CHROMATIC NUMBER OF THE PRODUCT OF GRAPHS, GRAPH HOMOMORPHISMS, ANTICHAINS AND COFINAL SUBSETS OF POSETS WITHOUT AC

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Abstract. In set theory without the Axiom of Choice (AC), we observe new relations of the following statements with weak choice principles.

- If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.
- If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.
- CS (Every partially ordered set without a maximal element has two disjoint cofinal subsets).
- CWF (Every partially ordered set has a cofinal well-founded subset).
- DT (Dilworth's decomposition theorem for infinite p.o.sets of finite width).

We also study a graph homomorphism problem and a problem due to András Hajnal without AC. Further, we study a few statements restricted to linearly-ordered structures without AC.

1. Introduction

Firstly, the first author observes the following in ZFA (Zermelo-Fraenkel set theory with atoms).

(1) In **Problem 15, Chapter 11 of** [KT06], applying Zorn's lemma, Komjáth and Totik proved the statement "Every partially ordered set without a maximal element has two disjoint cofinal subsets" (CS). In **Theorem 3.26 of** [THS16], Tachtsis, Howard and Saveliev proved that CS does not imply 'there are no amorphous sets' in ZFA. We observe that CS $\not\rightarrow AC^{\omega}_{fin}$ (the axiom of choice for countably infinite familes of non-empty finite sets), CS $\not\rightarrow AC^{-}_{n}$ ('Every infinite family of n-element sets has a partial choice function') [1] for every $2 \leq n < \omega$ and CS $\not\rightarrow LOKW^{-}_{4}$ (Every infinite linearly orderable family $\mathcal{A}$ of 4-element sets has a partial Kinna–Wegner selection function)[2] in ZFA.

(2) In **Problem 14, Chapter 11 of** [KT06], applying the well-ordering theorem, Komjáth and Totik proved the statement "Every partially ordered set has a cofinal well-founded subset" (CWF). In **Theorem 10(ii) of** [Tac17], Tachtsis proved that CWF holds in the basic Fraenkel model. Moreover, in **Lemma 5 of** [Tac17], Tachtsis proved that CWF is equivalent to AC in ZF. We observe that CWF $\not\rightarrow AC^{\omega}_{fin}$, CWF $\not\rightarrow AC^{-}_{n}$ for every $2 \leq n < \omega$ and CWF $\not\rightarrow LOKW^{-}_{4}$ in ZFA.

(3) In **Problem 7, Chapter 11 of** [KT06], applying Zorn's lemma, Komjáth and Totik proved that if in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable. We observe that *'If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable'* $\not\rightarrow AC^{-}_{n} (\forall n \geq 2)$, $\not\rightarrow$ 'There are no amorphous sets' in ZFA which are new results. Moreover, we prove that *'For any regular $\aleph_\alpha$, if in a partially ordered set, all chains are finite and all*

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[1] It is easy to see that AC^{-}_{n} follows from AC_n (Axiom of choice for n-element sets).

[2] We denote the principle 'Every infinite linearly orderable family $\mathcal{A}$ of n-element sets has a partial Kinna–Wegner selection function' by $LOKW^{-}_{n}$ (see [HT19]).

antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$' $\not\rightarrow AC_n^-(\forall n \geq 2)$, $\not\rightarrow$ 'There are no amorphous sets' in ZFA.

(4) Dilworth [Dil50] proved the following statement: *'If $\mathbb{P}$ is an arbitrary p.o.set, and k is a natural number such that $\mathbb{P}$ has no antichains of size $k+1$ while at least one k-element subset of $\mathbb{P}$ is an antichain, then $\mathbb{P}$ can be partitioned into k chains'*, we abbreviate by DT (see **Problem 4, Chapter 11 of** [KT06] also). Tachtsis [Tac19] investigated the possible placement of DT in the hierarchy of weak choice principles. He proved that DT does not imply AC^ω_{fin} as well as AC_2 (Every family of pairs has a choice function). We observe that DT does not imply AC_n^- for any $2 \leq n < \omega$ in ZFA. In particular, we observe that DT holds in the permutation model of **Theorem 8** of [HT19], due to Halbeisen and Tachtsis. We also observe that a weaker form of Łoś's lemma (Form 253 of [HR98]) fails in the permutation model of **Theorem 8** of [HT19].

(5) In **Theorem 4.5.2** of [Kom], Komjáth sketched the following generalization of the n-coloring theorem (For every graph $G = (V, E)$ if every finite subgraph of G is n-colorable then G is n-colorable) applying the Boolean prime ideal theorem (BPI): *'For an infinite graph $G = (V_G, E_G)$ and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G'* we abbreviate by $\mathcal{P}_{G,H}$. We observe that if $X \in \{AC_3, AC^\omega_{fin}\}$, then $\mathcal{P}_{G,H}$ restricted to finite graph H with 2 vertices does not imply X in ZFA.

Secondly, we study a weaker formulation of a problem due to András Hajnal in ZFA.

(1) In **Theorem 2** of [Haj85], Hajnal proved that if the chromatic number of a graph G_1 is finite (say $k < \omega$), and the chromatic number of another graph G_2 is infinite, then the chromatic number of $G_1 \times G_2$ is k using the Gödel's Compactness theorem. In the solution of **Problem 12, Chapter 23 of** [KT06], Komjáth provided another argument using the Ultrafilter lemma. For a natural number $k < \omega$, we denote by $\mathcal{P}_k$ the following statement.

$$\text{`}\chi(E_{G_1}) = k < \omega \textit{ and } \chi(E_{G_2}) \geq \omega \textit{ implies } \chi(E_{G_1 \times G_2}) = k.\text{'}$$

We observe that if $X \in \{AC_3, AC^\omega_{fin}\}$, then $\mathcal{P}_k \not\rightarrow X$ in ZFA when $k = 3$.

Lastly, we study a few algebraic and graph-theoretic statements restricted to linearly-ordered structures without AC. We abbreviate the statement 'The union of a well-orderable family of finite sets is well-orderable' by $UT(WO, fin, WO)$. In **Theorem 3.1 (i)** of [Tac19], Tachtsis proved DT for well-ordered infinite p.o.sets with finite width in ZF applying the following theorem.

Theorem 1.1. *(**Theorem 1** of* [Loeb65]*). Let $\{X_i\}_{i\in I}$ be a family of compact spaces which is indexed by a set I on which there is a well-ordering $\leq$. If I is an infinite set and there is a choice function F on the collection $\{C : C$ is closed, $C \neq \emptyset, C \subset X_i$ for some $i \in I\}$, then the product space $\Pi_{i\in I} X_i$ is compact in the product topology.*

Using the same technique from **Theorem 3.1** of [Tac19], we prove a few algebraic and graph-theoretic statements restricted to well-ordered sets, either in ZF or in ZF + $UT(WO, fin, WO)$. Consequently, those statements restricted to linearly ordered sets are true, in permutation models where LW (Every linearly ordered set can be well-ordered) holds. In particular, we observe the following.

(1) In **Theorem 18 of** [HT13], Howard and Tachtsis obtained that for every finite field $\mathcal{F} = \langle F, ...\rangle$, for every nontrivial vector space V over $\mathcal{F}$, there exists a non-zero linear functional $f : V \rightarrow F$ applying BPI. Fix an arbitrary $2 \leq n < \omega$. We observe that '*For every finite field $\mathcal{F} = \langle F, ...\rangle$, for every nontrivial linearly-ordered vector space V over $\mathcal{F}$, there exists a non-zero linear functional $f : V \rightarrow F$*' $\not\rightarrow AC^\omega_{fin}$, $\not\rightarrow LOKW_4^-$, and $\not\rightarrow AC_n^-$ in ZFA.

(2) Fix an arbitrary $2 \leq n < \omega$. We observe that '*For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite*

subgraph of G has a homomorphism into H, then so has G' $\not\rightarrow AC^{\omega}_{fin}$, $\not\rightarrow LOKW^{-}_{4}$, and $\not\rightarrow AC^{-}_{n}$ in ZFA.

(3) Fix an arbitrary $2 \leq n < \omega$. We prove that for every $3 \leq k < \omega$, the statement '$\mathcal{P}_k$ if the graph G_1 is on some linearly-orderable set of vertices' $\not\rightarrow AC^{\omega}_{fin}$, $\not\rightarrow LOKW^{-}_{4}$, and $\not\rightarrow AC^{-}_{n}$ in ZFA.

(4) Marshall Hall [Hal48] proved that if S is a set and $\{S_i\}_{i\in I}$ is an indexed family of **finite** subsets of S, then if the following property holds,

(P) for every finite $F \subseteq I$, there is an injective choice function for $\{S_i\}_{i\in F}$.

then there is an injective choice function for $\{S_i\}_{i\in I}$. We abbreviate the above assertion by MHT. We recall that BPI implies MHT and MHT implies the Axiom of choice for finite sets (AC_{fin}) in ZF (c.f. [HR98]). Fix an arbitrary $2 \leq n < \omega$. We prove that MHT restricted to a linearly-ordered collection of finite subsets of a set does not imply AC^{-}_{n} in ZFA.

2. A List of forms and definitions

(1) The ***Axiom of Choice*, AC (Form 1 in** [HR98]**)**: Every family of nonempty sets has a choice function.

(2) The ***Axiom of Choice for Finite Sets*, AC_n (Form 62 in** [HR98]**)**: Every family of non-empty nite sets has a choice function.

(3) AC_2 **(Form 88 in** [HR98]**)**: Every family of pairs has a choice function.

(4) AC_n **for each** $n \in \omega, n \geq 2$ **(Form 61 in** [HR98]**)**: Every family of n element sets has a choice function. We denote by AC^{-}_{n} the statement *'Every infinite family of n-element sets has a partial choice function' (**Form 342(n)** in* [HR98]*, denoted by* C^{-}_{n} *in **Definition 1 (2)** of* [HT19]*).* We denote by $LOKW^{-}_{n}$ the statement *'Every infinite linearly orderable family $\mathcal{A}$ of n-element sets has a partial Kinna–Wegner selection function' (c.f. **Definition 1 (2)** of* [HT19]*).*

(5) AC^{ω}_{n} **(Form 10 in** [HR98]**)**: Every countably innite family of non-empty nite sets has a choice function. We denote by PAC^{ω}_{fin} the statement *'Every countably innite family of non-empty nite sets has a partial choice function'.*

(6) The ***Principle of Dependent Choice*, DC (Form 43 in** [HR98]**)**: If S is a relation on a non-empty set A and $(\forall x \in A)(\exists y \in A)(xSy)$ then there is a sequence $a_0, a_1, ...$ of elements of A such that $(\forall n \in \omega)(a(n)Sa(n+1))$.

(7) **LW (Form 90 in** [HR98]**)**: Every linearly-ordered set can be well-ordered.

(8) **UT(WO, WO, WO) (Form 231 in** [HR98]**)**: The union of a well-ordered collection of well-orderable sets is well-orderable.

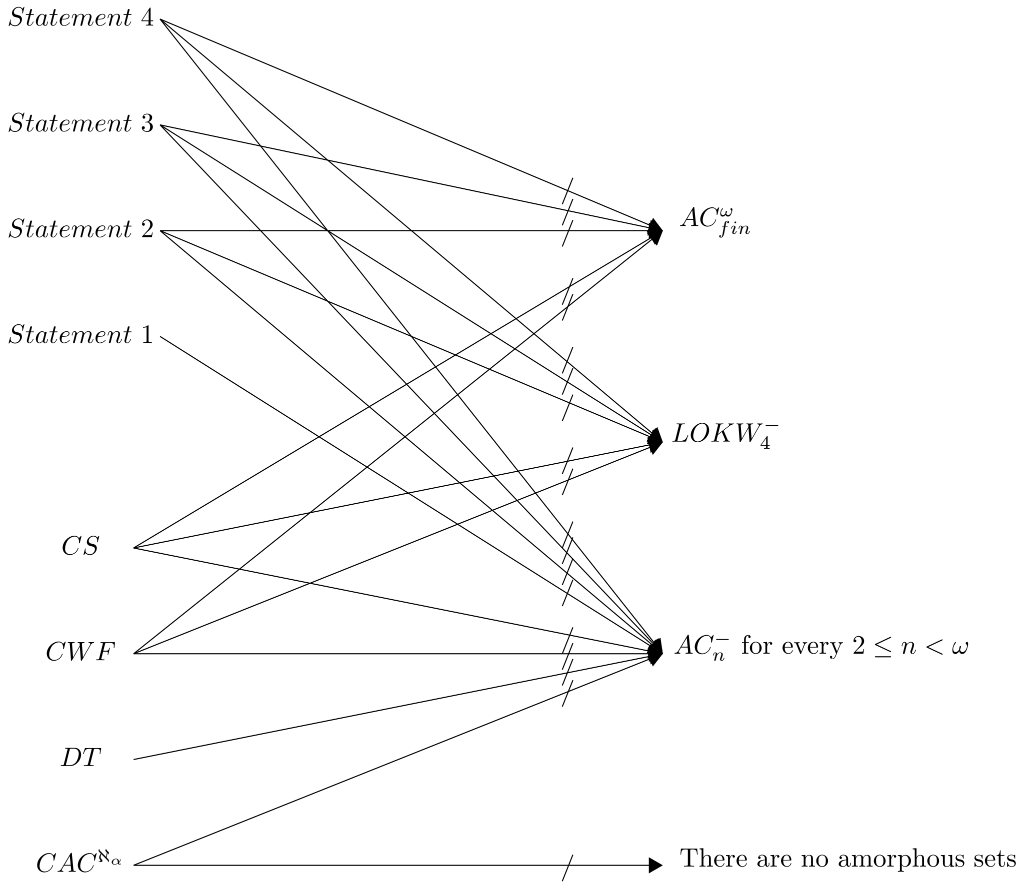
(9) **UT(WO, fin, WO) (Form 10N in** [HR98]**)**: The union of a well-orderable family of finite sets is well-orderable.

(10) $(\forall\alpha)UT(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ **(Form 23 in** [HR98]**)**: For every ordinal α, if A and every member of A has cardinality $\aleph_\alpha$, then $|\cup A| = \aleph_\alpha$.

(11) **UT($\aleph_0$, fin, $\aleph_0$) (Form 10A in** [HR98]**)**: The union of a denumerable collection of finite sets is countable.

(12) The ***Boolean Prime Ideal Theorem*, BPI (Form 14 in** [HR98]**)**: Every Boolean algebra has a prime ideal. We recall the following equivalent formulations of BPI.
   - **(Form 14AW in** [HR98]**)**: The Compactness theorem for propositional logic.
   - The ***Ultrafilter lemma*, UL (Form 14A in** [HR98]**)**: Every proper filter over a set S in $\mathcal{P}(S)$ can be extended to an ultrafilter.
   - The ***n-coloring theorem for*** $n \geq 3$, **(Form 14G(n)($n \in \omega, n \geq 3$) in** [HR98]**)**: For every graph $G = (V, E)$ if every finite subgraph of G is n-colorable then G is n-colorable. This is De Bruijn–Erdős theorem for $n \geq 3$ colorings.

(13) The ***Principle of consistent choice*, PCC (Form 14AH in** [HR98]**)**: Let $\mathcal{A} = \{A_i\}_{i\in I}$ be a family of finite sets and $\mathcal{R}$ is a symmetric binary relation on $\cup_{i\in I} A_i$. Suppose that for every finite $W \subset I$, there is an $\mathcal{R}$-consistent choice function for $\{A_i\}_{i\in W}$, then there is an $\mathcal{R}$-consistent choice function for $\{A_i\}_{i\in I}$.



$\mathcal{P}_{G,H}$ restricted to finite graph H with 2 vertices, $\mathcal{P}_3 \not\longrightarrow AC_3, AC^{\omega}_{fin}$

Figure 1. *In the above figure, we sketch the results of this note in ZFA. For each regular $\aleph_\alpha$, we denote by $CAC^{\aleph_\alpha}$ the statement 'if in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$'. We use* ***Statement 4*** *to denote '$\mathcal{P}_k$ for the graph G_1 on some linearly-orderable set of vertices' for a natural number $k \geq 3$. We use* ***Statement 3*** *to denote 'For every finite field $\mathcal{F} = \langle F, ...\rangle$, for every nontrivial linearly-ordered vector space V over $\mathcal{F}$, there exists a non-zero linear functional $f : V \rightarrow F$'. We use* ***Statement 2*** *to denote 'For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G'. We use* ***Statement 1*** *to denote Marshall Hall's theorem for linearly-ordered collection of finite subsets of a set.*

We note that Form 14AH in [HR98] is different than the above formulation. Łoś/Ryll-Nardzewski [LN51] introduced both the formulations where it was noted that they are equivalent. Let $n \in \omega \backslash \{0, 1\}$. We recall the notation F_n introduced by Cowen in [Cow77], which is PCC restricted to families $\mathcal{A} = \{A_i : i \in I\}$, where $|A_i| \leq n$ for all $i \in I$.

(14) ***Marshall Hall's theorem*, MHT (Form 107 in** [HR98]**)**: If S is a set and $\{S_i\}_{i\in I}$ is an indexed family of **finite** subsets of S, then if the following property holds,

(P) for every finite $F \subseteq I$, there is an injective choice function for $\{S_i\}_{i\in F}$.

then there is an injective choice function for $\{S_i\}_{i\in I}$.

Philip Hall's theorem states that the property (P) is equivalent to the Hall's condition which states that '$\forall F \in [I]^{<\omega}, |\cup_{i\in F} S_i| \geq |F|$'. We recall that Philip Hall's theorem or finite Hall's theorem can be proved in ZF without using any choice principles.

(15) **A weaker form of Łoś's lemma, LT (Form 253 in** [HR98]**)**: If $\mathcal{A} = \langle A, \mathcal{R}^{\mathcal{A}} \rangle$ is a non-trivial relational $\mathcal{L}$-structure over some language $\mathcal{L}$, and $\mathcal{U}$ be an ultrafilter on a non-empty set I, then the ultrapower $\mathcal{A}^I/\mathcal{U}$ and $\mathcal{A}$ are elementarily equivalent.

(16) **MCC** (c.f. **Definition 5** and **Definition 6** of [Tac17]): Every topological space with the minimal cover property is compact.

(17) **Bounded and unbounded amorphous sets**: An innite set X is called *amorphous* if X cannot be written as a disjoint union of two innite subsets. There are two types of amorphous sets, namely bounded amorphous sets and unbounded amorphous sets. Let $\mathcal{U}$ be a finitary partition of an amorphous set X. Then all but finitely many elements of $\mathcal{U}$ have the same cardinality, say $n(\mathcal{U})$. Let $\Pi(X)$ be the set of all finitary partitions of X and $n(X) = sup\{n(\mathcal{U}) : \mathcal{U} \in \Pi(X)\}$. If $n(X)$ is finite, then X is called *bounded amorphous* and if $n(X)$ is infinite, then X is called *unbounded amorphous*. We recall **Theorem 6 of** [Tac17] which states that MCC $\rightarrow$ "there are no bounded amorphous sets".

(18) **(Form 64 in** [HR98]**)**: There are no amorphous sets.

(19) **Martin's Axiom (c.f.** [Tac16b]**)**: If κ is a well-ordered cardinal, we denote by $MA(\kappa)$ the principle 'If $(P,<)$ is a nonempty, c.c.c. quasi order and $\mathcal{D}$ is a family of $\leq \kappa$ dense sets in P, then there is a filter $\mathcal{F}$ of P such that $\mathcal{F} \cap D \neq \emptyset$ for all $D \in \mathcal{D}$'. We recall from **Remark 2.7** of [Tac16b] that $AC^{\omega}_{fin} + MA(\aleph_0) \rightarrow$ 'for every infinite set X, 2^X is Baire' and 'for every infinite set X, 2^X is Baire' $\rightarrow$ 'there are no amorphous sets'.

(20) ***Dilworth's decomposition theorem* for infinite p.o.sets of finite width, DT (c.f.** [Tac19]**)**: If $\mathbb{P}$ is an arbitrary p.o.set, and k is a natural number such that $\mathbb{P}$ has no antichains of size $k+1$ while at least one k-element subset of $\mathbb{P}$ is an antichain, then $\mathbb{P}$ can be partitioned into k chains. We abbreviate the above formulation as DT. We recall **Theorem 3.1(i)** of [Tac19], which states that DT for well-ordered infinite p.o.sets with finite width is provable in ZF.

(21) The ***Chain/Antichain Principle*, CAC (Form 217 in** [HR98]**)**: Every infinite p.o.set has an infinite chain or an infinite antichain. We recall that CAC implies AC^{ω}_{fin} from **Lemma 4.4** of [Tac19a].

(22) **CS** (c.f. [THS16]): Every partially ordered set without a maximal element has two disjoint cofinal subsets.

(23) **CWF** (c.f. **Definition 6 (11)** of [Tac17]): Every partially ordered set has a cofinal well-founded subset.

(24) **Chromatic number of the product of graphs:** We recall a few basic terminologies of graphs. An *independent set* is a set of vertices in a graph, no two of which are connected by an edge. A *good coloring* of a graph $G = (V_G, E_G)$ with a color set C is a mapping $f : V_G \rightarrow C$ such that for every $\{x, y\} \in E_G$, $f(x) \neq f(y)$. The *chromatic number* $\chi(E_G)$ of a graph $G = (V_G, E_G)$ is the smallest cardinal κ such that the graph G can be colored by κ colors. We define the cartesian product of two graphs $G_1 = (V_{G_1}, E_{G_1})$ and $G_2 = (V_{G_2}, E_{G_2})$ as the graph $G_1 \times G_2 = (V_{G_1\times G_2}, E_{G_1\times G_2}) = (V_{G_1} \times V_{G_2}, \{\{(x_0, x_1), (y_0, y_1)\} : \{x_0, y_0\} \in E_{G_1}, \{x_1, y_1\} \in E_{G_2}\})$ where $V_{G_1} \times V_{G_2}$ is the cartesian product of the vertex sets V_{G_1} and V_{G_2}. It can be seen that $\chi(E_{G_1\times G_2}) \leq min(\chi(E_{G_1}), \chi(E_{G_2}))$. In particular, if $\chi(E_{G_1}) = k < \omega$ then $\chi(E_{G_1\times G_2}) = k$, since if $f : V_{G_1} \rightarrow \{1, ..., k\}$ is a good k-coloring of G_1, then $F(\langle x, y\rangle) = f(x)$ is a good k-coloring of $G_1 \times G_2$. In **Theorem 2** of [Haj85], Hajnal proved that if $\chi(E_{G_1})$ is finite (say $k < \omega$), and $\chi(E_{G_2})$ is infinite, then $\chi(E_{G_1\times G_2})$ is k. For a natural number $k < \omega$, we denote by $\mathcal{P}_k$ the following statement.

'$\chi(E_{G_1}) = k < \omega$ *and* $\chi(E_{G_2}) \geq \omega$ *implies* $\chi(E_{G_1\times G_2}) = k$.'

2.1. **Permutation models.** Let M be a model of $ZFA + AC$ where A is a set of atoms or ur-elements. Each permutation $\pi : A \rightarrow A$ extends uniquely to a permutation of $\pi' : M \rightarrow M$ by ϵ-induction. Let $\mathcal{G}$ be a group of permutations of A and $\mathcal{F}$ be a normal filter of subgroups of $\mathcal{G}$.

For $x \in M$, we denote the symmetric group with respect to $\mathcal{G}$ by $sym_{\mathcal{G}}(x) = \{g \in \mathcal{G} \mid g(x) = x\}$. We say x is *$\mathcal{F}$-symmetric* if $sym_{\mathcal{G}}(x) \in \mathcal{F}$ and x is *hereditarily $\mathcal{F}$-symmetric* if x is $\mathcal{F}$-symmetric and each element of transitive closure of x is symmetric. We define the permutation model $\mathcal{N}$ with respect to $\mathcal{G}$ and $\mathcal{F}$, to be the class of all hereditarily $\mathcal{F}$-symmetric sets. It is well-known that $\mathcal{N}$ is a model of ZFA (see **Theorem 4.1** of [Jec73]). If $\mathcal{I} \subseteq \mathcal{P}(A)$ is a normal ideal, then the set $\{\text{fix}_{\mathcal{G}} E : E \in \mathcal{I}\}$ generates a normal filter over $\mathcal{G}$. Let $\mathcal{I}$ be a normal ideal generating a normal filter $\mathcal{F}_{\mathcal{I}}$ over $\mathcal{G}$. Let $\mathcal{N}$ be the permutation model determined by $\mathcal{M}, \mathcal{G}$, and $\mathcal{F}_{\mathcal{I}}$. We say $E \in \mathcal{I}$ supports a set $\sigma \in \mathcal{N}$ if $\text{fix}_{\mathcal{G}} E \subseteq sym_{\mathcal{G}}(\sigma)$.

3. Well-ordered structures in ZF

3.1. **Applications of Loeb's theorem.** We recall the following fact from [Ker00].

Lemma 3.1. *(ZF). If X is well-orderable, then 2^X is compact.*

Remark. We can also prove **Lemma 3.1** applying **Theorem 1** of [Loeb65].

Observation 3.2. $UT(WO, fin, WO)$ implies *Marshall Hall's theorem for any well-ordered collection of finite subsets of a set.*

Proof. Let S be a set and $\{S_i\}_{i \in I}$ be a well-ordered indexed family of **finite** subsets of S such that the following property holds,

(P) for every finite $F \subseteq I$, there is an injective choice function for $\{S_i\}_{i \in F}$.

We work with the propositional language $\mathcal{L}$ with the following sentence symbols.

$$A'_{i,j} \text{ where } j \in S_i \text{ and } i \in I.$$

Let $\mathcal{F}$ be the set of all formulae of $\mathcal{L}$ and $\Sigma \subset \mathcal{F}$ be the collection of the following formulae.

(1) $\neg(A'_{i,m} \wedge A'_{j,m})$ for $i \neq j$, $m \in S_i \cap S_j$.
(2) $\neg(A'_{i,j} \wedge A'_{i,l})$ for any $l \neq j \in S_i$ where $i \in I$.
(3) $A'_{i,y_1} \vee A'_{i,y_2} ... \vee A'_{i,y_k}$ for each $i \in I$ where $S_i = \{y_1, ...y_k\}$.

We enumerate $Var = \{A'_{i,j} : i \in I, j \in S_i\}$ since each S_i is finite, I is well-orderable and $UT(WO, fin, WO)$ **is assumed**. For every $W \in [I]^{<\omega} \backslash \{\emptyset\}$, we let Σ_W be the subset of $\mathcal{F}$, which is defined as Σ except that the subscripts in the formulae are from the set $W \cup \bigcup_{i \in W} S_i$. Endow the discrete 2-element space $\{0, 1\}$ with the discrete topology and consider the product space 2^{Var} with the product topology. Let $F_W = \{f \in 2^{Var} : \forall \phi \in \Sigma_W (f'(\phi) = 1)\}$ where for $f \in 2^{Var}$, the element f' of $2^{\mathcal{F}}$ denotes the valuation mapping determined by f. By **Philip Hall's theorem** which is provable in ZF without using any choice principles, each F_W is non-empty and the family $\mathcal{X} = \{F_W : W \in [I]^{<\omega} \backslash \{\emptyset\}\}$ has the finite intersection property. Also for each $W \in [I]^{<\omega} \backslash \{\emptyset\}$, F_W is closed in the topological space 2^{Var}. By **Lemma 3.1** since 2^{Var} is compact in ZF, $\cap \mathcal{X}$ is non-empty. Pick an $f \in \cap \mathcal{X}$ and let $f' \in 2^{\mathcal{F}}$ be the unique valuation mapping that extends f. Clearly, $f'(\phi) = 1$ for all $\phi \in \Sigma$. Consequently, we obtain an injective choice function for $\{S_i\}_{i \in I}$ by the following claim.

claim 3.3. *If v is a truth assignment which satisfies Σ, then we can define a system of distinct representatives by*

$$y \in S_i \text{ if and only if } v(A'_{i,y}) = T.$$

Proof. By (2) and (3) for each $i \in I$, each collection S_i gets assigned a unique representative. By (1), distinct sets S_i and S_j gets assigned distinct representatives. □

□

Banaschewski [Bana92] proved the uniqueness of the algebraic closure of an arbitrary field applying BPI.[3]

Observation 3.4. $UT(WO, fin, WO)$ implies '*If a field $\mathcal{K}$ has an algebraic closure, and the ring of polynomials $\mathcal{K}[x]$ is well-orderable, then the algebraic closure is unique*'.

Proof. Let $\mathcal{K}$ be a field, and suppose $\mathcal{E}$ and $\mathcal{F}$ be two algebraic closures of $\mathcal{K}$. We prove that there is an isomorphisn from $\mathcal{E}$ onto $\mathcal{F}$ which fix $\mathcal{K}$ pointwise. Let $\mathcal{E}_u$ and $\mathcal{F}_u$ be the splitting fields of $u \in \mathcal{K}[x]$ inside $\mathcal{E}$ and $\mathcal{F}$ respectively. Let $\mathcal{H}_u$ be the set of all isomorphisms from $\mathcal{E}_u$ onto $\mathcal{F}_u$ which fix $\mathcal{K}$. Clearly, $\mathcal{H}_u$ is a non-empty, finite set. Also, we can see that $\cup_u \mathcal{E}_u = \mathcal{E}$ and $\cup_u \mathcal{F}_u = \mathcal{F}$. Let $\mathcal{H} = \Pi_{u\in\mathcal{K}[x]}\mathcal{H}_u$, and if $v|w$ define $H_{v,w} = \{(h_u) \in \mathcal{H} : h_v = h_w \restriction E_v\}$. Clearly, $H_{v,w}$ has finite intersection property and they are closed in the product topology of $\mathcal{H}$, where each $\mathcal{H}_u$ is discrete. Since $\mathcal{K}[x]$ **is well-orderable** as assumed and for each $u \in \mathcal{K}[x]$, $\mathcal{H}_u$ is finite, we have that $\cup_{u\in\mathcal{K}[x]}\mathcal{H}_u$ is well-orderable by $UT(WO, fin, WO)$. By **Theorem 1** of [Loeb65], $\mathcal{H}$ is compact. Consequently, $\bigcap_{v|w} \mathcal{H}_{v,w} \neq \emptyset$ and each (h_u) in this intersection determines a unique embedding $h : \cup_u\mathcal{E}_u \to \cup_u\mathcal{F}_u$ which is onto and fixes $\mathcal{K}$. □

Observation 3.5. The statement '*For an infinite graph $G = (V_G, E_G)$ on a well-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G*' is provable in ZF.

Proof. Fix a finite graph $H = (V_H, E_H)$ and a graph $G = (V_G, E_G)$ on a well-ordered set of vertices V_G. We consider $V_H = \{v_1, ...v_k\}$ for some $k < \omega$. We work with the propositional language $\mathcal{L}$ with the following sentence symbols.

$$A'_{x_i,v_j} \text{ where } v_j \in V_H \text{ and } x_i \in V_G.$$

Let $\mathcal{F}$ be the set of all formulae of $\mathcal{L}$ and $\Sigma \subset \mathcal{F}$ be the collection of the following formulae.

(1) $A'_{x_i,v_m} \wedge A'_{x_j,v_l}$ if and only if $\{x_i, x_j\} \in E_G$ implies $\{v_m, v_l\} \in E_H$.
(2) $\neg(A'_{x_i,v_j} \wedge A'_{x_i,v_l})$ for any $v_l, v_j \in V_H$ such that $v_l \neq v_j$ and each $x_i \in V_G$.
(3) $A'_{x_i,v_1} \vee A'_{x_i,v_2} ... \vee A'_{x_i,v_k}$ for each $x_i \in V_G$.

By our assumption V_G is well-orderable and V_H is finite. So V_H is well-orderable. Consequently, $V_G \times V_H$ is well-orderable in ZF. We enumerate $Var = \{A'_{x_i,v_j} : x_i \in V_G, v_j \in V_H\}$. By assumption, for every $s \in [V_G]^{<\omega}$ there is a homomorphism $f_s : G \restriction s \to H$ of $G \restriction s$ into H. Following the methods used in the proof of **Observation 3.2**, we may obtain a $f' \in 2^{\mathcal{F}}$ such that $f'(\phi) = 1$ for all $\phi \in \Sigma$. Consequently, we can obtain a homomorphism h from G to H. □

Observation 3.6. The statement '*For every finite field $\mathcal{F} = \langle F, ...\rangle$, for every nontrivial well-ordered vector space V over $\mathcal{F}$, there exists a non-zero linear functional $f : V \to F$*' is provable in ZF.

Proof. We follow the proof of **Theorem 18 of** [HT13] and modify it in the context of well-orderable vector space. Fix a finite field $\mathcal{F} = \langle F, ...\rangle$ where $F = \{v_1, ...v_k\}$ and a nontrivial well-ordered vector space V over $\mathcal{F}$. We work with the propositional language $\mathcal{L}$ with the following sentence symbols.

$$A'_{x_i,v_j} \text{ where } v_j \in F \text{ and } x_i \in V.$$

Let $\mathcal{F}'$ be the set of all formulae of $\mathcal{L}$ and $\Sigma \subset \mathcal{F}'$ be the collection of the following formulae.

(1) $A'_{a,1}$.
(2) $A'_{x_i,v_j} \to A'_{v_k x_i, v_k v_j}$ for $v_k, v_j \in F$ and $x_i \in V$.
(3) $A'_{x_i,v_j} \wedge A'_{x_{i'},v_{j'}} \to A'_{x_i+x_{i'},v_j+v_{j'}}$ for $x_i, x_{i'} \in V$ and $v_j, v_{j'} \in F$.

[3] c.f. the last paragraph of page 384 and page 385 of [Bana92].

(4) $\neg(A'_{x_i,v_j} \wedge A'_{x_i,v_l})$ for any $v_l, v_j \in F$ such that $v_l \neq v_j$ and each $x_i \in V$.
(5) $A'_{x_i,v_1} \vee A'_{x_i,v_2} ... \vee A'_{x_i,v_k}$ for each $x_i \in V$.

By our assumption V is well-orderable and F is finite. So F is well-orderable. Consequently, $V \times F$ is well-orderable. We enumerate $Var = \{A'_{x_i,v_j} : x_i \in V, v_j \in F\}$. Fix $V' \in [V]^{<\omega}$. Let W be the subspace of V generated by the finite set $V' \cup \{a\}$. We can see that W is finite since F is finite. Consequently, a linear functional $f : W \to F$ with $f(a) = 1$ can be constructed in ZF. Following the methods used in the proof of **Observation 3.2**, we can obtain a non-zero linear functional $f : V \to F$. □

Observation 3.7. For every $3 \leq k < \omega$, the statement $\mathcal{P}_k$ for the graph G_1 on some well-orderable set of vertices is provable under ZF.

Proof. Fix $3 \leq k < \omega$. Suppose $\chi(E_{G_1}) = k$, $\chi(E_{G_2}) \geq \omega$ and G_1 is a graph on some well-orderable set of vertices. First we observe that if $g : V_{G_1} \to \{1, ..., k\}$ is a good k-coloring of G_1, then $G(\langle x, y\rangle) = g(x)$ is a good k-coloring of $G_1 \times G_2$. So, $\chi(E_{G_1 \times G_2}) \leq k$. For the sake of contradiction assume that $F : V_{G_1} \times V_{G_2} \to \{1, ..., k-1\}$ is a good coloring of $G_1 \times G_2$. For each color $c \in \{1, ..., k-1\}$ and each vertex $x \in V_{G_1}$ we let $A_{x,c} = \{y \in V_{G_2} : F(x, y) = c\}$.

claim 3.8. (ZF). *For all finite $F \subset V_{G_1}$, there exists a mapping $i_F : F \to \{1, ..., k-1\}$ such that for any $x, x' \in F$, $A_{x,i_F(x)} \cap A_{x',i_F(x')}$ is not independent.*

Proof. Since any superset of non-independent set is non-independent, it is enough to show that for all finite $F \subset V_{G_1}$, there exists an $i_F : F \to \{1, ..., k-1\}$ such that $\cap_{x \in F} A_{x,i_F(x)}$ is not independent. For the sake of contradiction assume that there exist a finite $F \subset V_{G_1}$ such that for all $i_F : F \to \{1, ..., k-1\}$, $\cap_{x \in F} A_{x,i_F(x)}$ is independent. Now, $V_{G_2} = \cup_{i_F : F \to \{1,...,k-1\}} \cap_{x \in F} A_{x,i_F(x)}$. Thus V_{G_2} can be written as a finite union of independent sets which contradicts the fact that $\chi(E_{G_2})$ is infinite. Thus for all finite $F \subset V_{G_1}$, we can obtain a mapping $i_F : F \to \{1, ..., k-1\}$ such that $\cap_{x \in F} A_{x,i_F(x)}$ is not independent. □

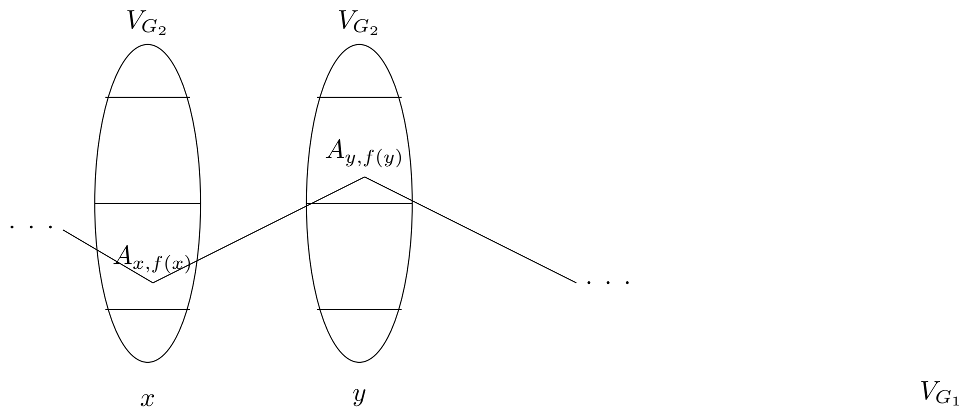


FIGURE 2. *A map $f : V_{G_1} \to \{1, ..., k-1\}$ such that intersection of any two elements in $\{A_{x,f(x)} : x \in V_{G_1}\}$ is not independent.*

Endow $\{1, 2, ..., k-1\}$ with the discrete topology. Since V_{G_1} is well-orderable, $\{1, 2, ..., k-1\} \times V_{G_1}$ is well-orderable under ZF. Applying **Theorem 1** of [Loeb65], $\{1, 2, ..., k-1\}^{V_{G_1}}$ is compact. For $s \in [V_{G_1}]^{<\omega}$, define $F_s = \{f \in \{1, 2, ..., k-1\}^{V_{G_1}} : x, y \in s, x \neq y \to A_{x,f(x)} \cap A_{y,f(y)}$ is not independent$\}$. By **claim 3.8**, for each $s \in [V_{G_1}]^{<\omega}$ we have that F_s is non-empty. We can see that $\{F_s : s \in [V_{G_1}]^{<\omega}\}$ has finite intersection property as $F_{s_0 \cup ... \cup s_k} \subseteq F_{s_0} \cap ... \cap F_{s_k}$. Thus by compactness of $\{1, 2, ..., k-1\}^{V_{G_1}}$, there is a $f \in \bigcap\{F_s : s \in [V_{G_1}]^{<\omega}\}$. Clearly, for any $x, x' \in V_{G_1}$, $A_{x,f(x)} \cap A_{x',f(x')}$ is not independent (see **figure 2**). Since $x \to f(x)$ is not a good coloring in G_1 as $\chi(E_{G_1}) = k$, there are $x, x' \in V_{G_1}$ with $f(x) = f(x') = j$ and $\{x, x'\} \in E_{G_1}$.

Consequently, $A' = A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. Pick $y, y' \in A'$ joined by an edge in E_{G_2}. Then (x, y) and (x', y') are joined in $E_{G_1} \times E_{G_2}$ and get the same color j which is a contradiction to the fact that F is a good coloring of $G_1 \times G_2$. □

3.2. **On partially ordered sets based on a well-ordered set of elements.** The first author modifies the arguments from **Claim 5** of [Tac16] and observes the following.

Observation 3.9. The following holds.

(1) $UT(\aleph_0, \aleph_0, \aleph_0)$ implies *'If in a partially ordered set based on a well-ordered set of elements, all chains are finite and all antichains are countable, then the set is countable'.*
(2) $UT(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ implies *'If in a partially ordered set based on a well-ordered set of elements, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$'* for any regular $\aleph_\alpha$.

Proof. We prove **Observation 3.9 (1)**. For the sake of contradiction, assume that all antichains of $(P, \leq)$ are countable, all chains of $(P, \leq)$ are finite, but the set P is uncountable and well-ordered. We construct an infinite chain in $(P, \leq)$ using $UT(\aleph_0, \aleph_0, \aleph_0)$ and obtain the desired contradiction.

claim 3.10. *$\leq$ is a well-founded relation on P i.e., every non-empty subset of P has a $\leq$-minimal element.*

Proof. Let P is well-orderable, say by $\preceq$. We claim that $\leq$ is a well-founded relation on P. Otherwise, there is a nonempty subset $P_1 \subseteq P$ with no minimal elements. Consequently, using the fact that $\preceq$ is a well-ordering in P, we can obtain a strictly $\leq$-decreasing sequence of elements of P_1. This contradicts the assumption that P has no infinite chains. □

Without loss of generality we may assume $P = \cup\{P_\alpha : \alpha < \kappa\}$ where κ is a well-ordered cardinal, P_0 is the set of minimal elements of P and for each $\alpha < \kappa$, P_α is the set of minimal elements of $P \backslash \cup \{P_\beta : \beta < \alpha\}$. For each $\alpha < \kappa$, P_α is countable since P_α is an antichain.

- We note that $P = \cup\{P_p : p \in P_0\}$ where $P_p = \{q \in P : p \leq q\}$. Since P is uncountable and P_0 is countable, P_p is uncountable for some $p \in P_0$. Otherwise for all $p \in P_0$, P_p is either countable or finite and $UT(\aleph_0, \aleph_0, \aleph_0) + UT(\aleph_0, fin, \aleph_0)$ implies P is countable which is a contradiction. Now $UT(\aleph_0, \aleph_0, \aleph_0)$ implies $UT(\aleph_0, fin, \aleph_0)$ in ZF, thus $UT(\aleph_0, \aleph_0, \aleph_0)$ suffices. Since $\{q \in P_0 : P_q \text{ is uncountable}\}$ is a non-empty subset of P, we can find a least $p_0 \in P_0$ with respect to $\preceq$ such that P_{p_0} is uncountable.
- Let us consider $P' = P_{p_0} \backslash \{p_0\}$. So, P' is uncountable. Again if P'_1 is the set of minimal elements of P', we can write $P' = \cup\{P_p : p \in P'_1\}$ where $P_p = \{q \in P : p \leq q\}$. Since P' is uncountable and P'_1 is countable (since all antichains of $(P, \leq)$ are countable by assumption), once again applying $UT(\aleph_0, \aleph_0, \aleph_0)$ as in the previous paragraph, P_p is uncountable for some $p \in P'_1$. Since $\{q \in P'_1 : P_q \text{ is uncountable}\}$ is a non-empty subset of P, we can find a least $p_1 \in P'_1$ with respect to $\preceq$ such that P_{p_1} is uncountable. We can see that $p_0 < p_1$.

Continuing this process step by step we obtain a sequence $\langle p_n : n \in \omega\rangle$ of elements of P such that $p_n < p_{n+1}$ for each $n \in \omega$. Consequently, we obtain an infinite chain.

Remark. Similarly for any regular $\aleph_\alpha$, assuming $UT(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ we can prove the following since alephs are well-ordered. *'If in a partially ordered set based on well-ordered set of elements, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$.'* Consequently, we can prove **Observation 3.9(2)**. □

4. Consistency results

Theorem 4.1. *For every natural number $n \geq 2$, there is a permutation model $\mathcal{N}$ of ZFA where CAC holds and AC_n^- fails. Moreover, we can observe the following in the model.*

(1) *CS, as well as CWF, holds.*
(2) *DT holds, $MA(\aleph_0)$ fails, MCC fails, LT fails.*
(3) *If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable. Moreover, if in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.*
(4) *The following statements hold.*
  (a) *Marshall Hall's theorem for linearly-ordered collection of finite subsets of a set.*
  (b) *For every $3 \leq k < \omega$, $\mathcal{P}_k$ holds for any graph G_1 on some linearly-orderable set of vertices.*
  (c) *For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G.*
  (d) *For every finite field $\mathcal{F} = \langle F, ...\rangle$, for every nontrivial linearly-orderable vector space V over $\mathcal{F}$, there exists a non-zero linear functional $f : V \to F$.*

Proof. In **Theorem 8** of [HT19], Halbeisen and Tachtsis constructed a permutation model $\mathcal{N}$ where for arbitrary $n \geq 2$, AC_n^- fails but CAC holds. We fix an arbitrary integer $n \geq 2$ and recall the model constructed in the proof of **Theorem 8** of [HT19] as follows.

- **Defining the ground model M.** We start with a ground model M of $ZFA + AC$ where A is a countably infinite set of atoms written as a disjoint union $\cup\{A_i : i \in \omega\}$ where for each $i \in \omega$, $A_i = \{a_{i_1}, a_{i_2}, ...a_{i_n}\}$.
- **Defining the group $\mathcal{G}$ of permutations and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$.**
  – **Defining $\mathcal{G}$.** $\mathcal{G}$ is defined in [HT19] in a way so that **if $\eta \in \mathcal{G}$, then η only moves finitely many atoms** and for all $i \in \omega$, $\eta(A_i) = A_k$ for some $k \in \omega$. We recall the details from [HT19] as follows. For all $i \in \omega$, let τ_i be the n-cycle $a_{i_1} \mapsto a_{i_2} \mapsto ...a_{i_n} \mapsto a_{i_1}$. For every permutation ψ of ω, which moves only finitely many natural numbers, let ϕ_ψ be the permutation of A defined by $\phi_\psi(a_{i_j}) = a_{\psi(i)_j}$ for all $i \in \omega$ and $j = 1, 2, ..., n$. Let $\eta \in \mathcal{G}$ if and only if $\eta = \rho\phi_\psi$ where ψ is a permutation of ω which moves only finitely many natural numbers and ρ is a permutation of A for which there is a finite $F \subseteq \omega$ such that for every $k \in F$, $\rho \restriction A_k = \tau_k^j$ for some $j < n$, and ρ fixes A_m pointwise for every $m \in \omega\backslash F$.
  – **Defining $\mathcal{F}$.** Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_\mathcal{G}(E) : E \in [A]^{<\omega}\}$.
- **Defining the permutation model.** Consider the permutation model $\mathcal{N}$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

Following **point 1** in the proof of **Theorem 8** of [HT19], both A and $\mathcal{A} = \{A_i\}_{i\in\omega}$ are amorphous in $\mathcal{N}$ and no infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ has a Kinna–Wegner selection function. Consequently, AC_n^- fails. The first author observes the following.

Lemma 4.2. *In $\mathcal{N}$, DT, CS as well as CWF holds. Moreover the following holds in $\mathcal{N}$.*

- *If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.*
- *'If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$' for any regular $\aleph_\alpha$.*

Proof. We follow the steps below.

(1) Let $(P, \leq)$ be a p.o.set in $\mathcal{N}$ and $E \in [A]^{<\omega}$ be a support of $(P, \leq)$. We can write P as a disjoint union of $\text{x}_\mathcal{G}(E)$-orbits, i.e., $P = \bigcup\{Orb_E(p) : p \in P\}$, where $Orb_E(p) = \{\phi(p) : \phi \in \text{x}_\mathcal{G}(E)\}$ for all $p \in P$. The family $\{Orb_E(p) : p \in P\}$ is well-orderable in $\mathcal{N}$ since $\text{x}_\mathcal{G}(E) \subseteq Sym_\mathcal{G}(Orb_E(p))$ for all $p \in P$.
(2) Since **if $\eta \in \mathcal{G}$, then η only moves finitely many atoms**, $Orb_E(p)$ is an antichain in P for each $p \in P$. Otherwise there is a $p \in P$, such that $Orb_E(p)$ is not an antichain in $(P, \leq)$. Thus, for some $\phi, \psi \in \text{fix}_\mathcal{G}(E)$, $\phi(p)$ and $\psi(p)$ are comparable. Without loss of

generality we may assume $\phi(p) < \psi(p)$. Since **if $\eta \in \mathcal{G}$, then η only moves finitely many atoms**, there exists some $k < \omega$ such that $\phi^k = 1_A$. Let $\pi = \psi^{-1}\phi$. Consequently, $\pi(p) < p$ and $\pi^k = 1_A$ for some $k \in \omega$. Thus, $p = \pi^k(p) < \pi^{k-1}(p) < ... < \pi(p) < p$. By transitivity of $<$, $p < p$, which is a contradiction.

(3) We prove that in $\mathcal{N}$, DT holds. Let $E \subset A$ be a finite support of an infinite p.o.set $\mathbb{P} = (P, <)$ with finite width. Then $P = \bigcup\{Orb_E(p) : p \in P\}$. Following (2), $Orb_E(p)$ is an antichain in $\mathbb{P}$. Consequently, $Orb_E(p)$ is finite for each $p \in P$ since the width of $\mathbb{P}$ is finite. Following (1), $\{Orb_E(p) : p \in P\}$ is well-orderable in $\mathcal{N}$. Following **point 4** in the proof of **Theorem 8** of [HT19] and **Lemma 3** of [Tac16], UT(WO,WO,WO) holds in $\mathcal{N}$, and so P is well-orderable in $\mathcal{N}$. Applying **Theorem 3.1(i) of** [Tac19], DT holds in $\mathcal{N}$.

(4) To see that CS as well as CWF holds in $\mathcal{N}$ we follow **Theorem 3.26** of [THS16] and **Theorem 10(ii) of** [Tac17] respectively. We sketch the important steps below.
   (a) We follow **Theorem 3.26** of [THS16] to see that CS holds in $\mathcal{N}$ as follows. Let $(P, \leq)$ be a poset without maximal elements supported by E. Following (1), $\mathcal{O} = \{Orb_E(p) : p \in P\}$ is a well-ordered partition of P. Define $\preceq$ on $\mathcal{O}$, as $X \preceq Y \leftrightarrow \exists x \in X, \exists y \in Y$ such that $x \leq y$. Since $(P, \leq)$ has no maximal element, $(\mathcal{O}, \preceq)$ has no maximal element following (2). Since $\mathcal{O}$ is well-ordered there exists a partition $\mathcal{U}_{\mathcal{O}} = \{\mathcal{Q}, \mathcal{R}\}$ of $\mathcal{O}$ in 2 cofinal subsets. Consequently, $\mathcal{U}_P = \{\cup\mathcal{Q}, \cup\mathcal{R}\}$ is a partition of P in 2 cofinal subsets.
   (b) We follow **Theorem 10 (ii)** of [Tac17] to see that CWF holds in $\mathcal{N}$ as follows. Let $(P, \leq)$ be a poset supported by $\mathcal{N}$. Since $\mathcal{O} = \{Orb_E(p) : p \in P\}$ is well-orderable, it has a cofinal well-founded subset $\mathcal{W} = \{W_\alpha : \alpha < \gamma\}$ such that for $\beta < \alpha$, $W_\alpha \not\preceq W_\beta$ for all $\beta, \alpha < \gamma$. Consequently, $C = \cup\mathcal{W}$ is a cofinal well-founded subset of P.

(5) We show the following in $\mathcal{N}$.

'If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.'

It is known that in every FM-model $UT(WO, WO, WO)$ implies $(\forall\alpha)UT(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ (c.f. page 176 of [HR98]). Consequently, $UT(\aleph_0, \aleph_0, \aleph_0)$ holds in $\mathcal{N}$. Let $(P, <)$ be an uncountable p.o.set in $\mathcal{N}$ where all antichains are countable and $E \in [A]^{<\omega}$ be a support of $(P, <)$. Following (1), $\mathcal{O} = \{Orb_E(p) : p \in P\}$ is a well-ordered partition of P since for all $p \in P$, E is a support of $Orb_E(p)$. Following (2), $Orb_E(p)$ is an antichain and hence countable. Consequently, $Orb_E(p)$ is well-orderable. Since $UT(WO, WO, WO)$ holds in $\mathcal{N}$, P is well-orderable. By **Observation 3.9(1)**, since $UT(\aleph_0, \aleph_0, \aleph_0)$ holds in $\mathcal{N}$, there is an infinite chain in $\mathcal{N}$.

(6) Following (5) and **Observation 3.9(2)**, we can prove the following in $\mathcal{N}$.

'If in a partially ordered set $(P, <)$, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$'.

□

Remark. The referee pointed out that the statements If in a partially ordered set based on a well-ordered set of elements all chains are finite and all antichains are countable then the set is countable and If in a partially ordered set based on a well-ordered set of elements all chains are finite and all antichains have size $\aleph_\alpha$ then the set has size $\aleph_\alpha$ are true in all Fraenkel-Mostowski permutation models. So **Observation 3.9(1)** and **Observation 3.9(2)** are not needed in the proofs of parts (5) and (6) of the proof of **Lemma 4.2**.

Lemma 4.3. *In $\mathcal{N}$, $MA(\aleph_0)$ fails.*

Proof. Since A is amorphous, the statement 'for all infinite X, 2^X is Baire' is false following **Remark 2.7** of [Tac16b]. Since CAC holds in $\mathcal{N}$, AC^{ω}_{fin} holds as well (c.f. **Lemma 4.4** of [Tac19a]). Consequently, $MA(\aleph_0)$ fails following **Remark 2.7** of [Tac16b]. □

Lemma 4.4. *In $\mathcal{N}$, MCC fails.*

Proof. Modifying the proof of **Theorem 8 (ii) of** [Tac17], we can see that $n(A) = n$. Thus there is a bounded amorphous set A. Consequently, MCC fails by **Theorem 6** of [Tac17]. □

Lemma 4.5. *In $\mathcal{N}$, LT fails.*

Proof. Since $\mathcal{A}$ is an amorphous set of non-empty sets which has no choice function in $\mathcal{N}$, following **Lemma 4.1**(i)[Tac19a], LT fails in $\mathcal{N}$. □

Lemma 4.6. *In $\mathcal{N}$, the following statements hold for linearly-ordered structures.*

(1) *Marshall Hall's theorem for linearly-ordered collection of finite subsets of a set.*
(2) *For every $3 \leq k < \omega$, $\mathcal{P}_k$ holds for any graph G_1 on some linearly-orderable set of vertices.*
(3) *For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G.*
(4) *For every finite field $\mathcal{F} = \langle F, ...\rangle$, for every nontrivial linearly-orderable vector space V over $\mathcal{F}$, there exists a non-zero linear functional $f : V \to F$.*

Proof. Since $UT(WO, WO, WO)$ and LW holds in $\mathcal{N}$ (c.f. pt 4 and pt 3 in the proof of **Theorem 8** in [HT19]), (1), (2), (3) and (4) hold in $\mathcal{N}$ following the observations in **section 3**. □

□

Remark 1. In **Theorem 7** of [Tac19a], Tachtsis generalized the above construction and proved that $AC^{LO} + LW \not\to LT$ by constructing a permutation model $\mathcal{N}$. Since AC^{WO} holds in $\mathcal{N}$, DC holds in $\mathcal{N}$ as well (c.f. **Theorem 8.2** of [Jec73]). We observe another standard argument to see that DC holds in $\mathcal{N}$. Since $\mathcal{I}$ is closed under countable unions in the model, we can see that DC holds in $\mathcal{N}$. Let $\mathcal{R}$ is a relation in $\mathcal{N}$ such that if $x \in dom(\mathcal{R})$, there exists a y such that xRy. Consequently, there is a sequence $\langle x_n : n \in \omega\rangle$ in the ground model M such that for each $n \in \omega$, $x_n R x_{n+1}$. If x_n is supported by E_n for every $n \in \omega$, then $\langle x_n : n \in \omega\rangle$ is supported by $\cup_{n\in\omega} E_n$. Since $\mathcal{I}$ is closed under countable unions, the sequence $\langle x_n : n \in \omega\rangle$ is in $\mathcal{N}$.

A class of models $M_{\aleph_\alpha}$ for any regular cardinal $\aleph_\alpha$ (similar to the model $M_{\aleph_1}$ constructed in **Theorem 7** of [Tac19a]) can be defined where AC^{LO} and LW holds but LT fails, by replacing $\aleph_1$ by $\aleph_\alpha$. Moreover in $M_{\aleph_\alpha}$, $DC_{<\aleph_\alpha}$ holds since $\mathcal{I}$ is closed under $< \aleph_\alpha$ unions.

Remark 2. In the permutation model $\mathcal{N}$ of [Tac16], CS, as well as CWF, holds following the work in this section. Moreover, the following statement holds in $\mathcal{N}$, following the work in this section.

'If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.'

Theorem 4.7. *There is a permutation model $\mathcal{N}$ of ZFA, where there is an amorphous set. Moreover, the following holds in $\mathcal{N}$.*

(1) *If in a partially ordered set, all chains are finite and all antichains are countable, then the set is countable.*
(2) *If in a partially ordered set, all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$ for any regular $\aleph_\alpha$.*

Proof. We consider the basic Fraenkel model (labeled as Model $\mathcal{N}_1$ in [HR98]) where 'there are no amorphous sets' is false and $UT(WO, WO, WO)$ holds (c.f. [HR98]). Let $(P, \leq)$ be a p.o.set in $\mathcal{N}_1$, and E be a nite support of $(P, \leq)$. By (1) in the proof of **Lemma 4.2**, $\mathcal{O} = \{Orb_E(p) : p \in P\}$ is a well-ordered partition of P. Now for each $p \in P$, $Orb_E(p)$ is an antichain (c.f. the proof of **Lemma 9.3** in [Jec73]). Thus, by methods of **Lemma 4.2**, (1) and (2) hold in $\mathcal{N}_1$. □

Theorem 4.8. *There is a permutation model of ZFA where CS, as well as CWF, holds, but AC^{ω}_{fin} fails. Moreover, the following statements hold in the model.*

1. *For every $3 \leq k < \omega$, $\mathcal{P}_k$ holds for any graph G_1 on some linearly-orderable set of vertices.*
2. *For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G.*
3. *For every finite field $\mathcal{F} = \langle F, ...\rangle$, for every nontrivial linearly-orderable vector space V over $\mathcal{F}$, there exists a non-zero linear functional $f : V \to F$.*

Proof. We recall the Lévy's permutation model (labeled as Model $\mathcal{N}_6$ in [HR98]).

- **Defining the ground model M.** We start with a ground model M of $ZFA + AC$ where A is a countably infinite set of atoms written as a disjoint union $\cup\{P_n : n \in \omega\}$, where $P_n = \{a^n_1, ...a^n_{p_n}\}$ such that p_n is the n^{th}-prime number.
- **Defining the group $\mathcal{G}$ of permutations and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$.**
  - **Defining $\mathcal{G}$.** $\mathcal{G}$ be the group generated by the following permutations π_n of A.
    $\pi_n : a^n_1 \mapsto a^n_2 \mapsto ...a^n_{p_n} \mapsto a^n_1$ and $\pi_n(x) = x$ for all $x \in A \backslash P_n$.
  - **Defining $\mathcal{F}$.** $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$.
- **Defining the permutation model.** Consider the permutation model $\mathcal{N}_6$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

It is well-known that in $\mathcal{N}_6$, AC^{ω}_{fin} fails since $\{P_i : i \in \omega\}$ has no (partial) choice function (c.f. [Jec73]). Consequently, following **Lemma 4.4** of [Tac19a], CAC fails in $\mathcal{N}_6$. Since **every permutation $\phi \in \mathcal{G}$ moves only finitely many atoms**, following the arguments in **Lemma 4.2**, we can observe that CS, as well as CWF, holds in $\mathcal{N}_6$.

Lemma 4.9. *In $\mathcal{N}_6$, LW holds.*

Proof. Let $(X, \leq)$ be a linearly ordered set in $\mathcal{N}_6$ supported by E. We show $\text{fix}_{\mathcal{G}}E \subseteq \text{fix}_{\mathcal{G}}X$ which implies that X is well-orderable in $\mathcal{N}_6$. For the sake of contrary assume $\text{fix}_{\mathcal{G}}E \nsubseteq \text{fix}_{\mathcal{G}}X$. So there is an element $y \in X$ which is not supported by E and there is a $\phi \in \text{fix}_{\mathcal{G}}E$ such that $\phi(y) \neq y$. Since $\phi(y) \neq y$ and $\leq$ is a linear order on X, we obtain either $\phi(y) < y$ or $y < \phi(y)$. Let $\phi(y) < y$. Since **every permutation $\phi \in \mathcal{G}$ moves only finitely many atoms** there exists some $k < \omega$ such that $\phi^k = 1_A$. Thus, $p = \phi^k(p) < \phi^{k-1}(p) < ... < \phi(p) < p$ which is a contradiction. Similarly we can arrive at a contradiction if we assume $y < \phi(y)$. □

Since LW holds in $\mathcal{N}_6$, we can observe (1), (2) and (3) in $\mathcal{N}_6$ by observations in **section 3**. □

Theorem 4.10. *There is a permutation model of ZFA where CS, as well as CWF, holds, but $LOKW^-_4$ fails. Moreover, the following statements hold in the model.*

1. *For every $3 \leq k < \omega$, $\mathcal{P}_k$ holds for any graph G_1 on some linearly-orderable set of vertices.*
2. *For an infinite graph $G = (V_G, E_G)$ on a linearly-ordered set of vertices V_G and a finite graph $H = (V_H, E_H)$, if every finite subgraph of G has a homomorphism into H, then so has G.*
3. *For every finite field $\mathcal{F} = \langle F, ...\rangle$, for every nontrivial linearly-orderable vector space V over $\mathcal{F}$, there exists a non-zero linear functional $f : V \to F$.*

Proof. We recall the permutation model $\mathcal{M}$ from the second assertion of **Theorem 10(ii) of** [HT19].

- **Defining the ground model M.** Let κ be any infinite well-ordered cardinal number. We start with a ground model M of $ZFA+AC$ where A is a κ-sized set of atoms written as a disjoint union $\cup\{A_\alpha : \alpha < \kappa\}$, where $A_\alpha = \{a_{\alpha,1}, a_{\alpha,2}, a_{\alpha,3}, a_{\alpha,4}\}$ such that $|A_\alpha| = 4$ for all $\alpha < \kappa$.

- **Defining the group $\mathcal{G}$ of permutations and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$.**
  - **Defining $\mathcal{G}$.** Let $\mathcal{G}$ be the weak direct product of $\mathcal{G}_\alpha$'s where $\mathcal{G}_\alpha$ is the alternating group on $\mathcal{A}_\alpha$ for each $\alpha < \kappa$.
  - **Defining $\mathcal{F}$.** Let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$.
- **Defining the permutation model.** Consider the permutation model $\mathcal{M}$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

In $\mathcal{M}$, $LOKW_4^-$ fails (c.f. **Theorem 10(ii) of** [HT19]). Since **every permutation,** $\phi \in \mathcal{G}$ **moves only finitely many atoms**, following the arguments in **Lemma 4.2** we can observe that CS, as well as CWF, holds in $\mathcal{M}$. Since LW holds in $\mathcal{M}$ (c.f. **Theorem 10(ii) of** [HT19]), we can observe (1), (2) and (3) in $\mathcal{M}$ by observations in **section 3**. □

5. Observations in Howard's model

Theorem 5.1. *For any $3 \leq k < \omega$, $\mathcal{P}_k$ follows from F_{k-1} in ZF. Moreover, if $X \in \{AC_3, AC_{fin}^{\omega}\}$, then the statement $\mathcal{P}_k$ does not imply X in ZFA when $k = 3$.*

Proof. Fix $3 \leq k < \omega$. Suppose $\chi(E_{G_1}) = k$, $\chi(E_{G_2}) \geq \omega$ and G_1 is a graph on some well-orderable set of vertices. First we observe that if $g : V_{G_1} \to \{1, ..., k\}$ is a good k-coloring of G_1, then $G(\langle x, y\rangle) = g(x)$ is a good k-coloring of $G_1 \times G_2$. So, $\chi(E_{G_1 \times G_2}) \leq k$. For the sake of contradiction assume that $F : V_{G_1} \times V_{G_2} \to \{1, ..., k-1\}$ is a good coloring of $G_1 \times G_2$. For each color $c \in \{1, ..., k-1\}$ and each vertex $x \in V_{G_1}$ we let $A_{x,c} = \{y \in V_{G_2} : F(x, y) = c\}$. Define a relation R on $\{1, ..., k-1\}$ as $(v_1, i)R(v_2, j)$ if and only if '*$v_1 \neq v_2$ implies $A_{v_1,i} \cap A_{v_2,j}$ is not independent*' for $v_1, v_2 \in V_{G_1}$. By F_{k-1} and **claim 3.8** there exist a choice function f such that for any $x, x' \in V_{G_1}$, $A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. Since $x \to f(x)$ is not a good coloring in G_1 as $\chi(E_{G_1}) = k$, there are $x, x' \in V_{G_1}$ with $f(x) = f(x') = j$ and $\{x, x'\} \in E_{G_1}$. Consequently, $A' = A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. Pick $y, y' \in A'$ joined by an edge in E_{G_2}. Then (x, y) and (x', y') are joined in $E_{G_1} \times E_{G_2}$ and get the same color j which is a contradiction to the fact that F is a good coloring of $G_1 \times G_2$.

For the second assertion, we consider the permutation model $\mathcal{N}$ from **section 3** of [How84] where AC_3 fails, and F_2 holds. Consequently, $\mathcal{P}_3$ holds in $\mathcal{N}$. In $\mathcal{N}$, there is a countable family $\mathcal{A} = \{A_i : i \in \omega\}$ which has no partial choice function. Consequently, PAC_{fin}^{ω} fails. Since PAC_{fin}^{ω} is equivalent to AC_{fin}^{ω} (see the proof of **Lemma 4.4** of [Tac19a]), AC_{fin}^{ω} fails in $\mathcal{N}$. □

Question 5.2. *If $k > 3$, does UL follow from $\mathcal{P}_k$? Otherwise is there any model of ZF or ZFA, where $\mathcal{P}_k$ holds for $k > 3$, but UL fails?*

Theorem 5.3. *For any $2 \leq k < \omega$, $\mathcal{P}_{G,H}$ restricted to finite graph H with k vertices follows from F_k in ZF. Moreover, if $X \in \{AC_3, AC_{fin}^{\omega}\}$, then $\mathcal{P}_{G,H}$ restricted to finite graph H with 2 vertices does not imply X in ZFA.*

Proof. Fix $2 \leq k < \omega$. Let $V_H = \{v_1, ...v_k\}$. For each $x \in V_G$, let $A_x = \{(x, v_1), ...(x, v_k)\}$. Define a relation R on $\cup_{x \in V_G} A_x$ by $(x, v_i)R(x', v_j)$ if and only if '*$\{x, x'\} \in E_G$ implies $\{v_i, v_j\} \in E_H$*' for $(x, v_i) \in A_x, (x', v_j) \in A_{x'}$. By assumption, for all finite $F \subset V_G$, there exists a homomorphism $h_F : G \restriction F \to H$. For any finite $F \subset V_G$, and an homomorphism h_F of F, let ${h_F}^*(j) = (j, h_F(j))$ for $j \in F$. Clearly, ${h_F}^*$ is an R-consistent choice function for $\{A_x\}_{x \in F}$. By F_k, there is a R-consistent choice function ${h_F}^*$ for $\{A_x\}_{x \in V_G}$. Define h_{V_G} on V_G by ${h_{V_G}}^*(j) = (j, h_{V_G}(j))$ for $j \in V_G$. Let $(j, j') \in E_G$ such that $j, j' \in V_G$. Since ${i_I}^*$ is R-consistent, $(j, h_{V_G}(j))R(j', h_{V_G}(j'))$. By the definition of R, $(h_{V_G}(j), h_{V_G}(j')) \in E_H$.

For the second assertion, we once more consider the permutation model $\mathcal{N}$ from **section 3** of [How84] where AC_3 and AC_{fin}^{ω} fails, and F_2 holds. Consequently, '$\mathcal{P}_{G,H}$ for a finite graph H with 2 vertices' holds in $\mathcal{N}$. □

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