

The domination game played on diameter 2 graphs

Csilla Bujtás^a, Vesna Iršič^{a,b}, Sandi Klavžar^{a,b,c}, Kexiang Xu^d

^a Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

^b Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

^c Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

^d College of Science, Nanjing University of Aeronautics & Astronautics,
Nanjing, Jiangsu 210016, PR China

csilla.bujtas@fmf.uni-lj.si, vesna.irsic@fmf.uni-lj.si

sandi.klavzar@fmf.uni-lj.si, kexxu1221@126.com

February 3, 2021

Abstract

Let $\gamma_g(G)$ be the game domination number of a graph G . It is proved that if $\text{diam}(G) = 2$, then $\gamma_g(G) \leq \left\lceil \frac{n(G)}{2} \right\rceil - \left\lfloor \frac{n(G)}{11} \right\rfloor$. The bound is attained: if $\text{diam}(G) = 2$ and $n(G) \leq 10$, then $\gamma_g(G) = \left\lceil \frac{n(G)}{2} \right\rceil$ if and only if G is one of seven sporadic graphs with $n(G) \leq 6$ or the Petersen graph, and there are exactly ten graphs of diameter 2 and order 11 that attain the bound.

Keywords: domination game; diameter 2 graph; computer experiment

AMS Math. Subj. Class. (2020): 05C57, 05C69

1 Introduction

The domination game has been investigated in depth by now, hence let us very quickly recall its definition [3]. The game is played on a graph G by Dominator and Staller who alternately select their vertices. Each selected vertex is required to dominate at least one new vertex. The game ends when the vertices selected form a dominating set; Dominator's goal is to finish the game as soon as possible,

Staller's goal is the opposite. If Dominator is the first to play, we speak of a *D-game*, otherwise it is an *S-game*. The number of vertices selected in a D-game under the assumption that both players follow optimal strategies is the *game domination number* $\gamma_g(G)$ of G . The corresponding invariant for the S-game is denoted by $\gamma'_g(G)$.

A central theme in the investigation of the game domination number is its upper bounds in terms of the order $n(G)$ of a graph G . It all started with the 3/5-Graph Conjecture [19] asserting that if G is an isolate-free graph, then $\gamma_g(G) \leq \frac{3}{5}n(G)$. A strong support for the conjecture is [15, Theorem 2.7] which asserts that it is true for all graphs with minimum degree at least 2. The conjecture is still open in general, the best upper bound that holds for all graphs is $\gamma_g(G) \leq \frac{5}{8}n(G)$ [7, Theorem 2.25]. Another appealing conjecture, first stated in [18], is Rall's 1/2-Conjecture which asserts that if G is a traceable graph, then $\gamma_g(G) \leq \lceil \frac{1}{2}n(G) \rceil$, cf. [8]. For additional topics of interest related to the domination game and its variants see [2, 4, 6, 11, 16, 17, 22, 23, 25, 26].

In this paper we focus on the domination game played on graphs with diameter 2 and proceed as follows. In the next section we present some preliminary results and introduce a new proof technique (Lemma 2.5) to be used in the rest of the paper. In Section 3 we prove that if G is a graph with $\text{diam}(G) = 2$, then $\gamma_g(G) \leq \lceil \frac{n(G)}{2} \rceil$. Moreover, we show that the equality holds for precisely eight graphs, including the Petersen graph. Based on Section 3, in the subsequent section we prove that if G is a graph with $\text{diam}(G) = 2$, then $\gamma_g(G) \leq \lceil \frac{n(G)}{2} \rceil - \lfloor \frac{n(G)}{11} \rfloor$. All equality graphs of order 11 are also discovered. In the concluding section we relate our results to Rall's 1/2-Conjecture.

2 Preliminaries

We follow the standard graph terminology and notation from [24]. In particular, if G is a graph, then its minimum degree, maximum degree, and domination number are denoted by $\delta(G)$, $\Delta(G)$, and $\gamma(G)$, respectively. Also, if $v \in V(G)$, then $N_G(v)$ and $N_G[v]$ denote the open and the closed neighborhood of v , respectively.

First observe that a nontrivial graph G satisfies $\text{diam}(G) \leq 2$ if and only if the open neighborhood $N(v)$ is a dominating set for every vertex $v \in V(G)$. Therefore, $\text{diam}(G) = 2$ implies $\gamma(G) \leq \delta(G)$. By [3, Theorem 1], we have $\gamma_g(G) \leq 2\gamma(G) - 1$, and thus $\gamma_g(G) \leq 2\delta(G) - 1$. We formulate this observation as a lemma.

Lemma 2.1 *If G is a graph of diameter 2, then $\gamma_g(G) \leq 2\delta(G) - 1$.*

Recall that if $S \subseteq V(G)$, then $G|S$ denotes a *partially dominated graph*, so a graph G , where vertices from S are already dominated. The number of moves remaining in the game on $G|S$ under optimal play when Dominator, resp. Staller, has the next move is denoted by $\gamma_g(G|S)$, resp. $\gamma'_g(G|S)$.

Lemma 2.2 *If $G = (V, E)$ is a graph of diameter 2, and $X \subseteq V$ is a non-empty set of (undominated) vertices with $|X| = x$, then the partially dominated graph $G|(V \setminus X)$ satisfies $\gamma_g(G|(V \setminus X)) \leq \lfloor \frac{2}{3}x + \frac{1}{3} \rfloor$ and $\gamma'_g(G|(V \setminus X)) \leq \lfloor \frac{2}{3}x + \frac{2}{3} \rfloor$.*

Proof. First, we claim that Dominator can play a vertex which dominates at least two new vertices in each of his moves, except maybe in the last one. Assume that it is Dominator's turn and that he cannot finish the game with a single move. Hence at least two vertices of G are not yet dominated, say u and v . Since $\text{diam}(G) = 2$, either $d(u, v) = 1$ or $d(u, v) = 2$. In the first case, Dominator can play u (or v) to dominate at least two vertices. And if $d(u, v) = 2$, then Dominator can play a common neighbor of u and v , thus again dominating at least two vertices.

Let us now consider the D-game. Assume that the game is played on $G|(V \setminus X)$, and that Dominator uses the above strategy of dominating at least two new vertices at each of his moves (except maybe in his last one), and Staller plays optimally. We distinguish the following two cases.

Case 1: The last move of the game is played by Staller.

In this case the number of moves played is even, say $2k$, $k \geq 1$. The strategy of Dominator assures that during the game at least $2k + k$ different vertices are dominated. Since in this counting the vertices are pairwise different, we infer that

$$2k + k \leq x.$$

Since Staller plays optimally, but Dominator maybe not, we can estimate the game domination number as follows:

$$\gamma_g(G|(V \setminus X)) \leq 2k \leq \left\lfloor \frac{2}{3}x \right\rfloor \leq \left\lfloor \frac{2}{3}x + \frac{1}{3} \right\rfloor.$$

Case 2: The last move of the game is played by Dominator.

Now the number of moves played is odd, say $2k + 1$. If $k = 0$, then Dominator's first move finishes the game. Thus $\gamma_g(G|(V \setminus X)) = 1 \leq \lfloor \frac{2}{3}x + \frac{1}{3} \rfloor$. So from now on, we can assume that $k \geq 1$, and hence Dominator cannot finish the game with a single move. By the strategy of Dominator, at least $2k + k + 1$ different vertices are dominated. In this sum, the last 1 corresponds to the last move of Dominator in which it is possible that he dominates only one new vertex. It follows that $3k + 1 \leq x$. Again, as Staller plays optimally but Dominator maybe not, we can estimate that

$$\gamma_g(G|(V \setminus X)) \leq 2k + 1 \leq \left\lfloor \frac{2}{3}(x - 1) + 1 \right\rfloor = \left\lfloor \frac{2}{3}x + \frac{1}{3} \right\rfloor,$$

and we are done also in this case.

Similar reasoning shows the result for the S-game. □

Corollary 2.3 *If G is a graph with $\text{diam}(G) = 2$, then*

$$\gamma_g(G) \leq \left\lfloor \frac{2}{3} \left(n(G) - \Delta(G) \right) \right\rfloor + 1.$$

Moreover, equality holds if $\Delta(G) \in \{n(G) - 1, n(G) - 2\}$.

Proof. If $\Delta(G) = n(G) - 1$, then $\gamma_g(G) = 1$, and if $\Delta(G) = n(G) - 2$, then $\gamma_g(G) = 2$. Hence in both cases the equality holds. In the rest we may thus assume that $\Delta(G) \leq n(G) - 3$.

Suppose that Dominator starts the game by playing a vertex v of degree $\Delta(G)$. After this move, we are observing an S-game on $G|N[v]$, thus, by Lemma 2.2, we have $\gamma'_g(G|N[v]) \leq \lfloor \frac{2}{3}x + \frac{2}{3} \rfloor$, where $x = n(G) - (\Delta(G) + 1)$. This immediately gives $\gamma_g(G) \leq 1 + \gamma'_g(G|N[v]) \leq 1 + \lfloor \frac{2}{3}(n(G) - \Delta(G)) \rfloor$. \square

In [3] it was observed that $\gamma_g(P) = 5$, where P is the Petersen graph. Hence the equality in Corollary 2.3 is also sharp for some graphs with $\Delta(G) < n(G) - 2$. The following result yields an infinite family of this kind of sharpness examples. Recall that vertices u and v of a graph G are *twins* if $N[u] = N[v]$.

Proposition 2.4 *If G is a twin-free graph with $\text{diam}(G) = 2$ and $\Delta(G) \in \{n(G) - 3, n(G) - 4\}$, then $\gamma_g(G) = 3$.*

Proof. Under the given conditions, Corollary 2.3 directly implies $\gamma_g(G) \leq 3$. To prove that $\gamma_g(G) \geq 3$, we need to describe an appropriate strategy of Staller. Assume that Dominator plays a vertex w as his first move. Let $Y = V(G) - N[w]$ and note that $|Y| \geq 2$. If there exist vertices $u, v \in Y$ such that $uv \notin E(G)$, then Staller can play u (or v) as her first move, forcing Dominator to play his second move. So assume that Y induces a complete subgraph of G and consider arbitrary vertices $u, v \in Y$. Since G is twin-free, $N(u) \cap N(w) \neq N(v) \cap N(w)$. Let $x \in N(w)$ be a vertex with $xu \in E(G)$ and $xv \notin E(G)$. If Staller plays x as her first move, then she again forces Dominator to play one more move. We conclude that $\gamma_g(G) \geq 3$. \square

Let G_k , $k \geq 2$, be the graph obtained from the disjoint union of $K_{1,k}$ with leaves u_1, \dots, u_k , and from K_k with vertices v_1, \dots, v_k , by adding the edges $v_i u_j$ for all $i, j \in [k]$, $i \neq j$. (We note in passing that G_k is the Mycielskian $M(K_k)$ of K_k [13, 21].) It is straightforward to check that G_k satisfies all the assumptions of Proposition 2.4, hence constituting an infinite family of equality cases in Proposition 2.4.

To see that the twin-free condition in Proposition 2.4 is needed, consider the following example. Let H_k , $k \geq 4$, be constructed as follows. Start with $K_{k,k}$, and select two adjacent vertices u and v from it. Add a vertex w and connect it to all vertices of $K_{k,k}$. Finally, add two new vertices x and y , the edge xy , and connect

both x and y to both u and v . Note that x and y are twins, $\text{diam}(H_k) = 2$, and $\Delta(H_k) = n(H_k) - 3$. If Dominator plays w as his first move, then Staller is forced to finish the game with her first move. Hence, $\gamma_g(H_k) = 2$. To have such examples of G with $\Delta(G) = n(G) - 4$, proceed similarly as above, with the only difference that instead of adding the edge xy , we add a triangle.

We conclude this section by proving another lemma, which seems rather technical, but is extremely useful. In a D-game, let U_i and U'_i denote the set of undominated vertices after the move d_i of Dominator and the move s_i of Staller, respectively. Set further $u_i = |U_i|$, $u'_i = |U'_i|$, and $S'_i = \{d_1, \dots, d_i, s_1, \dots, s_i\}$. A *greedy strategy* of Dominator means that, for every $i \geq 1$, he plays a vertex d_i that makes u_i as small as possible, that is, Dominator will always select a vertex v that maximizes $|N[v] \setminus N[S'_i]|$.

Lemma 2.5 *If G is a graph on n vertices with minimum degree δ and $i \geq 1$, then*

$$\begin{aligned} u_1 &\leq n - \delta - 1, \\ u_{i+1} &\leq u'_i \left(1 - \frac{\delta + 1}{n - 2i}\right) = u'_i \cdot \frac{n - 2i - \delta - 1}{n - 2i}, \\ u'_i &\leq u_i - 1, \end{aligned}$$

if Dominator follows a greedy strategy.

Proof. It is clear that at least $\delta + 1$ vertices become dominated with the first move d_1 of Dominator, and therefore, $u_1 \leq n - (\delta + 1)$. Since Staller must dominate at least one new vertex on each move, we also have $u'_i \leq u_i - 1$ for every $i \geq 1$.

It remains to prove the upper bound for u_{i+1} . If a vertex v is undominated after the move s_i , then $N[v] \subseteq V \setminus S'_i$. As $|N[v]| \geq \delta + 1$, each vertex from U'_i can be dominated by at least $\delta + 1$ different vertices from $V \setminus S'_i$. Since $|V \setminus S'_i| = n - 2i$, we may assume $\delta + 1 \leq n - 2i$ as otherwise the game would be over before the move d_{i+1} . A double counting argument shows that $\sum_{v \in U'_i} |N[v]| = \sum_{u \in V \setminus S'_i} |N[u] \cap U'_i|$. As $\sum_{v \in U'_i} |N[v]| \geq u'_i(\delta + 1)$ and $|V \setminus S'_i| = n - 2i$, a vertex from $V \setminus S'_i$ dominates at least

$$\frac{u'_i(\delta + 1)}{n - 2i}$$

new vertices on average. Thus, according to his greedy strategy, Dominator plays such a vertex that

$$u_{i+1} \leq u'_i - \frac{u'_i(\delta + 1)}{n - 2i} = u'_i \left(1 - \frac{\delta + 1}{n - 2i}\right),$$

which concludes the proof. □

3 One-half of the order upper bound

In this section we first prove the following result and close the section by discussing an alternative approach to its proof.

Theorem 3.1 *If G is a graph with $\text{diam}(G) = 2$, then*

$$\gamma_g(G) \leq \left\lceil \frac{n(G)}{2} \right\rceil.$$

Moreover, the equality holds if and only if G is one of the graphs from Fig. 1 or the Petersen graph.

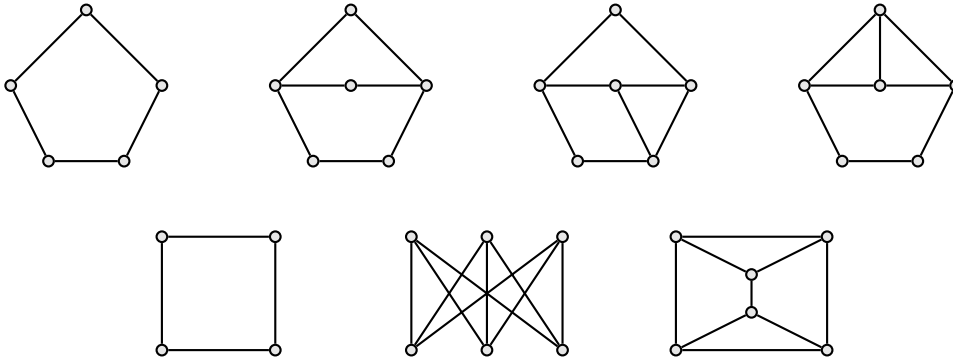


Figure 1: Sporadic graphs with $\gamma_g(G) = \lceil n(G)/2 \rceil$.

Proof. Let G be a graph with $\text{diam}(G) = 2$. Set for this proof $V = V(G)$, $n = n(G)$, $\delta = \delta(G)$, and $\Delta = \Delta(G)$.

Assume first that $\delta \geq n/4 + 1$. By Corollary 2.3, keeping in mind that $\delta \leq \Delta$, we get:

$$\begin{aligned} \gamma_g(G) &\leq \frac{2}{3}(n - \Delta) + 1 \leq \frac{2}{3}(n - n/4 - 1) + 1 \\ &= \frac{n}{2} + \frac{1}{3}. \end{aligned}$$

Since both n and $\gamma_g(G)$ are integers, we may infer $\gamma_g(G) \leq \lceil \frac{n}{2} \rceil$.

Assume next that $\delta < n/4 + 1$. By Lemma 2.1, $\gamma_g(G) < n/2 + 1$. Since both n and $\gamma_g(G)$ are integers, we get $\gamma_g(G) \leq \lceil \frac{n}{2} \rceil$. This proves the inequality.

For the equality, we have first performed a computer search over all graphs of diameter 2 and order at most 10, and found the graphs listed in the statement of the theorem. It thus remains to prove that if $n \geq 11$, then $\gamma_g(G) < \lceil n/2 \rceil$.

First assume that $\delta \leq \lfloor \frac{n+1}{4} \rfloor$. Using Lemma 2.1 again we obtain

$$\gamma_g(G) \leq 2\delta - 1 \leq 2 \left\lfloor \frac{n+1}{4} \right\rfloor - 1 < 2 \frac{n+2}{4} - 1 = \frac{n}{2} \leq \left\lceil \frac{n}{2} \right\rceil.$$

For the remaining cases, we now assume $\delta \geq \lfloor \frac{n+1}{4} \rfloor + 1 = \lfloor \frac{n+5}{4} \rfloor$. Using this bound on the results from Lemma 2.5, we have that

$$u_1 \leq n - \left\lfloor \frac{n+5}{4} \right\rfloor - 1,$$

and

$$u_{i+1} \leq (u_i - 1) \left(1 - \frac{\lfloor \frac{n+9}{4} \rfloor}{n - 2i} \right).$$

Applying these formulas for small cases, we obtain the following conclusions. If $n = 11$, then $u_2 \leq 2$ and so Staller's move s_2 leaves at most one vertex undominated. Therefore, under the greedy strategy of Dominator, the game finishes within 5 moves that is smaller than $\lceil \frac{11}{2} \rceil$. If $n = 12$ or 13 , then $u_2 \leq 3$. This gives $u'_2 \leq 2$. Since G is of diameter 2, any two vertices are adjacent or share a neighbor, they can be dominated by one move. We conclude $\gamma_g(G) \leq 5 < \lceil \frac{12}{2} \rceil < \lceil \frac{13}{2} \rceil$, thus establishing the statement for $n = 12$ and 13 . In a similar way we may show $u_2 \leq 4$, $u'_2 \leq 3$ and $u_3 \leq 1$ for $n = 14$, thus proving $\gamma_g(G) \leq 6 < \lceil \frac{14}{2} \rceil$.

From now on, we assume that $n \geq 15$ and, instead of $\delta \geq \lfloor \frac{n+5}{4} \rfloor$, we use the weaker estimation $\delta \geq \frac{n+2}{4}$. Then, $u_1 \leq n - (\delta + 1) \leq \frac{3n-6}{4}$ and $u'_1 \leq \frac{3n-10}{4}$. By these inequalities and Lemma 2.5, we get

$$u_2 \leq \frac{3n-10}{4} \left(1 - \frac{n+6}{4(n-2)} \right) = \frac{(3n-10)(3n-14)}{16(n-2)}.$$

Lemma 2.2 implies that we need at most $1 + 2u_2/3$ further moves to dominate all vertices from U_2 . Thus, Dominator can ensure that the game finishes in at most

$$4 + \frac{2}{3} \cdot \frac{(3n-10)(3n-14)}{16(n-2)}$$

moves. Now it is enough to consider the strict inequality

$$4 + \frac{2}{3} \cdot \frac{(3n-10)(3n-14)}{16(n-2)} < \frac{n}{2}.$$

Since it is equivalent to

$$0 < 3n^2 - 48n + 52,$$

which is valid for all $n \geq 15$, we may conclude that $\gamma_g(G) < n/2$ holds also for every integer $n \geq 15$. \square

One can also think about an alternative approach for proving the upper bound in Theorem 3.1 using the following two known results.

Theorem 3.2 [14, Theorem 3.4] *If G is a graph with $\text{diam}(G) = 2$, then $\gamma(G) \leq \lfloor n/4 \rfloor + 1$.*

Theorem 3.3 [20, Theorem 5] *Let G be a graph of order n and diameter 2. If $n = 4p + r$ with integers $p \geq 1$ and $0 \leq r \leq 3$, then $\gamma(G) \leq \lfloor n/4 \rfloor = p$, when $r = 0$, $p \geq 4$ or $r = 1$, $p \geq 5$, or $r \in \{2, 3\}$, $p \geq 6$.*

Since $\gamma_g(G) \leq 2\gamma(G) - 1$, see [3, Theorem 1], it follows from Theorem 3.2 that for a graph G on n vertices, where $n \not\equiv 0 \pmod{4}$, and with diameter 2, we have $\gamma_g(G) \leq \lfloor n/2 \rfloor$. The same conclusion follows from Theorem 3.3 for $n \equiv 0 \pmod{4}$ and $n \geq 16$. The remaining cases of graphs on 4, 8, and 12 vertices could then be handled by computer. As stated in the proof of Theorem 3.1, we have done this for graphs of order at most 10. However, the computation for all diameter 2 graphs on 12 vertices would require a lot of computer time, hence we did not do it.

4 A stronger upper bound

In this section we improve Theorem 3.1 as follows.

Theorem 4.1 *If G is a graph with $\text{diam}(G) = 2$, then*

$$\gamma_g(G) \leq \left\lceil \frac{n(G)}{2} \right\rceil - \left\lfloor \frac{n(G)}{11} \right\rfloor. \quad (1)$$

Proof. Let G be a graph of diameter 2 and order $n = n(G)$. First recall that, by Theorem 3.1, we have $\gamma_g(G) \leq \lfloor n/2 \rfloor$ and consequently, (1) holds if $n < 11$. By the same theorem, the strict inequality $\gamma_g(G) < \lfloor n/2 \rfloor$ is valid whenever $n \geq 11$. The latter implies that (1) is true for every graph G with $11 \leq n \leq 21$. Therefore, in the following, we assume $n \geq 22$.

The general result [5, Theorem 4] directly implies that, in the case of $\delta(G) \geq 11$, we have $\gamma_g(G) < 0.404n$. Since

$$0.404n < \frac{9}{22}n = \frac{n}{2} - \frac{n}{11} \leq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{11} \right\rfloor,$$

we may infer that the theorem holds if $\delta(G) \geq 11$.

Assume now that $\delta(G) \leq 5$. As G is a graph of diameter 2, we have $\gamma_g(G) \leq 2\delta(G) - 1 \leq 9$ by Lemma 2.1. On the other hand, $9 \leq 9n/22 \leq \lfloor n/2 \rfloor - \lfloor n/11 \rfloor$ holds under the condition $n \geq 22$, thus proving the theorem for the case of $\delta(G) \leq 5$.

If $\delta(G) = 6$ and $n \geq 27$, we apply the inequalities $\gamma_g(G) \leq 2\delta(G) - 1$ and $9n/22 > 11$, which results in

$$\gamma_g(G) \leq 2\delta(G) - 1 = 11 < \frac{9}{22}n \leq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{11} \right\rfloor.$$

In addition, one can check that $11 = \lceil n/2 \rceil - \lfloor n/11 \rfloor$ holds, and therefore (1) is valid for $n = 25$ and $n = 26$. The remaining cases are $n = 22, 23$ and 24 . Now, Lemma 2.5 moves in. Namely, following the greedy strategy, we observe that Dominator dominates at least $\delta(G) + 1 = 7$ vertices with his first move and therefore, $u_1 \leq n - 7$ and $u'_1 \leq n - 8$. Before the second move of Dominator, there is a vertex that dominates at least $7u'_1/(n - 2)$ vertices. By playing such a vertex, he achieves

$$u_2 \leq u'_1 \left(1 - \frac{7}{n-2} \right) \leq \frac{(n-8)(n-9)}{n-2}.$$

After the second move of Staller, we then have $u'_2 \leq u_2 - 1$ and may calculate that the number of remaining moves is at most $\lfloor \frac{2}{3}u'_2 + \frac{1}{3} \rfloor$ by Lemma 2.2. Thus we conclude

$$\gamma_g(G) \leq 4 + \left\lfloor \frac{2}{3} \left(\frac{(n-8)(n-9)}{n-2} - 1 \right) + \frac{1}{3} \right\rfloor.$$

The right-hand side formula equals 9, 10, 10, respectively, for $n = 22, 23, 24$, which are exactly the corresponding values of $\lceil n/2 \rceil - \lfloor n/11 \rfloor$. This finishes the proof for $\delta(G) = 6$.

Assuming that $\delta(G) = 7$, we give a similar reasoning as for $\delta(G) = 6$. If $n \geq 32$, then

$$\gamma_g(G) \leq 2\delta(G) - 1 = 13 < \frac{9}{22}n \leq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{11} \right\rfloor.$$

If $29 \leq n \leq 31$, the inequality $13 \leq \lceil n/2 \rceil - \lfloor n/11 \rfloor$ is still valid and so (1) is true. Assume finally that $22 \leq n \leq 28$. It is clear that $u_1 \leq n - 8$ and $u'_1 \leq n - 9$. Applying Lemma 2.5, we get

$$u_2 \leq u'_1 \left(1 - \frac{8}{n-2} \right) \leq \frac{(n-9)(n-10)}{n-2}.$$

Then, by Lemma 2.2,

$$\gamma_g(G) \leq 4 + \left\lfloor \frac{2}{3} \left(\frac{(n-9)(n-10)}{n-2} - 1 \right) + \frac{1}{3} \right\rfloor \leq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{11} \right\rfloor,$$

where the last estimation can be verified by calculating the values for each integer $22 \leq n \leq 28$. This completes the proof for $\delta(G) = 7$.

For the next case, suppose $\delta(G) = 8$. If $n \geq 37$, then

$$\gamma_g(G) \leq 2\delta(G) - 1 = 15 < \frac{9}{22}n \leq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{11} \right\rfloor$$

which proves (1). If $n = 35$ or $n = 36$, then $15 = \lceil n/2 \rceil - \lfloor n/11 \rfloor$ holds and (1) is true. Suppose now that $22 \leq n \leq 33$. By Lemma 2.5, $u_1 \leq n - 9$ and $u'_1 \leq n - 10$ hold. Moreover, Dominator can ensure that

$$u_2 \leq u'_1 \left(1 - \frac{9}{n-2} \right) \leq \frac{(n-10)(n-11)}{n-2}.$$

Then, taking into account Lemma 2.2, we get

$$\gamma_g(G) \leq 4 + \left\lfloor \frac{2}{3} \left(\frac{(n-10)(n-11)}{n-2} - 1 \right) + \frac{1}{3} \right\rfloor \leq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{11} \right\rfloor$$

by checking the last inequality for each integer between 22 and 33. The only remaining case for $\delta(G) = 8$ is thus $n = 34$. Here, we get $u'_2 \leq \lfloor \frac{24 \cdot 23}{32} - 1 \rfloor = 16$ and continue the process with estimating u_3 by using Lemma 2.5 again. This yields

$$u_3 \leq \left\lfloor u'_2 \left(1 - \frac{9}{30} \right) \right\rfloor \leq \left\lfloor \frac{16 \cdot 21}{30} \right\rfloor = 11$$

and $u'_3 \leq 10$. From this point, by Lemma 2.2, Dominator can ensure that the game finishes within seven moves. This establishes $\gamma_g(G) \leq 13 < \lceil 34/2 \rceil - \lfloor 34/11 \rfloor = 14$ and completes the proof for $\delta(G) = 8$.

If $\delta(G) = 9$ and $n \geq 42$, we have $\gamma_g(G) \leq 2\delta(G) - 1 = 17 < \frac{9}{22}n$, which implies inequality (1). For $n = 39, 40, 41$, simple calculation shows $17 \leq \lceil n/2 \rceil - \lfloor n/11 \rfloor$ and we infer $\gamma_g(G) \leq \lceil n/2 \rceil - \lfloor n/11 \rfloor$ again. To prove the theorem for the remaining cases, $22 \leq n \leq 38$, we first observe that $u_1 \leq n - 10$ and $u'_1 \leq n - 11$. Then, by playing greedily, Dominator can ensure (Lemma 2.5)

$$u_2 \leq u'_1 \left(1 - \frac{10}{n-2} \right) \leq \frac{(n-11)(n-12)}{n-2}$$

and, by Lemma 2.2, we get

$$\gamma_g(G) \leq 4 + \left\lfloor \frac{2}{3} \left(\frac{(n-11)(n-12)}{n-2} - 1 \right) + \frac{1}{3} \right\rfloor.$$

For each integer $22 \leq n \leq 38$, the value of the right-hand side formula is bounded from above by $\lceil \frac{n}{2} \rceil - \lfloor \frac{n}{11} \rfloor$. This completes the proof of (1) for $\delta(G) = 9$.

The last case we have to consider is $\delta(G) = 10$. If $n \geq 47$, then $\gamma_g(G) \leq 2\delta(G) - 1 = 19 < \frac{9}{22}n$ holds, thus proving (1). For $n = 45$ and $n = 46$, the inequality $\gamma_g(G) \leq 19 = \lceil n/2 \rceil - \lfloor n/11 \rfloor$ holds, thus implying the statement. If $22 \leq n \leq 44$, we consider the greedy strategy of Dominator which, by Lemma 2.5, results in $u_1 \leq n - 11$, $u'_1 \leq n - 12$,

$$u_2 \leq u'_1 \left(1 - \frac{11}{n-2} \right) \leq \frac{(n-12)(n-13)}{n-2}.$$

Then, for $22 \leq n \leq 43$, we can estimate (Lemma 2.2) the length of the game as

$$\gamma_g(G) \leq 4 + \left\lfloor \frac{2}{3} \left(\frac{(n-12)(n-13)}{n-2} - 1 \right) + \frac{1}{3} \right\rfloor \leq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{11} \right\rfloor,$$

where the last inequality can be easily checked for each integer n in the interval $[22, 43]$. If $n = 44$, the previous argumentation gives $u_2 \leq 32 \cdot 31/42$. Since u_2 is an integer, we have $u_2 \leq 23$ and $u'_2 \leq 22$. It follows from Lemma 2.5 that

$$u_3 \leq \left\lfloor u'_2 \left(1 - \frac{11}{40} \right) \right\rfloor \leq \left\lfloor \frac{22 \cdot 29}{40} \right\rfloor = 15,$$

and so, by Lemma 2.2, the game will be finished in at most 10 additional moves. Thus, we conclude $\gamma_g(G) \leq 15$, which implies $\gamma_g(G) < \lceil 44/2 \rceil - \lfloor 44/11 \rfloor = 18$. This completes the proof of Theorem 4.1. \square

The bound in Theorem 4.1 is attained. In fact, there are exactly 10 graphs on 11 vertices with the game domination number equal to 5, see Figure 2. They were obtained using a computer. Let G be a graph with $n(G) = 11$. If $\delta(G) \leq 2$, then Lemma 2.1 implies $\gamma_g(G) \leq 3$. And if $\Delta(G) \geq 6$, then Corollary 2.3 yields $\gamma_g(G) \leq 4$. Hence, in our computer search we only had to check the connected graphs G on 11 vertices with $\delta(G) \geq 3$ and $\Delta(G) \leq 5$.

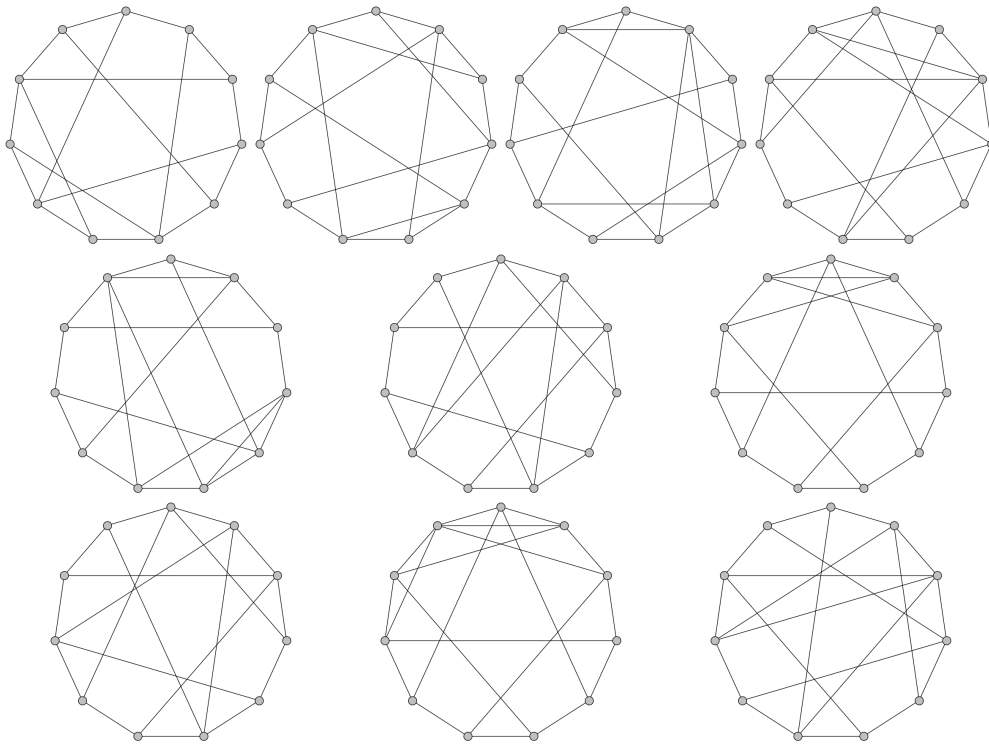


Figure 2: Graphs on 11 vertices with diameter 2 and game domination number 5.

5 Concluding remarks

Note also that the upper bound in Theorem 3.1 is asymptotically not tight, and for big enough n it follows from known upper bounds on (total) domination number, see for example [1, 10, 12]. In particular, the strongest known result asserts that $\gamma_t(G) < \sqrt{\frac{n \log n}{2}} + \sqrt{\frac{n}{2}}$ for all graphs G of diameter 2 and $n \geq 3$ vertices [12, Theorem 1]. Using the well-known bounds $\gamma_g(G) \leq 2\gamma(G) - 1$ and $\gamma(G) \leq \gamma_t(G)$, this yields

$$\gamma_g(G) \leq 2 \left\lfloor \sqrt{\frac{n \log n}{2}} + \sqrt{\frac{n}{2}} \right\rfloor - 1.$$

The latter value is smaller than $\lceil \frac{n}{2} \rceil$ for all $n \geq 65$, and smaller than $\lceil \frac{n}{2} \rceil - \lfloor \frac{n}{11} \rfloor$ for all $n \geq 111$.

Recall that for any fixed positive real number $p < 1$ (which is the probability with which the edges of a random graph are selected mutually independently), almost all graphs are connected with diameter 2, cf. [9, Theorem 13.6]. Hence Theorem 4.1 (or Theorem 3.1 for that matter) imply that

$$\gamma_g(G) < \frac{n(G)}{2}$$

holds for almost all graphs G .

Theorem 3.1 and/or Theorem 4.1 offer another support for Rall's 1/2-conjecture. That is, the conjecture holds for all graphs with diameter 2 and consequently for almost all graphs. In this direction, we have tried a different approach than in [8], and proved, using a computer, that the 1/2-conjecture holds for all Hamiltonian graphs on $n \leq 10$ vertices. Here, we have made use of [5, Corollary 5(ii)] to avoid checking graphs with $\delta(G) \geq 5$.

Acknowledgements

We are grateful to Gašper Košmrlj for providing us with his software that computes game domination invariants. We acknowledge the financial support from the Slovenian Research Agency (research core funding No. P1-0297 and projects J1-9109, J1-1693, N1-0095, N1-0108). Kexiang Xu is also supported by NNSF of China (grant No. 11671202) and China-Slovene bilateral grant 12-9.

References

- [1] S.M. Al-Yakoob, Zs. Tuza, Domination number of graphs with bounded diameter, *J. Combin. Math. Combin. Comput.* 40 (2002) 183–191.

- [2] M. Borowiecki, A. Fiedorowicz, E. Sidorowicz, Connected domination game, *Appl. Anal. Discrete Math.* 13 (2019) 261–289.
- [3] B. Brešar, S. Klavžar, D.F. Rall, Domination game and an imagination strategy, *SIAM J. Discrete Math.* 24 (2010) 979–991.
- [4] B. Brešar, Cs. Bujtás, T. Gologranc, S. Klavžar, G. Košmrlj, T. Marc, B. Patkós Zs. Tuza, M. Vizer, The variety of domination games, *Aequationes Math.* 93 (2019) 1085–1109.
- [5] Cs. Bujtás, On the game domination number of graphs with given minimum degree, *Electron. J. Combin.* 22 (2015) Paper 3.29, 18 pp.
- [6] Cs. Bujtás, Zs. Tuza, Fractional domination game, *Electron. J. Combin.* 26 (2019) Paper 4.3, 17 pp.
- [7] Cs. Bujtás, General upper bounds on the game domination number, *Discrete Appl. Math.* 285 (2020) 530–538.
- [8] Cs. Bujtás, V. Iršič, S. Klavžar, K. Xu, On Rall’s $1/2$ -conjecture on the domination game, *Quaest. Math.* (2020) DOI: 10.2989/16073606.2020.1822945.
- [9] G. Chartrand, L. Lesniak, *Graphs & Digraphs. Fourth Edition*, Chapman & Hall/CRC, Boca Raton, FL, 2005.
- [10] W.J. Desormeaux, T.W. Haynes, M.A. Henning, A. Yeo, Total domination in graphs with diameter 2, *J. Graph Theory* 75 (2014) 91–103.
- [11] P. Dorbec, G. Košmrlj, G. Renault, The domination game played on unions of graphs, *Discrete Math.* 338 (2015) 71–79.
- [12] A. Dubickas, Graphs with Diameter 2 and Large Total Domination Number, *Graphs Combin.* 37 (2021) 271–279.
- [13] D.C. Fisher, P.A. McKenna, E.D. Boyer, Biclique parameters of Mycielskians, *Congr. Numer.* 111 (1995) 136–142.
- [14] A. Hellwig, L. Volkmann, Some upper bounds for the domination number, *J. Combin. Math. Combin. Comput.* 57 (2006) 187–209.
- [15] M.A. Henning, W.B. Kinnersley, Domination game: a proof of the $3/5$ -conjecture for graphs with minimum degree at least two, *SIAM J. Discrete Math.* 30 (2016) 20–35.
- [16] M.A. Henning, S. Klavžar, D.F. Rall, The $4/5$ upper bound on the game total domination number, *Combinatorica* 37 (2017) 223–251.

- [17] Y. Jiang, M. Lu, Game total domination for cyclic bipartite graphs, *Discrete Appl. Math.* 265 (2019) 120–127.
- [18] T. James, S. Klavžar, A. Vijayakumar, The domination game on split graphs, *Bull. Aust. Math. Soc.* 99 (2019) 327–337.
- [19] W.B. Kinnersley, D.B. West, R. Zamani, Extremal problems for game domination number, *SIAM J. Discrete Math.* 27 (2013) 2090–2107.
- [20] D. Meierling, L. Volkmann, Upper bounds for the domination number in graphs of diameter two, *Util. Math.* 93 (2014) 267–277.
- [21] J. Mycielski, Sur le coloriage des graphes, *Colloq. Math.* 3 (1955) 161–162.
- [22] M. J. Nadjafi-Arani, M. Siggers, H. Soltani, Characterisation of forests with trivial game domination numbers, *J. Comb. Optim.* 32 (2016) 800–811.
- [23] W. Ruksasakchai, K. Onphaeng, C. Worawannotai, Game domination numbers of a disjoint union of paths and cycles, *Quaest. Math.* 42 (2019) 1357–1372.
- [24] D.B. West, *Introduction to Graph Theory*, 2nd ed., Prentice-Hall, NJ, 2001.
- [25] K. Xu, X. Li, On domination game stable graphs and domination game edge-critical graphs, *Discrete Appl. Math.* 250 (2018) 47–56.
- [26] K. Xu, X. Li, S. Klavžar, On graphs with largest possible game domination number, *Discrete Math.* 341 (2018) 1768–1777.