_{j,1}, \dots, \mathcal{S}_{j,q^2+q}\}.$$

We claim that the lines contained in the elements of the set

$$\mathcal{P} = \bigcup_{j=1}^{q^4+q^2+1} (\mathcal{P}_j \setminus \{\mathcal{L}_j\}) \cup \mathcal{S}$$

is equal to the line set of $\text{PG}(n, q)$. The lines of \mathcal{S} obviously appear in \mathcal{P} exactly once. If a line $\ell \notin \mathcal{S}$, then ℓ lies in a unique subspace of type \mathcal{L}_j . Namely, if the lines $e, f, g, h \in \mathcal{S}$ meet ℓ then $\ell \subset \langle e, f \rangle$ and $\ell \subset \langle g, h \rangle$, but this means that $g \cap \langle e, f \rangle \neq \emptyset$ and $h \cap \langle e, f \rangle \neq \emptyset$. Since \mathcal{S} is geometric this implies that g and h are contained in $\langle e, f \rangle$ and therefore $\langle e, f \rangle = \langle g, h \rangle$. But \mathcal{P} contains exactly one packing of \mathcal{L}_j , hence each line of $\text{PG}(n, q)$ appears in \mathcal{P} exactly once.

Now, we prove the statement of the lemma by induction on i .

If $i = 1$ then it follows from Theorem 2.6 that $\text{PG}(5, q)$ admits a geometric line spread \mathcal{S} . The elements of \mathcal{S} and the 3-spaces $\langle X, Y | X, Y \in \mathcal{S}, X \neq Y \rangle$ can be considered as points and lines of a plane $\Pi_{\mathcal{S}}$ of order q^2 , respectively. Denote the 3-spaces of $\text{PG}(5, q)$ corresponding to the lines of $\Pi_{\mathcal{S}}$ by \mathcal{L}_j , where $j = 1, \dots, q^4 + q^2 + 1$. Since \mathcal{S} is a geometric spread, Lemma 2.7 gives that the elements X of \mathcal{S} with $X \cap \mathcal{L}_j \neq \emptyset$ form a regular line spread of \mathcal{L}_j which will be denoted by $\mathcal{S}_{\mathcal{L}_j}$. Because of Theorem 2.5 the 3-space \mathcal{L}_j admits a packing \mathcal{P}_j such that $\mathcal{S}_{\mathcal{L}_j} \in \mathcal{P}_j$. For $i = 1, 2, \dots, q^2 + q$ let $\mathcal{S}_{j,1}$ be the other spreads of \mathcal{P}_j , hence

$$\mathcal{P}_j = \{\mathcal{S}_{\mathcal{L}_j}, \mathcal{S}_{j,1}, \dots, \mathcal{S}_{j,q^2+q}\}.$$

Consider now the case $i > 1$ and let us assume that the assertion of Lemma 2.8 is proved for all $i' < i$. Since $n + 1 = 3 \cdot 2^i = (1 + 1)(3 \cdot 2^{i-1} - 1 + 1)$, by Theorem 2.6, $\text{PG}(n, q)$ admits a geometric 1-spread \mathcal{S} . As before, we consider the elements X of \mathcal{S} and the 3-subspaces $\langle X, Y | X, Y \in \mathcal{S}, X \neq Y \rangle$ as points and lines of a $(3 \cdot 2^{i-1} - 1)$ -space $\Pi_{\mathcal{S}}$ of order q^2 , respectively. Denote the lines of $\Pi_{\mathcal{S}}$ by \mathfrak{V}_k ($k = 1, \dots, M$) where M is the number of lines of $\Pi_{\mathcal{S}}$. The spread \mathcal{S} is geometric, therefore the elements X of \mathcal{S} with $X \cap \mathfrak{V}_k \neq \emptyset$ form a spread of \mathfrak{V}_k which will be denoted by \mathcal{S}_k . By Lemma 2.7, \mathcal{S}_k is a regular spread of \mathfrak{V}_k . According to Theorem 2.5, in all \mathfrak{V}_k there exists a packing $\mathcal{P}_{\mathfrak{V}_k}$ of \mathfrak{V}_k which contains \mathcal{S}_k as one

of its spreads. Let this packing be

$$\mathcal{P}_{\mathfrak{U}_k} = \{\mathcal{S}_{k,0}, \dots, \mathcal{S}_{k,q^2+q}\}.$$

with $\mathcal{S}_{k,0} = \mathcal{S}_k$.

Hence -by induction- $\Pi_{\mathcal{S}}$ admits a basic construction \mathcal{C}_{i-1} with the property that any finite projective subspace \mathfrak{U}_j admits a packing \mathcal{P}_j such that the lines contained in the elements of \mathcal{S}_j^t are the union of elements of \mathcal{P}_j . Let $\mathcal{S}_j^t = \{\mathcal{T}_{j,1}, \dots, \mathcal{T}_{j,u}\}$ then $\mathcal{P}_j = \mathcal{S}_j^t \cup \{\mathcal{T}_{j,u+1}, \dots, \mathcal{T}_{j,v}\}$ where v is the number of 1-spreads in \mathfrak{U}_j of $\Pi_{\mathcal{S}}$.

Recall that each line of $\Pi_{\mathcal{S}}$ is a 3-subspace of $\text{PG}(n, q)$. If $\mathcal{T}_{j,l} = \{\mathfrak{u}_{l(1)}, \dots, \mathfrak{u}_{l(w)} : 1 \leq l \leq v\}$ where w is the number of lines in a 1-spread of \mathfrak{U}_j (as a subspace of $\Pi_{\mathcal{S}}$), then $\mathcal{T}_{j,l,m} = \{\mathcal{S}_{l(1),m}, \dots, \mathcal{S}_{l(w),m} : 0 \leq m \leq q^2 + q\}$ is a 1-spread of \mathfrak{U}_j (as a subspace of $\text{PG}(n, q)$).

We construct the following packing \mathcal{P}_j of \mathfrak{U}_j (as subspaces of $\text{PG}(n, q)$):

$$\mathcal{P}_j = \bigsqcup_{l=1}^u \bigsqcup_{m=0}^{q^2+q} \mathcal{T}_{j,l,m} \cup \bigsqcup_{l=u+1}^v \bigsqcup_{m=1}^{q^2+q} \mathcal{T}_{j,l,m}.$$

By construction, the set $\bigsqcup_{l=1}^u \bigsqcup_{m=0}^{q^2+q} \mathcal{T}_{j,l,m}$ contains all the lines of \mathcal{S}_j^t and the lemma follows. \square

3 On line colorings of projective spaces

First, we introduce some notions that we use to prove our results. Let \mathcal{L} be the set of lines of $\text{PG}(n, q)$ and \mathcal{P} be its set of points. Given a coloring $\varsigma: \mathcal{L} \rightarrow [k]$ with k colors, we say that a point $p \in \mathcal{P}$ is an *owner* of a set of colors $C \subseteq [k]$ whenever for every $c \in C$ there is a $q \in \mathcal{P} \setminus \{p\}$ such that $\varsigma(\langle p, q \rangle) = c$. Therefore, ς is a complete coloring if for every pair of colors in $[k]$ there is a point in \mathcal{P} which is an owner of both colors.

3.1 Lower bound

Now we are ready to prove the lower bound in Theorem 1.1.

Proof of Theorem 1.1, Part 1. Throughout the proof we use the notations of Section 2. Consider the geometric t -spread \mathcal{S}^t constructed in Lemma 2.8. Let $N = q^{2(t+1)} + q^{t+1} + 1$ denote the number of large lines of the corresponding projective plane $\Pi_{\mathcal{S}^t}$. The space $\text{PG}(n, q)$ admits a basic construction \mathcal{C}_i with the property that any finite projective subspace \mathfrak{L}_j admits a packing \mathcal{P}_j such that the set of lines contained in the elements of \mathcal{S}_j^t is the union of elements of \mathcal{P}_j .

Let $r = \binom{t}{1}_q$, and $\mathcal{U}_j = \{\mathcal{S}_{j,1}, \dots, \mathcal{S}_{j,r}\}$ denote the set of 1-spreads from \mathcal{P}_j whose union is the set of all lines that are contained in the elements of \mathcal{S}_j^t . Then $\mathcal{P}_j = \mathcal{U}_j \sqcup \{\mathcal{S}_{j,r+1}, \dots, \mathcal{S}_{j,s}\}$ where $s = \binom{2t+1}{1}_q$ is the number of 1-spreads of \mathcal{P}_j . Note that the number of 1-spreads in $\mathcal{P}_j^* := \mathcal{P}_j \setminus \mathcal{U}_j$ is $s - r = q^t \binom{t+1}{1}_q$. Every element X of \mathcal{S}^t is a t -subspace, hence, by Theorem 2.3, admits a packing $\mathcal{P}_X = \{\mathcal{S}_{1,X}, \dots, \mathcal{S}_{r,X}\}$. Then the set of lines contained in the elements of the set

$$\left(\bigcup_{j=1}^N \mathcal{P}_j^* \right) \cup \left(\bigcup_{X \in \mathcal{S}^t} \mathcal{P}_X \right)$$

is equal to \mathcal{L} .

Now we define the coloring. We distinguish the two types of spreads. For a fixed $1 \leq j \leq N$ and $r+1 \leq k \leq s$ let the lines of $\mathcal{S}_{j,k}$ be colored with the color $c_{j,k} = (k - r - 1)N + j$. This implies that each point of \mathfrak{L}_j is owner of the colors $c_{j,k}$ for all k . For a fixed $1 \leq m \leq r$ let the lines of $\mathcal{S}_{m,X}$ be colored with the color $c_m = (s - r)N + m$ for all X . As $\mathcal{S}_{m,X}$ is a 1-spread of $X \in \mathcal{S}^t$, the set of points on the lines of the set $\cup_{X \in \mathcal{S}^t} \mathcal{S}_{m,X}$ is equal to \mathcal{P} . Thus each point of $\text{PG}(n, q)$ is owner of the colors c_m for all m .

Observe that the coloring is proper by definition. We claim that it is also complete. We have to show that for every pair of colors $\{c, c'\}$ there is a point x of $\text{PG}(n, q)$, which is an owner of both c and c' . This is obvious when at least one of c and c' is of type c_m . Suppose that $c = c_{j,k}$ and $c' = c_{j',k'}$. Take the subspace $\mathfrak{L}_j \cap \mathfrak{L}_{j'}$. Its dimension is $2t + 1$ or t , according as $j = j'$ or $j \neq j'$, so it is not the empty set. Any point $x \in \mathfrak{L}_j \cap \mathfrak{L}_{j'}$ is the owner of both colors.

In the coloring we use

$$(s-r)N + r = q^t \frac{q^{n+1} - 1}{q-1} + \frac{q^t - 1}{q-1} = \frac{q^{n+t+1} - 1}{q-1}$$

colors. Let $h = \frac{4n+1}{3n}$. Since $n+t+1 = \frac{4n+1}{3} = hn$ and $2q^n > \frac{q^{n+1}-1}{q-1} = v$, we have

$$\frac{q^{n+t+1} - 1}{q-1} = \frac{q^{hn} - 1}{q-1} > \frac{1}{2^h} \frac{(2q^n)^h}{q} > \frac{1}{2^h} \frac{v^h}{q},$$

hence Inequality (2) of Theorem 1.1 holds with $c_n = \frac{1}{2^h}$, and the theorem follows, because $5 \leq n$ implies $\frac{4}{3} < h \leq \frac{7}{5}$. \square

3.2 Upper bound

Now we prove the upper bound for the pseudoachromatic index of $\text{PG}(n, q)$.

Proof of Theorem 1.1, Part 2. If r denotes the number of lines through a fixed point, then the total number of unordered line-line incidences is $v \binom{r}{2}$. Hence $v \binom{r}{2} \geq (\psi'(\text{PG}(n, q)))^2$. Solving this quadratic inequality we get

$$\psi'(\text{PG}(n, q)) \leq \frac{1 + \sqrt{1 + 4vr(r-1)}}{2}.$$

Since $\sqrt{1 + 4vr(r-1)} < \sqrt{4vr^2} - 1$ and $r = \frac{v-1}{q}$, this gives

$$\psi'(\text{PG}(n, q)) < \sqrt{vr} = \frac{1}{q} \sqrt{v(v-1)}$$

and the result follows. \square

4 The case of $\text{PG}(3, 2)$

In this section, we determine the pseudoachromatic index of the smallest finite projective space, $\text{PG}(3, 2)$, in a pure combinatorial way, without using any computer aided calculations. To do this, we need some lemmas about pencils and null polarities.

Definition 4.1. *Let Π be a plane and $P \in \Pi$ be a point in $\text{PG}(3, q)$. A pencil with carrier P in Π is the set of the $q+1$ lines of $\text{PG}(3, q)$ through P that are contained in Π .*

Lemma 4.2. *Let \mathcal{E} be a set of five lines in $\text{PG}(3, 2)$. If any two lines of \mathcal{E} have a point in common then \mathcal{E} contains a pencil.*

Proof. Any two lines of \mathcal{E} meet, hence, all the lines in \mathcal{E} are either coplanar or all of them have a point in common. It follows from Proposition 2.1 that in $\text{PG}(3, 2)$ there are seven lines through each point, and dually, each plane contains seven lines. Because of the duality, we may assume without loss of generality that the five lines of \mathcal{E} are coplanar. As each plane contains seven points and $\binom{5}{2} > 7$, at least three lines of \mathcal{E} have a point in common, thus \mathcal{E} contains a pencil. \square

Definition 4.3. *Let $\text{PG}(3, q)'$ denote the dual space of $\text{PG}(3, q)$, and let A be a 4×4 non-singular matrix over $\text{GF}(q)$ that satisfies the equation $A = -A^T$ and whose all diagonal elements are 0.*

A null polarity $\pi: \text{PG}(3, q) \rightarrow \text{PG}(3, q)'$ is a collineation which maps the point with coordinate vector \mathbf{x} to the point with coordinate vector $\mathbf{x}A$.

As the points, lines and planes of the dual space are planes, lines and points of the original space, respectively, a null polarity maps lines of $\text{PG}(3, q)$ to lines of $\text{PG}(3, q)$. A null polarity maps intersecting lines to intersecting lines, hence the proof of the following statement is straightforward.

Lemma 4.4. *Let π be a null polarity and ς be a line-coloring of $\text{PG}(3, q)$. Then ς is complete if and only if $\varsigma \circ \pi^{-1}$ is a complete line-coloring of $\text{PG}(3, q)'$.*

Theorem 4.5. *The pseudoachromatic index of $\text{PG}(3, 2)$ is equal to 18, i.e.,*

$$\psi'(\text{PG}(3, 2)) = 18.$$

Proof. In $\text{PG}(3, 2)$ there are three points on each line and there are seven lines through each point, hence the total number of lines intersecting a fixed line is $3 \cdot (7 - 1) = 18$. Thus if a complete line coloring contains a color class of size one then the coloring cannot contain more than $1 + 18 = 19$ color classes. There are 35 lines in $\text{PG}(3, 2)$, so the number of color classes containing at least two lines is at most $\lfloor 35/2 \rfloor = 17$. Hence $\psi'(\text{PG}(3, 2)) \leq 19$.

Now, we prove that $\psi'(\text{PG}(3,2)) \leq 18$. Suppose to the contrary that there exists a complete coloring ς of $\text{PG}(3,2)$ with 19 color classes.

We claim that ς contains three or four color classes of size one and no three of the corresponding lines form a pencil. By the pigeonhole principle, there are at least 3 color classes of size one. If there were at least five color classes of size one in ς then, by Lemma 4.2, we could choose three color classes such that the corresponding lines would form a pencil. Suppose that three lines, ℓ_1, ℓ_2 and ℓ_3 form a pencil with carrier P in the plane Π , and each of these lines forms a color class of size one. Consider the other 16 classes. At most 4 of them contain lines through P and at most 4 of them contain lines in Π . Each of the remaining at least 8 classes must have size at least 3, because they have to meet each ℓ_i for $i = 1, 2, 3$. This implies that these color classes contain altogether $8 \times 3 = 24$ or more lines. As the total number of lines is 35, this means that each of the remaining 11 color classes contains exactly one line. Hence each of the seven lines through P , and each of the 4 lines in Π not through P are color classes of size one, but they do not meet, so ς is not complete. This contradiction proves the statement.

Choose three color classes of size one and let ℓ_1, ℓ_2 and ℓ_3 be the lines in these color classes. Any two of these lines have a point in common, but they do not form a pencil, hence either they form a triangle, or they have a point in common but they are not coplanar. In the latter case apply Lemma 4.4. If the three lines meet in the point P then after a null polarity π , the lines ℓ_1^π, ℓ_2^π and ℓ_3^π form a triangle in the plane P^π . As $\text{PG}(3,2)$ is isomorphic to its dual space, it is enough to consider the first case.

From now on, we suppose that ℓ_1, ℓ_2 and ℓ_3 form a triangle ABC in the plane Π . Let A', B' and C' be the third points of the sides of the triangle, respectively, and let $D = AA' \cap BB' \cap CC'$ be the seventh point of the plane Π . Take Π as the plane at infinity and consider the remaining eight points as $\text{AG}(3,2)$. The coordinates of the points in Π can be chosen as follow.

$A = (0 : 1 : 0 : 0)$, $B = (0 : 0 : 1 : 0)$, $C = (0 : 0 : 0 : 1)$, $A' = (0 : 0 : 1 : 1)$,
 $B' = (0 : 1 : 0 : 1)$, $C' = (0 : 1 : 1 : 0)$ and $D = (0 : 1 : 1 : 1)$.

First, suppose that there is a 4th color class of size one and let ℓ_4 denote the line in this class. Then ℓ_4 must be in Π . If it contains one of the points

A, B or C , then a pencil appears, hence the coloring is not complete. So we may assume that ℓ_4 is the line $A'B'C'$. Among the other 15 color classes there are 14 classes of size 2 and one of size 3. Consider the four lines, say ℓ_5, ℓ_6, ℓ_7 and ℓ_8 , through D but not in Π . If two or three of them formed a color class, then this class would have empty intersection with each of ℓ_1, ℓ_2 and ℓ_3 , contradiction. So these four lines are distributed among at least three color classes and each class of size two must contain a line of Π . Thus there are two possibilities for these color classes:

- (a) $\{\ell_5, \ell_8, AA'\}, \{\ell_6, BB'\}, \{\ell_7, CC'\}$,
- (b) $\{\ell_5, AA'\}, \{\ell_6, BB'\}, \{\ell_7, CC'\}, \{\ell_8, \ell_9, \ell_{10}\}$, where ℓ_9 is a line through A and ℓ_{10} is a line through A' .

Each of the remaining classes contains two lines whose points at infinity cover ℓ_1, ℓ_2, ℓ_3 and ℓ_4 . Since no three of these lines have a point in common, each of the remaining classes is incident to ℓ_1, ℓ_2, ℓ_3 and ℓ_4 if and only if two points at infinity of these color classes coincide with one of the sets $\{A, A'\}, \{B, B'\}$ and $\{C, C'\}$.

If there is no more color class of size one, then each of the remaining 16 classes has size 2. The pairs of the four lines through D must be the four lines of Π distinct from ℓ_1, ℓ_2, ℓ_3 . If an affine line passes on A' , then its pair must pass on A , and the same is true for the lines through $\{B, B'\}$ and $\{C, C'\}$.

We can summarize these possibilities as follow.

- Each of the lines ℓ_1, ℓ_2 and ℓ_3 forms a class of size one.
- There are 12 classes such that two points at infinity of these color classes coincide with one of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}$.
- Each of the pairs $\{\ell_5, AA'\}, \{\ell_6, BB'\}$ and $\{\ell_7, CC'\}$ belong to one color class.
- The nineteenth color class contains the line $A'B'C'$.
- The line ℓ_8 is “free”.

We can choose the system of reference such that the pair of AA' is the line DOE where $O = (1 : 0 : 0 : 0)$ and $E = (1 : 1 : 1 : 1)$. Let $X = (1 : 1 : 0 : 0)$, $Y = (1 : 0 : 1 : 0)$, $Z = (1 : 0 : 0 : 1)$, $K = (1 : 1 : 1 : 0)$, $L = (1 : 1 : 0 : 1)$ and $M = (1 : 0 : 1 : 1)$ be the other affine points of $\text{PG}(3, 2)$, see Figure 1. The pair of the line CXL is either the line $C'O$ or $C'E$. As the roles of O and E were symmetric previously, we may assume without loss of generality that $C'OK$ is the pair of CXL .

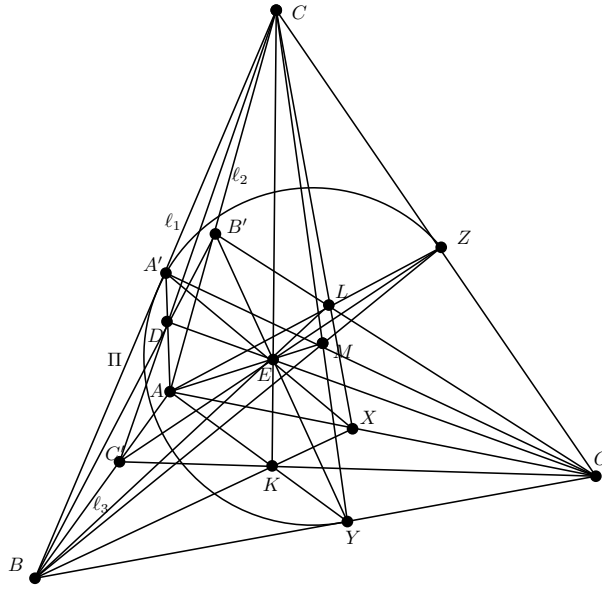


Figure 1: $\text{PG}(3, 2)$, not all lines shown.

First, consider the three other classes whose two points in Π are C and C' . The affine part of the three lines through C are OZ , MY , KE , while the affine part of the three lines through C' are EZ , XY , LM . Each of these classes must meet the line DOE . Hence, we need a matching between these two line-triples such that each pair contains at least one of the points O and E . So the pair of MY must be EZ . There are two possibilities for the remaining two pairs, so the four possible pairs through C and C' are:

- i) (XL, OK) , (MY, EZ) , (OZ, XY) , (KE, LM) ,

ii) $(XL, OK), (MY, EZ), (KE, XY), (OZ, LM)$.

In Case i) take the four classes whose two points in Π are B and B' . The affine parts of the four lines through B are OY, EL, XK, ZM , while the affine parts of the four lines through B' are OL, EY, XZ, MK . Again, we need a matching such that each pair contains at least one of the points O and E , and each class must meet the four classes belonging to $\{C, C'\}$. So the pair of XK is EY , because (XK, OL) has empty intersection with (MY, EZ) . Hence the pair of ZM is OL . The pair of MK is OY , because (EL, MK) has empty intersection with (OZ, XY) . So the affine parts of the four pairs belonging to $\{B, B'\}$ are

$$(XK, EY), (ZM, OL), (OY, MK), (EL, XZ).$$

Take the four classes whose two points in Π are A and A' . The affine parts of the four lines through A are OX, EM, YK, ZL , while the affine part of the four lines through A' are OM, EX, YZ, LK . At least three classes consist of only two lines. Let us look for these classes. None of the pairs $(OX, LK), (OX, YZ), (EM, YZ), (EM, LK)$ is good, because its intersection is empty with $(YM, EZ), (KE, LM), (XL, OK), (OZ, XY)$, respectively. In the same way none of the pairs $(KY, OM), (KY, EX), (LZ, OM), (LZ, EX)$ is good because their intersections are empty with $(EL, XZ), (ZM, LO), (XK, YE), (OY, MK)$, respectively. This means that in the matching there are only four possible pairs containing OX or EM , namely

$$(OX, OM), (OX, EX), (EM, OM), (EM, EX), \quad (3)$$

and four possible pairs containing KY or LZ , namely

$$(KY, LK), (KY, YZ), (LZ, YZ), (LZ, LK). \quad (4)$$

Thus at least one pair from (3) and at least one pair from (4) form a color class.

Now consider the affine part of the two color classes containing $\{B, B'\}$ and $\{C, C'\}$. These are the lines through D , except DOE . So they consist

of the points X and M , Z and K , L and Y . At least one of the two classes contains only one pair of points. But the pair $\{X, M\}$ has empty intersection with any class from (4), while both pairs $\{Z, K\}$ and $\{L, Y\}$ have empty intersection with any class from (3). Hence the coloring cannot be complete in Case i).

Now consider Case ii). Take the four classes whose two points in Π are B, B' . The affine parts of the four lines through B are OY, EL, XK, ZM , while the affine parts of the four lines through B' are OL, EY, XZ, MK . Again we need a matching such that each pair contains at least one of the points O and E , and each class must meet the four classes belonging to $\{C, C'\}$.

So the pair of XK is OL , because (XK, EY) has empty intersection with (OZ, LM) . Hence the pair of ZM is EY . We distinguish two cases, according to the pair of OY . So the affine parts of the four pairs belonging to $\{B, B'\}$ are

$$(a) (XK, LO), (ZM, EY), (OY, XZ), (EL, MK),$$

$$(b) (XK, LO), (ZM, EY), (OY, MK), (EL, XZ).$$

Take the four classes whose two points at infinity are A and A' . The affine parts of the four lines through A are OX, EM, YK, ZL , while the affine part of the four lines through A' are OM, EX, YZ, LK . At least three classes consist of only two lines. Let us look for these classes. In both cases none of the pairs $(OX, LK), (KY, EX), (EM, YZ), (LZ, OM)$ is good, because it has empty intersection with $(YM, EZ), (OZ, LM), (XK, LO), (KZ, XY)$, respectively.

Furthermore, in Case (a) none of the pairs $(OX, YZ), (EM, LK)$ is good, because its intersection is empty with $\{O, X, Y, Z\} \cap \{E, L, M, K\} = \{E, M, L, K\} \cap \{O, Y, X, Z\} = \emptyset$. Thus we get four possible pairs containing OX or EM :

$$(OX, OM), (OX, EX), (EM, OM), (EM, EX), \quad (5)$$

and six possible pairs containing KY or LZ :

$$(KY, OM), (LZ, EX),$$

$$(KY, LK), (KY, YZ), (LZ, LK), (LZ, YZ). \quad (6)$$

If either (KY, OM) or (LZ, EX) belongs to the matching, then only one more pair from (5) can be in it, hence at least one more pair from (6) also belongs to the matching. Thus at least one pair from (5) and at least one pair from (6) form a color class.

In Case (b) none of the pairs (LZ, EX) , (KY, OM) is good, because it has empty intersection with (OY, MK) , (EL, XZ) , respectively. Thus we get six possible pairs containing OX or EM :

$$(OX, YZ), (EM, LK), \\ (OX, OM), (OX, EX), (EM, OM), (EM, EX), \quad (7)$$

and four possible pairs containing KY or LZ :

$$(KY, LK), (KY, YZ), (LZ, LK), (LZ, YZ). \quad (8)$$

If either (OX, YZ) or (EM, LX) belongs to the matching, then only one more pair from (7) can be in it, hence at least one more pair from (8) also belongs to the matching. Thus at least one pair from (7) and at least one pair from (8) form a color class.

Finally, in both Cases (a) and (b), consider the affine part of the two color classes containing $\{B, B'\}$ and $\{C, C'\}$. These are the lines through D , except DOE . So they consist of the points X and M , Z and K , L and Y . At least one of the two classes contains only one pair of points. But the pair $\{X, M\}$ has empty intersection with any class from (6) and from (8), while both of the pairs $\{Z, K\}$ and $\{L, Y\}$ have empty intersection with any class from (5) and from (7). Hence the coloring cannot be complete in Case ii).

Now, we present a complete coloring with 18 color classes.

Let the lines ℓ_1, ℓ_2, ℓ_3 and $\ell_4 = A'B'C'$ form color classes of size one. These classes are denoted by $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 , respectively. The class \mathcal{C}_5 consists of the lines $AA'D$ and OED , while the class \mathcal{C}_6 consists of the remaining five lines through D . Any two of these six classes obviously have non-empty intersection. The remaining twelve classes of size two are formed

by the 3×4 pairs of lines whose points at infinity are $\{A, A'\}$, $\{B, B'\}$ and $\{C, C'\}$, respectively. The affine parts of these classes are the following:

$$\begin{aligned} \mathcal{C}_{A1}: (OX, EX), \quad \mathcal{C}_{A2}: (OM, EM), \quad \mathcal{C}_{A3}: (YK, YZ), \quad \mathcal{C}_{A4}: (LZ, LK); \\ \mathcal{C}_{B1}: (OY, MX), \quad \mathcal{C}_{B2}: (XK, EY), \quad \mathcal{C}_{B3}: (ZM, OL), \quad \mathcal{C}_{B4}: (EL, XZ); \\ \mathcal{C}_{C1}: (OZ, XY), \quad \mathcal{C}_{C2}: (XL, OK), \quad \mathcal{C}_{C3}: (YM, EZ), \quad \mathcal{C}_{C4}: (KE, LM). \end{aligned}$$

If $1 \leq i \leq 6$ then \mathcal{C}_i contains at least one element of each of the pairs $\{A, A'\}$, $\{B, B'\}$ and $\{C, C'\}$, and any two color classes belonging to the same quadruple of classes of type \mathcal{C}_{Q_i} also intersect each other. Hence it is enough to show that \mathcal{C}_{Q_i} and \mathcal{C}_{R_j} have non-empty intersection if $Q \neq R$. The three parts of Table 1 give one point of intersection in each case.

	\mathcal{C}_{B1}	\mathcal{C}_{B2}	\mathcal{C}_{B3}	\mathcal{C}_{B4}		\mathcal{C}_{C1}	\mathcal{C}_{C2}	\mathcal{C}_{C3}	\mathcal{C}_{C4}
\mathcal{C}_{A1}	O	X	O	X	\mathcal{C}_{A1}	O	O	E	E
\mathcal{C}_{A2}	M	E	M	E	\mathcal{C}_{A2}	O	O	M	M
\mathcal{C}_{A3}	Y	Y	Z	Z	\mathcal{C}_{A3}	Y	K	Y	K
\mathcal{C}_{A4}	K	K	L	l	\mathcal{C}_{A4}	Z	L	Z	l

	\mathcal{C}_{C1}	\mathcal{C}_{C2}	\mathcal{C}_{C3}	\mathcal{C}_{C4}
\mathcal{C}_{B1}	O	O	M	M
\mathcal{C}_{B2}	X	X	E	E
\mathcal{C}_{B3}	O	O	M	M
\mathcal{C}_{B4}	X	X	E	E

Table 1: Points of intersections of \mathcal{C}_{Q_i} and \mathcal{C}_{R_j} .

This proves that the coloring is complete. \square

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