# Quantum Rényi divergences and the strong converse exponent of state discrimination in operator algebras 

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#### Abstract

The sandwiched Rényi $\alpha$-divergences of two finite-dimensional quantum states play a distinguished role among the many quantum versions of Rényi divergences as the tight quantifiers of the trade-off between the two error probabilities in the strong converse domain of state discrimination. In this paper we show the same for the sandwiched Rényi divergences of two normal states on an injective von Neumann algebra, thereby establishing the operational significance of these quantities. Moreover, we show that in this setting, again similarly to the finite-dimensional case, the sandwiched Rényi divergences coincide with the regularized measured Rényi divergences, another distinctive feature of the former quantities. Our main tool is an approximation theorem (martingale convergence) for the sandwiched Rényi divergences, which may be used for the extension of various further results from the finite-dimensional to the von Neumann algebra setting.

We also initiate the study of the sandwiched Rényi divergences of pairs of states on a $C^{*}$-algebra, and show that the above operational interpretation, as well as the equality to the regularized measured Rényi divergence, holds more generally for pairs of states on a nuclear $C^{*}$-algebra.


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## 1 Introduction

Rényi's $\alpha$-divergences [63] give a one-parameter family of pseudo-distances on probability measures, which play a central role in information theory as quantifiers of the trade-off between the relevant operational quantities in many information theoretic tasks; see, e.g., [15]. In quantum information theory, the non-commutativity of density operators allows infinitely many different extensions of the classical Rényi divergences to pairs of finite-dimensional quantum states; among others, the standard (or Petz-type) Rényi divergences [58], the sandwiched Rényi divergences [52, 76], their common generalization, the Rényi $(\alpha, z)$-divergences [5, 36], or the maximal (or geometric) Rényi divergences [44, 60]. Many of these notions have also been extended to pairs of density operators on infinite-dimensional Hilbert spaces, or even to pairs of normal states on a von Neumann algebra [9, 27, 28, 29, 38, 39, 45, 57]. The study of these quantities has been motivating extensive research in the fields of matrix analysis and operator algebras.

On the other hand, in quantum information theory the relevant problem is to identify the quantum Rényi divergences with operational significance, i.e., those which appear as natural quantifiers of the trade-off relations between the quantities describing a problem, like error probabilities or coding rates. This has been established for the standard Rényi divergences with parameter values $\alpha \in(0,1)$ in the context of binary state discrimination (hypothesis testing) of finite-dimensional quantum states in a series of works [4, 23, 24, 32, 46, 54, 56], and was also extended to the von Neumann algebra setting in [37]. Complementary to this, the sandwiched Rényi divergences were shown to have operational significance for the
parameter values $\alpha>1$ in the strong converse problem of binary state discrimination of finite-dimensional quantum states [24, 48, 49], and in the strong converse problem of classicalquantum channel coding [50,51]. Apart from the standard and the sandwiched Rényi $\alpha$ divergences mentioned above, no other quantum Rényi divergence has been shown to have a direct operational interpretation so far.

It is also a problem of central importance how much the distinguishability of two states, as measured by a quantum divergence, changes under quantum operations, in particular, under quantum measurements. While there is no known explicit formula for the optimal postmeasurement Rényi $\alpha$-divergence (called the measured Rényi divergence), it is known to be strictly smaller than the standard Rényi $\alpha$-divergence [8,31]. Interestingly, if the measured Rényi $\alpha$-divergence is evaluated on several copies of the states, and normalized by the number of copies, then the asymptotic limit of these quantities, called the regularized measured Rényi $\alpha$-divergence, turns out to coincide with the sandwiched Rényi $\alpha$-divergence for all $\alpha \in[1 / 2,+\infty)$. This is another feature distinguishing the sandwiched Rényi divergences among the multitude of different quantum Rényi divergences.

The proof of the hypothesis testing interpretation of the sandwiched Rényi divergences in [48] goes via replacing the quantum i.i.d. problem with a non-i.i.d. classical hypothesis testing problem by block-diagonalizing (pinching) large tensor powers of the first state by the spectral projections of the same tensor powers of the second state. It can be shown that the resulting classical problem has the same optimal error asymptotics as the original quantum problem, by using the pinching inequality [22] and the fact that the number of distinct eigenvalues of the $n$th tensor power of a density operator grows only polynomially in $n$, even though the dimension of the underlying Hilbert space grows exponentially. The same technique can be used to show the equality of the regularized measured Rényi divergences and the sandwiched Rényi divergences [48]. While the pinching technique is very simple, it is also very powerful (see, e.g., [66] for further applications), but its applicability is obviously limited to the finite-dimensional case. Hence, even though the sandwiched Rényi divergences have been defined for pairs of normal states on a von Neumann algebra already some time ago [9, 38, 39], it has been an open problem (as proposed in [9]) whether they have an operational significance similar to the finite-dimensional case. This has been confirmed very recently in [45] in the simplest case where the von Neumann algebra is the space of bounded operators on an infinite-dimensional Hilbert space, using a finite-dimensional approximation technique, in particular, by showing that the Rényi divergences of the restrictions of the states to finite-dimensional subspaces converge to the Rényi divergences of the original states as the subspaces increase to the whole space.

In this paper we extend the above results about the sandwiched Rényi divergences to considerably more general settings, including injective, i.e., approximately finite-dimensional (AFD) von Neumann algebras. Our main tool is again finite-dimensional approximation. More generally, we show in Theorem 3.1 that the sandwiched Rényi divergences have the martingale convergence property for every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$, i.e., if an increasing net of von Neumann subalgebras generates the whole algebra then the sandwiched Rényi $\alpha$ divergences of the restrictions of two states converge to the sandwiched Rényi $\alpha$-divergence of the original states. The proof is based on variational representations of the sandwiched Rényi divergences [29, 39], and the martingale convergence of generalized conditional expectations from [34]. Using this result, we show in Theorem 3.7 that the strong converse exponents of discriminating two normal states of an injective algebra are equal to their Hoeffding antidivergences, analogously to the finite-dimensional case [48] and the case where the algebra is
$\mathcal{B}(\mathcal{H})$ [45]. Based on this result, in Theorem 3.11 we give a direct operational representation of the sandwiched Rényi $\alpha$-divergences as generalized cutoff rates, following Csiszár's approach [15]. Finally, using again the martingale convergence property, in Proposition 3.13 we show that for any $\alpha \in[1 / 2,+\infty) \backslash\{1\}$, the sandwiched Rényi $\alpha$-divergence of two states on an injective algebra coincides with their regularized measured Rényi $\alpha$-divergence.

Moreover, we also initiate the study of the sandwiched Rényi divergences for states of a $C^{*}$-algebra. In Theorem 4.3 we show that for any two states on a $C^{*}$-algebra, and any representation of the algebra that admits normal extensions of the states to the generated von Neumann algebra, the sandwiched Rényi $\alpha$-divergence of the extensions is independent of the specific representation for any $\alpha \in[1 / 2,+\infty)$, and hence it gives a well-defined notion of sandwiched Rényi $\alpha$-divergence of the original states. We also show the same statement for the standard $\alpha$-divergence and every $\alpha \in[0,+\infty) \backslash\{1\}$. In Proposition 4.5 we establish the basic properties of these extensions: joint lower semi-continuity, monotonicity under composition with unital positive maps (Schwarz maps in the case of the standard Rényi divergences), the inequality between the sandwiched and the standard Rényi divergences, and the martingale convergence property for both. In Theorem 4.12 we show that the sandwiched Rényi divergences on nuclear $C^{*}$-algebras have the same operational interpretation as in the case of injective von Neumann algebras, i.e., we show the equality of the strong converse exponents and the Hoeffding anti-divergences, from which the generalized cutoff rate representation also follows immediately. Finally, in Proposition 4.14 we show that the sandwiched Rényi divergences coincide with the regularized measured Rényi divergences on nuclear $C^{*}$-algebras.

The main text is accompanied by eight appendices. In Appendices A-E, we give brief overviews of the notions and concepts in von Neumann algebra theory that we use in the paper. Our general reference for this part is [30]. In Appendix F, we fill a gap in the proof of the equality of the strong converse exponents and the Hoeffding anti-divergences in the finite-dimensional case [48], and show the same equality for a slightly modified definition of the strong converse exponent. Appendix G contains a simple observation about the boundary values of convex functions, needed in the proof of Theorem 3.7. Finally, Appendix H contains the rather technical proof of Theorem 4.3.

## 2 Sandwiched and standard Rényi divergences

In this section we briefly review the notions of sandwiched and standard Rényi divergences in von Neumann algebras. For a more detailed exposition, see, e.g., [29]. We refer the reader to [30] for the necessary concepts in von Neumann algebra theory, some of which we will also briefly explain here and in the Appendices, for the convenience of the reader.

The notion of sandwiched Rényi divergences with parameter $\alpha \in[1 / 2,+\infty) \backslash\{1\}$, introduced first in $[52,76]$ for pairs of finite-dimensional density operators, was generalized by Berta-Scholz-Tomamichel [9] and Jenčová [38, 39] to the general von Neumann algebra setting. The definitions are different from each other between the three papers [9, 38, 39] but their equivalence was proved in $[38,39]$ (also [29, Sec. 3.3]). Here we work with the definition in [38] based on Kosaki's interpolation $L^{p}$-spaces.

For a von Neumann algebra $\mathcal{M}$, let $\mathcal{M}_{*}^{+}$denote the set of positive normal functionals on $\mathcal{M}$. The identity of $\mathcal{M}$ is denoted by 1. If $\mathcal{M}=\mathcal{B}(\mathcal{H})$ is the von Neumann algebra of all bounded operators on a finite-dimensional Hilbert space, then any $\psi \in \mathcal{M}_{*}^{+}$can be
represented by a positive semi-definite operator $\hat{\psi} \in \mathcal{B}(\mathcal{H})_{+}$such that $\psi(x)=\operatorname{Tr}(x \hat{\psi})$ for any $x \in \mathcal{M}$. The sandwiched Rényi $\alpha$-divergence of $\rho, \sigma \in \mathcal{M}_{*}^{+}$can then be defined for any $\alpha \in(0,1) \cup(1,+\infty)$ as [52, 76]

$$
\begin{equation*}
D_{\alpha}^{*}(\rho \| \sigma):=\frac{1}{\alpha-1} \log Q_{\alpha}^{*}(\rho \| \sigma), \tag{2.1}
\end{equation*}
$$

where

$$
Q_{\alpha}^{*}(\rho \| \sigma):= \begin{cases}\operatorname{Tr}\left(\hat{\sigma}^{\frac{1-\alpha}{2 \alpha}} \hat{\rho} \hat{\sigma}^{\frac{1-\alpha}{2 \alpha}}\right)^{\alpha}, & \text { if } s(\rho) \leq s(\sigma) \text { or } \alpha \in(0,1),  \tag{2.2}\\ +\infty, & \text { otherwise }\end{cases}
$$

Here, $s(\rho)$ is the smallest projection $p \in \mathcal{M}$ such that $\rho(p)=1$ (the support projection of $\rho$ ), and $s(\sigma)$ is defined similarly. Real powers of a positive semi-definite operator $A \in \mathcal{B}(\mathcal{H})_{+}$are defined as $A^{x}:=\sum_{\lambda>0} \lambda^{x} P_{\lambda}, x \in \mathbb{R}$, where $P_{\lambda}$ is the spectral projection of $A$ corresponding to $\{\lambda\} \subseteq \mathbb{R}$. The logarithm can be taken in any base that is larger than 1 , and it is extended to $[0,+\infty]$ by $\log 0:=-\infty, \log (+\infty):=+\infty$.
Remark 2.1. It is customary to define $D_{\alpha}^{*}$ with a normalization like

$$
D_{\alpha}^{*}(\rho \| \sigma)=\frac{1}{\alpha-1} \log \frac{Q_{\alpha}^{*}(\rho \| \sigma)}{\rho(\mathbf{1})}
$$

(or with restricting $\rho$ to states) but we use $D_{\alpha}^{*}$ without this normalization because that is the natural choice for the study of the strong converse exponent in Section 3.2; see, e.g., (3.7).

In the case of a general von Neumann algebra $\mathcal{M}$, there need not be a trace functional on $\mathcal{M}$, and a useful representation of states as operators is not at all straightforward. To this end, one may use Haagerup's construction (see Appendix B for details) to obtain a larger von Neumann algebra $\mathcal{N}$ with a faithful normal semifinite trace $\tau$ on it, with the corresponding *-algebra $\widetilde{\mathcal{N}}$ of $\tau$-measurable operators affiliated with $\mathcal{N}$, and Banach spaces $L^{p}(\mathcal{M}) \subseteq \widetilde{\mathcal{N}}$, with corresponding norm $\|\cdot\|_{p}, p \in[1,+\infty]$, such that

- $L^{\infty}(M)$ is identical to the von Neumann algebra $\mathcal{M}$;
- there exists an order isomorphic linear bijection $\psi \mapsto h_{\psi}$ from $\mathcal{M}_{*}$ onto $L^{1}(\mathcal{M})$;
- for every $\psi \in \mathcal{M}_{*}^{+}$and every $p \in[1,+\infty), h_{\psi}^{1 / p} \in L^{p}(\mathcal{M})_{+}\left(=L^{p}(\mathcal{M}) \cap \widetilde{\mathcal{N}}_{+}\right)$, where $h_{\psi}^{1 / p}$ is defined via standard functional calculus;
- for every $p, q \in[1,+\infty]$ with $1 / p+1 / q=1$, and every $a \in L^{p}(\mathcal{M}), b \in L^{q}(\mathcal{M})$, $a b \in L^{1}(\mathcal{M})$.

Moreover, the order isomorphism above defines a functional $\operatorname{tr}$ on $L^{1}(\mathcal{M})$ as $\operatorname{tr} h_{\psi}:=\psi(\mathbf{1})$, and for every $\psi \in \mathcal{M}_{*}$ and $x \in \mathcal{M}$,

$$
\psi(x)=\operatorname{tr}\left(x h_{\psi}\right),
$$

in complete analogy with the finite-dimensional case. For any $\sigma \in \mathcal{M}_{*}^{+}$, Kosaki's (symmetric) interpolation $L^{p}$-spaces $L^{p}(\mathcal{M}, \sigma)$ with respect to $\sigma$ for $p \in[1,+\infty]$ with $1 / p+1 / q=1$ are defined as

$$
\begin{aligned}
& L^{p}(\mathcal{M}, \sigma):=h_{\sigma}^{\frac{1}{2 q}} L^{p}(\mathcal{M}) h_{\sigma}^{\frac{1}{2 q}}\left(\subseteq L^{1}(\mathcal{M})\right), \\
& \left\|h_{\sigma}^{\frac{1}{2 q}} a h_{\sigma}^{\frac{1}{2 q}}\right\|_{p, \sigma}:=\|a\|_{p}, \quad a \in L^{p}(\mathcal{M}),
\end{aligned}
$$

that is, $L^{p}(\mathcal{M}) \cong L^{p}(\mathcal{M}, \sigma)$ by the isometry $a \mapsto h_{\sigma}^{\frac{1}{2 q}} a h_{\sigma}^{\frac{1}{2 q}}$ (see Appendix C).
In this general setting, the sandwiched Rényi $\alpha$-divergence $D_{\alpha}^{*}(\rho \| \sigma)$ of $\rho, \sigma \in \mathcal{M}_{*}^{+}$is defined by the same formula as in (2.1), with $Q_{\alpha}^{*}(\rho \| \sigma)$ in (2.2) replaced by

$$
Q_{\alpha}^{*}(\rho \| \sigma):= \begin{cases}\operatorname{tr}\left(h_{\sigma}^{\frac{1-\alpha}{2 \alpha}} h_{\rho} h_{\sigma^{\frac{1-\alpha}{2 \alpha}}}^{2^{\alpha}}=\left\|h_{\sigma}^{\frac{1-\alpha}{2 \alpha}} h_{\rho}^{1 / 2}\right\|_{2 \alpha}^{2 \alpha},\right. & \text { if } \alpha \in[1 / 2,1),  \tag{2.3}\\ \left\|h_{\rho}\right\|_{\alpha, \sigma}^{\alpha}, & \text { if } \alpha>1 \text { and } h_{\rho} \in L^{\alpha}(\mathcal{M}, \sigma), \\ +\infty, & \text { otherwise }\end{cases}
$$

according to [38] for $\alpha>1$, and [39, Theorem 3.1] and [29, Theorem 3.11] for $\alpha \in[1 / 2,1)$. Note here that the condition $h_{\rho} \in L^{\alpha}(\mathcal{M}, \sigma)$ contains, in particular, $s(\rho) \leq s(\sigma)$.

Example 2.2. Consider the simple case where $\mathcal{M}=B(\mathcal{H})$ with a finite-dimensional Hilbert space $\mathcal{H}$. Then the larger von Neumann algebra $\mathcal{N}$ mentioned above is given as $\mathcal{N}=$ $B(\mathcal{H}) \bar{\otimes} L^{\infty}(\mathbb{R})$ and Haagerup's $L^{p}$-spaces are given by

$$
L^{p}(\mathcal{M})= \begin{cases}B(\mathcal{H}) \otimes e^{-t / p}, & p \in[1,+\infty), \\ B(\mathcal{H}) \otimes \mathbf{1}=B(\mathcal{H}), & p=+\infty,\end{cases}
$$

with norms

$$
\begin{cases}\left\|X \otimes e^{-t / p}\right\|_{p}=\|X\|_{p}=\left(\operatorname{Tr}|X|^{p}\right)^{1 / p}, & p \in[1,+\infty) \\ \|X \otimes \mathbf{1}\|_{\infty}=\|X\|_{\infty}(\text { operator norm }), & p=+\infty\end{cases}
$$

where $e^{-t / p}$ is a shorthand notation for the multiplication operator on $L^{2}(\mathbb{R})$ with the function $t \mapsto e^{-t / p}$ on $\mathbb{R}$. Then $h_{\psi}=\hat{\psi} \otimes e^{-t}$ for any $\psi \in \mathcal{M}_{*}^{+}$, and a straightforward computation yields that the definitions in (2.2) and in (2.3) give the same values when $\alpha \in[1 / 2,1)$. On the other hand, for any $\sigma \in \mathcal{M}_{*}^{+}$with $e:=s(\sigma)$, Kosaki's interpolation $L^{p}$-spaces are given by

$$
L^{p}(\mathcal{M}, \sigma)=\hat{\sigma}^{\frac{1}{2 q}} B(\mathcal{H}) \hat{\sigma}^{\frac{1}{2 q}}=e B(\mathcal{H}) e, \quad p \in[1,+\infty], 1 / p+1 / q=1,
$$

with norms

$$
\|X\|_{p, \sigma}=\left\|\hat{\sigma}^{-\frac{1}{2 q}} X \hat{\sigma}^{-\frac{1}{2 q}}\right\|_{p}, \quad X \in e B(\mathcal{H}) e .
$$

Then it immediately follows that the definitions in (2.2) and in (2.3) give the same values when $\alpha>1$ too. In this way, (2.3) does indeed give an extension of the definition (2.1) from the finite-dimensional to the most general case; see Appendices B and C for more about Haagerup's and Kosaki's $L^{p}$-spaces in the case $\mathcal{M}=B(\mathcal{H})$.

When $\rho \in \mathcal{M}_{*}^{+}$is a state, $\alpha \mapsto D_{\alpha}^{*}(\rho \| \sigma)$ is monotone increasing on $[1 / 2,1) \cup(1,+\infty)$, and

$$
\begin{equation*}
\lim _{\alpha \nearrow 1} D_{\alpha}^{*}(\rho \| \sigma)=D(\rho \| \sigma) ; \tag{2.4}
\end{equation*}
$$

if, in addition, $D_{\alpha}^{*}(\rho \| \sigma)<+\infty$ for some $\alpha>1$, then

$$
\begin{equation*}
\lim _{\alpha \searrow 1} D_{\alpha}^{*}(\rho \| \sigma)=D(\rho \| \sigma), \tag{2.5}
\end{equation*}
$$

where $D(\rho \| \sigma)$ is the relative entropy of $\rho$ and $\sigma[2,3,43,74]$. Moreover, for any $\rho, \sigma \in \mathcal{M}_{*}^{+}$,

$$
\begin{equation*}
\lim _{\alpha \rightarrow+\infty} D_{\alpha}^{*}(\rho \| \sigma)=D_{\max }(\rho \| \sigma), \tag{2.6}
\end{equation*}
$$

where

$$
D_{\max }(\rho \| \sigma):=\log \min \{\lambda \geq 0: \rho \leq \lambda \sigma\}
$$

$(=+\infty$ if no such $\lambda$ exists), is the max-relative entropy $[16,62]$. For these properties of $D_{\alpha}^{*}$, see $[52,76]$ for the finite-dimensional case and $[9,38,39]$ (also a concise survey in [29, Sec. 3.3]) for the von Neumann algebra case.

The next variational formulas shown in [29, Lemma 3.19] and [39, Proposition 3.4] are the von Neumann algebra versions of [20, Lemma 4], which will play a crucial role in the next section. Here, $\mathcal{M}_{+}$is the set of positive operators in $\mathcal{M}$ and $\mathcal{M}_{++}$is the set of invertible $x \in \mathcal{M}_{+}$.

Proposition 2.3 ([29, 39]). For any $\rho, \sigma \in \mathcal{M}_{*}^{+}$the following hold:
(i) For every $\alpha \in(1,+\infty)$,

$$
\begin{equation*}
Q_{\alpha}^{*}(\rho \| \sigma)=\sup _{x \in \mathcal{M}_{+}}\left[\alpha \rho(x)-(\alpha-1) \operatorname{tr}\left(h_{\sigma}^{\frac{\alpha-1}{2 \alpha}} x h_{\sigma^{\frac{\alpha-1}{2 \alpha}}}\right)^{\frac{\alpha}{\alpha-1}}\right] \tag{2.7}
\end{equation*}
$$

(ii) For every $\alpha \in[1 / 2,1)$,

$$
\begin{equation*}
Q_{\alpha}^{*}(\rho \| \sigma)=\inf _{x \in \mathcal{M}_{++}}\left[\alpha \rho(x)+(1-\alpha) \operatorname{tr}\left(h_{\sigma^{\frac{1-\alpha}{2 \alpha}}} x^{-1} h_{\sigma^{\frac{1-\alpha}{2 \alpha}}}\right)^{\frac{\alpha}{1-\alpha}}\right] . \tag{2.8}
\end{equation*}
$$

A different quantum extension of the classical Rényi divergences is given by the standard (or Petz type) Rényi divergences $D_{\alpha}(\rho \| \sigma)$ defined for every $\rho, \sigma \in \mathcal{M}_{*}^{+}$and any $\alpha \in[0,+\infty) \backslash$ $\{1\}$ in terms of the relative modular operator $\Delta_{\rho, \sigma}$ (see Appendix A), which is a particular case of standard $f$-divergences developed first in [41, 57]. The following brief overview is based on [29]. When $0 \leq \alpha<1$, note that $h_{\sigma}^{1 / 2}$ is in the domain $\mathcal{D}\left(\Delta_{\rho, \sigma}^{\alpha / 2}\right)$ of $\Delta_{\rho, \sigma}^{\alpha / 2}$, and define

$$
\begin{equation*}
Q_{\alpha}(\rho \| \sigma):=\left\|\Delta_{\rho, \sigma}^{\alpha / 2} h_{\sigma}^{1 / 2}\right\|^{2} \tag{2.9}
\end{equation*}
$$

When $\alpha>1$,

$$
Q_{\alpha}(\rho \| \sigma):= \begin{cases}\left\|\Delta_{\rho, \sigma}^{\alpha / 2} h_{\sigma}^{1 / 2}\right\|^{2} & \text { if } s(\rho) \leq s(\sigma) \text { and } h_{\sigma}^{1 / 2} \in \mathcal{D}\left(\Delta_{\rho, \sigma}^{\alpha / 2}\right),  \tag{2.10}\\ +\infty & \text { otherwise. }\end{cases}
$$

Then $D_{\alpha}(\rho \| \sigma)$ is defined as

$$
D_{\alpha}(\rho \| \sigma):=\frac{1}{\alpha-1} \log Q_{\alpha}(\rho \| \sigma) .
$$

Example 2.4. Assume that $\mathcal{M}=B(\mathcal{H})$ with $\operatorname{dim} \mathcal{H}<+\infty$. For any $\rho, \sigma \in B(\mathcal{H})^{+}$, since $\Delta_{\rho, \sigma}=L_{\rho} R_{\sigma^{-1}}$ (see Appendix A) and hence $\Delta_{\rho, \sigma}^{\alpha / 2} \sigma^{1 / 2}=\rho^{\alpha / 2} \sigma^{\frac{1-\alpha}{2}}$, where $\sigma^{\frac{1-\alpha}{2}}$ is defined with restriction to the support $s(\sigma) \mathcal{H}$ when $\alpha>1$. Thus, the expressions of $Q_{\alpha}(\rho \| \sigma)$ in (2.9) and (2.10) give the well-known formulas [58]

$$
Q_{\alpha}(\rho \| \sigma)= \begin{cases}\operatorname{Tr}\left(\rho^{\alpha} \sigma^{1-\alpha}\right), & \text { if } 0 \leq \alpha<1 \text { or } s(\rho) \leq s(\sigma), \\ +\infty, & \text { if } \alpha>1 \text { and } s(\rho) \not \leq s(\sigma)\end{cases}
$$

Properties of $D_{\alpha}(\rho \| \sigma)$ in the von Neumann algebra case were summarized in [27, Proposition 5.3], and a handy description of $D_{\alpha}(\rho \| \sigma)$ in terms of $h_{\rho}, h_{\sigma}$ was given in [29, Theorem $3.6]$; in particular, when $\alpha \in[0,1)$,

$$
D_{\alpha}(\rho \| \sigma)=\frac{1}{\alpha-1} \log \operatorname{tr}\left(h_{\rho}^{\alpha} h_{\sigma}^{1-\alpha}\right) .
$$

(Compare this with (2.3).) When $\rho$ is a state, $\alpha \mapsto D_{\alpha}(\rho \| \sigma)$ is monotone increasing on $[0,1) \cup(1,+\infty), \lim _{\alpha} \nearrow_{1} D_{\alpha}(\rho \| \sigma)=D(\rho \| \sigma)$, and if $D_{\alpha}(\rho \| \sigma)<+\infty$ for some $\alpha>1$, then $\lim _{\alpha \searrow 1} D_{\alpha}(\rho \| \sigma)=D(\rho \| \sigma)$. The $\alpha=0$ case is the min-relative entropy $[16,62]$

$$
D_{\min }(\rho \| \sigma):=D_{0}(\rho \| \sigma)=-\log \operatorname{tr}\left(s(\rho) h_{\sigma}\right)
$$

According to [9, Theorem 1.2] and [38, Corollary 3.6], the inequality

$$
D_{\alpha}^{*}(\rho \| \sigma) \leq D_{\alpha}(\rho \| \sigma)
$$

holds for every $\rho, \sigma \in \mathcal{M}_{*}^{+}$and any $\alpha \in[1 / 2,+\infty) \backslash\{1\}$, while equality holds here when $\rho, \sigma$ "commute" (see [29, Remark 3.18 (2)] for the precise statement). Apart from $D_{\alpha}$ and $D_{\alpha}^{*}$, the two extreme cases $D_{\min }$ and $D_{\max }$ are also useful in some quantum information problems such as resource theory; see, e.g., [12, 64, 75].

## 3 The von Neumann algebra case

### 3.1 Martingale convergence for sandwiched Rényi divergences

Let $\mathcal{M}$ be a von Neumann algebra and $\left\{\mathcal{M}_{i}\right\}_{i \in \mathcal{I}}$ be an increasing net (on a directed set $\mathcal{I}$ ) of von Neumann subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$, such that $\mathcal{M}$ is generated by $\bigcup_{i \in \mathcal{I}} \mathcal{M}_{i}$, i.e.,

$$
\mathcal{M}=\left(\bigcup_{i \in \mathcal{I}} \mathcal{M}_{i}\right)^{\prime \prime}
$$

Let $\rho, \sigma \in \mathcal{M}_{*}^{+}$and $\rho_{i}:=\left.\rho\right|_{\mathcal{M}_{i}}, \sigma_{i}:=\left.\sigma\right|_{\mathcal{M}_{i}}$ for each $i \in \mathcal{I}$.
The next theorem provides the martingale convergence for the sandwiched Rényi divergence $D_{\alpha}^{*}$. It will play an essential role repeatedly in our later discussions.

Theorem 3.1. Let $\left\{\mathcal{M}_{i}\right\}_{i \in \mathcal{I}}$ be as stated above, and assume that $\mathcal{M}$ is $\sigma$-finite. Then for every $\rho, \sigma \in \mathcal{M}_{*}^{+}$and for every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ we have

$$
\begin{equation*}
D_{\alpha}^{*}(\rho \| \sigma)=\lim _{i} D_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}\right) \quad \text { increasingly } . \tag{3.1}
\end{equation*}
$$

To prove the theorem, we first give two lemmas. First, we state the martingale convergence for the generalized conditional expectations in [34, Theorem 3] as a lemma, which was given in [34] in a slightly more general setting.

Lemma 3.2 ([34]). In the situation stated above, assume that $\sigma$ is faithful and for each $i \in \mathcal{I}$ let $\mathcal{E}_{\mathcal{M}_{i}, \sigma}: \mathcal{M} \rightarrow \mathcal{M}_{i}$ be the generalized conditional expectation with respect to $\sigma$ (see Section $D)$. Then for every $x \in M$ we have $\mathcal{E}_{\mathcal{M}_{i}, \sigma}(x) \rightarrow x$ strongly.

The next lemma is indeed a special case of [38, Proposition 3.12]. The argument is also found in $[40$, Sec. 2.7$]$. We supply a sketchy proof for the convenience of the reader.

Lemma 3.3 ([38]). Assume that $\sigma \in \mathcal{M}_{*}^{+}$is faithful. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ containing the unit of $\mathcal{M}$, and $\sigma_{0}:=\left.\sigma\right|_{\mathcal{N}}$. Let $\mathcal{E}_{\mathcal{N}, \sigma}: \mathcal{M} \rightarrow \mathcal{N}$ be the generalized conditional expectation with respect to $\sigma$. Then for every $p \in[1,+\infty)$ and $x \in \mathcal{M}_{+}$we have

$$
\operatorname{tr}\left(h_{\sigma_{0}}^{\frac{1}{2 p}} \mathcal{E}_{\mathcal{N}, \sigma}(x) h_{\sigma_{0}}^{\frac{1}{2 p}}\right)^{p} \leq \operatorname{tr}\left(h_{\sigma}^{\frac{1}{2 p}} x h_{\sigma}^{\frac{1}{2 p}}\right)^{p},
$$

where $h_{\sigma_{0}}$ is the element of $L^{1}(\mathcal{N})_{+}$corresponding to $\sigma_{0}$, and $h_{\sigma} \in L^{1}(\mathcal{M})_{+}$corresponds to $\sigma$.

Proof (sketch). We utilize Kosaki's (symmetric) interpolation $L^{p}$-space $L^{p}(\mathcal{M}, \sigma)$ with the norm $\|\cdot\|_{p, \sigma}$ for $p \in[1,+\infty]$; see Appendix C. Let $\Psi=\Phi_{*}: L^{1}(\mathcal{M}) \rightarrow L^{1}(\mathcal{N})$ be the predual map of the injection $\mathcal{N} \hookrightarrow \mathcal{M}$, i.e., $\Psi\left(h_{\omega}\right)=h_{\left.\omega\right|_{\mathcal{N}}}$ for $\omega \in \mathcal{M}_{*}$, so that $\Psi$ is contractive with respect to $\|\cdot\|_{1}$. In the present situation, the description of $\Phi_{\omega}^{*}$ in (D.3) of Appendix D shows that $\Psi$ restricted to $L^{\infty}(\mathcal{M}, \sigma)$ is given as

$$
\Psi\left(h_{\sigma}^{1 / 2} x h_{\sigma}^{1 / 2}\right)=h_{\sigma_{0}}^{1 / 2} \mathcal{E}_{\mathcal{N}, \sigma}(x) h_{\sigma_{0}}^{1 / 2}, \quad x \in \mathcal{M}
$$

(so that the map $\mathcal{E}_{\mathcal{N}, \sigma}$ coincides with $\Phi_{\sigma}: \mathcal{M} \rightarrow \mathcal{N}$ given in [38]). Hence $\Psi$ is contractive from $L^{\infty}(\mathcal{M}, \sigma)$ to $L^{\infty}\left(\mathcal{N}, \sigma_{0}\right)$ with respect to $\|\cdot\|_{\infty, \sigma}$ (this can be seen more directly by (D.4)). It follows from the complex interpolation method (the Riesz-Thorin theorem) that $\Psi$ is a contraction from $L^{p}(\mathcal{M}, \sigma)$ to $L^{p}\left(\mathcal{N}, \sigma_{0}\right)$ with respect to $\|\cdot\|_{p, \sigma}$ for any $p \in(1,+\infty)$. For $h_{\sigma}^{1 / 2} x h_{\sigma}^{1 / 2} \in L^{\infty}(\mathcal{M}, \sigma) \subseteq L^{p}(\mathcal{M}, \sigma)$ we have

$$
\left\|h_{\sigma_{0}}^{1 / 2} \mathcal{E}_{\mathcal{N}, \sigma}(x) h_{\sigma_{0}}^{1 / 2}\right\|_{p, \sigma} \leq\left\|h_{\sigma}^{1 / 2} x h_{\sigma}^{1 / 2}\right\|_{p, \sigma}
$$

Noting by (C.1) and (C.2) that

$$
\left\|h_{\sigma}^{1 / 2} x h_{\sigma}^{1 / 2}\right\|_{p, \sigma}=\left\|h_{\sigma}^{\frac{1}{2 p}} x h_{\sigma}^{\frac{1}{2 p}}\right\|_{p}=\left[\operatorname{tr}\left(h_{\sigma}^{\frac{1}{2 p}} x h_{\sigma}^{\frac{1}{2 p}}\right)^{p}\right]^{1 / p}
$$

and similarly for $\left\|h_{\sigma_{0}}^{1 / 2} \mathcal{E}_{\mathcal{N}, \sigma}(x) h_{\sigma_{0}}^{1 / 2}\right\|_{p, \sigma}$, we have

$$
\operatorname{tr}\left(h_{\sigma_{0}}^{\frac{1}{2 p}} \mathcal{E}_{\mathcal{N}, \sigma}(x) h_{\sigma_{0}}^{\frac{1}{2 p}}\right)^{p} \leq \operatorname{tr}\left(h_{\sigma}^{\frac{1}{2 p}} x h_{\sigma}^{\frac{1}{2 p}}\right)^{p}
$$

as desired.
Proof of Theorem 3.1. From the monotonicity property of $D_{\alpha}^{*}$ proved in [9, 38] it follows that $D_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}\right) \leq D_{\alpha}^{*}(\rho \| \sigma)$ and $i \in \mathcal{I} \mapsto D_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}\right)$ is increasing. Hence, to show (3.1), it suffices to prove that

$$
\begin{equation*}
D_{\alpha}^{*}(\rho \| \sigma) \leq \sup _{i \in \mathcal{I}} D_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}\right) . \tag{3.2}
\end{equation*}
$$

To do this, we may assume that $\sigma$ is faithful. Indeed, assume that (3.2) has been shown when $\sigma$ is faithful. For general $\sigma \in \mathcal{M}_{*}^{+}$, since $\mathcal{M}$ is $\sigma$-finite, there exists a $\sigma_{0} \in \mathcal{M}_{*}^{+}$with $s\left(\sigma_{0}\right)=1-s(\sigma)$ and let $\sigma^{(n)}:=\sigma+n^{-1} \sigma_{0}, \sigma_{i}^{(n)}:=\left.\sigma^{(n)}\right|_{\mathcal{M}_{i}}$. From the lower semi-continuity and the order relation of $D_{\alpha}^{*}$ (see [38], [29, Theorem 3.16]) it follows that

$$
D_{\alpha}^{*}(\rho \| \sigma) \leq \liminf _{n \rightarrow \infty} D_{\alpha}^{*}\left(\rho \| \sigma^{(n)}\right) \leq \liminf _{n \rightarrow \infty} \sup _{i} D_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}^{(n)}\right) \leq \sup _{i} D_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}\right)
$$

proving (3.2) for general $\sigma$. Below we assume the faithfulness of $\sigma$ and divide the proof into two cases $1<\alpha<+\infty$ and $1 / 2 \leq \alpha<1$.

Case $1<\alpha<+\infty$. We need to prove that

$$
\begin{equation*}
Q_{\alpha}^{*}(\rho \| \sigma) \leq \sup _{i \in \mathcal{I}} Q_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}\right) \tag{3.3}
\end{equation*}
$$

For every $x \in \mathcal{M}_{+}$and $i \in \mathcal{I}$, by Lemma 3.3 we have

$$
\begin{aligned}
& \alpha \rho(x)-(\alpha-1) \operatorname{tr}\left(h_{\sigma^{\frac{\alpha-1}{2 \alpha}}}^{x} h_{\sigma^{\frac{\alpha-1}{2 \alpha}}}^{\frac{\alpha}{\alpha-1}}\right. \\
& \quad \leq \alpha \rho(x)-(\alpha-1) \operatorname{tr}\left(h_{\sigma_{i}}^{\frac{\alpha-1}{2 \alpha}} \mathcal{E}_{\mathcal{M}_{i}, \sigma}(x) h_{\sigma_{i}}^{\frac{\alpha-1}{2 \alpha}}\right)^{\frac{\alpha}{\alpha-1}} \\
& \quad=\alpha \rho\left(x-\mathcal{E}_{\mathcal{M}_{i}, \sigma}(x)\right)+\alpha \rho_{i}\left(\mathcal{E}_{\mathcal{M}_{i}, \sigma}(x)\right)-(\alpha-1) \operatorname{tr}\left(h_{\sigma_{i}}^{\frac{\alpha-1}{2 \alpha}} \mathcal{E}_{\mathcal{M}_{i}, \sigma}(x) h_{\sigma_{i}}^{\frac{\alpha-1}{2 \alpha}}\right)^{\frac{\alpha}{\alpha-1}} \\
& \quad \leq \alpha \rho\left(x-\mathcal{E}_{\mathcal{M}_{i}, \sigma}(x)\right)+\sup _{j \in \mathcal{I}} Q_{\alpha}^{*}\left(\rho_{j} \| \sigma_{j}\right),
\end{aligned}
$$

where the last inequality is due to the variational formula (2.7). Lemma 3.2 gives

$$
\alpha \rho(x)-(\alpha-1) \operatorname{tr}\left(h_{\sigma}^{\frac{\alpha-1}{2 \alpha}} x h_{\sigma}^{\frac{\alpha-1}{2 \alpha}}\right)^{\frac{\alpha}{\alpha-1}} \leq \sup _{j \in \mathcal{I}} Q_{\alpha}^{*}\left(\rho_{j} \| \sigma_{j}\right), \quad x \in M_{+} .
$$

By (2.7) again we have (3.3).
Case $1 / 2 \leq \alpha<1$. We need to prove that

$$
\begin{equation*}
Q_{\alpha}^{*}(\rho \| \sigma) \geq \inf _{i \in \mathcal{I}} Q_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}\right) \tag{3.4}
\end{equation*}
$$

For every $x \in \mathcal{M}_{++}$and $i \in \mathcal{I}$, by Lemma 3.3 we have

$$
\begin{aligned}
& \alpha \rho(x)+(1-\alpha) \operatorname{tr}\left(h_{\sigma}^{\frac{1-\alpha}{2 \alpha}} x^{-1} h_{\sigma}^{\frac{1-\alpha}{2 \alpha}}\right)^{\frac{\alpha}{1-\alpha}} \\
& \quad \geq \alpha \rho(x)+(1-\alpha) \operatorname{tr}\left(h_{\sigma_{i}^{2 \alpha}}^{\frac{1-\alpha}{2 \alpha}} \mathcal{E}_{\mathcal{M}_{i}, \sigma}\left(x^{-1}\right) h_{\sigma_{i}^{2-\alpha}}^{\frac{1-\alpha}{2 \alpha}}\right)^{\frac{\alpha}{1-\alpha}} \\
& \quad \geq \alpha \rho(x)+(1-\alpha) \operatorname{tr}\left(h_{\sigma_{i}}^{\frac{1-\alpha}{2 \alpha}} \mathcal{E}_{\mathcal{M}_{i}, \sigma}(x)^{-1} h_{\sigma_{i}}^{\frac{1-\alpha}{2 \alpha}}\right)^{\frac{\alpha}{1-\alpha}} \\
& \quad=\alpha \rho\left(x-\mathcal{E}_{\mathcal{M}_{i}, \sigma}(x)\right)+\alpha \rho_{i}\left(\mathcal{E}_{\mathcal{M}_{i}, \sigma}(x)\right)+(1-\alpha) \operatorname{tr}\left(h_{\sigma_{i}}^{\frac{1-\alpha}{2 \alpha}} \mathcal{E}_{\mathcal{M}_{i}, \sigma}(x)^{-1} h_{\sigma_{i}}^{\frac{1-\alpha}{2 \alpha}}\right)^{\frac{\alpha}{1-\alpha}} \\
& \quad \geq \alpha \rho\left(x-\mathcal{E}_{\mathcal{M}_{i}, \sigma}(x)\right)+\inf _{j \in \mathcal{I}} Q_{\alpha}^{*}\left(\rho_{j} \| \sigma_{j}\right),
\end{aligned}
$$

where the second inequality above follows from the Jensen inequality $\mathcal{E}_{\mathcal{M}_{i}, \sigma}\left(x^{-1}\right) \geq \mathcal{E}_{\mathcal{M}_{i}, \sigma}(x)^{-1}$ (see [13, Corollary 2.3]), and the last inequality is due to (2.8). By Lemma 3.2 and (2.8) we have (3.4).

From Theorem 3.1 we can easily obtain the following martingale type convergence for $D_{\alpha}^{*}$ under the restriction to reduced subalgebras $e_{i} \mathcal{M} e_{i}$ with $e_{i} \nearrow 1$. See [45] for related results in the case $\mathcal{M}=\mathcal{B}(\mathcal{H})$.

Proposition 3.4. Assume that $\mathcal{M}$ is $\sigma$-finite. Let $\left\{e_{i}\right\}_{i \in \mathcal{I}}$ be an increasing net of projections in $\mathcal{M}$ such that $e_{i} \nearrow 1$. Then for every $\rho, \sigma \in \mathcal{M}_{*}^{+}$and every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ we have

$$
D_{\alpha}^{*}(\rho \| \sigma)=\lim _{i} D_{\alpha}^{*}\left(e_{i} \rho e_{i} \| e_{i} \sigma e_{i}\right) \quad \text { increasingly },
$$

where $e_{i} \rho e_{i}$ is the restriction of $\rho$ to the reduced von Neumann algebra $e_{i} \mathcal{M} e_{i}$ and similarly for $e_{i} \sigma e_{i}$.

Proof. Let $\mathcal{M}_{i}:=e_{i} \mathcal{M} e_{i} \oplus \mathbb{C}\left(1-e_{i}\right)$; then $\left\{\mathcal{M}_{i}\right\}_{i \in \mathcal{I}}$ is an increasing net of von Neumann subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$ with $\mathcal{M}=\left(\bigcup_{i} \mathcal{M}_{i}\right)^{\prime \prime}$. Note that $\rho_{i}:=\left.\rho\right|_{\mathcal{M}_{i}}=$ $e_{i} \rho e_{i} \oplus \rho\left(1-e_{i}\right)$ and similarly for $\sigma_{i}:=\left.\sigma\right|_{\mathcal{M}_{i}}$. Hence by the definition of $Q_{\alpha}^{*}$ or by using the variational formulas in Proposition 2.3, for any $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ we have

$$
Q_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}\right)=Q_{\alpha}^{*}\left(e_{i} \rho e_{i} \| e_{i} \sigma e_{i}\right)+\rho\left(1-e_{i}\right)^{\alpha} \sigma\left(1-e_{i}\right)^{1-\alpha} .
$$

Below we will give the proof for the case $\alpha>1$ (that for $1 / 2 \leq \alpha<1$ is similar by using the reverse monotonicity of $\left.Q_{\alpha}^{*}\right)$. Let $i, j \in \mathcal{I}$ be such that $i \leq j$. Noting that $e_{i} \mathcal{M} e_{i} \oplus \mathbb{C}\left(e_{j}-e_{i}\right) \subseteq$ $e_{j} \mathcal{M} e_{j}$, by the monotonicity of $Q_{\alpha}^{*}$ we have

$$
\begin{aligned}
Q_{\alpha}^{*}\left(e_{j} \rho e_{j} \| e_{j} \sigma e_{j}\right) & \geq Q_{\alpha}^{*}\left(e_{i} \rho e_{i} \| e_{i} \sigma e_{i}\right)+\rho\left(e_{j}-e_{i}\right)^{\alpha} \sigma\left(e_{j}-e_{i}\right)^{1-\alpha} \\
& \geq Q_{\alpha}^{*}\left(e_{i} \rho e_{i} \| e_{i} \sigma e_{i}\right) .
\end{aligned}
$$

Hence, $i \in \mathcal{I} \mapsto Q_{\alpha}^{*}\left(e_{i} \rho e_{i} \| e_{i} \sigma e_{i}\right)$ is increasing. Since Theorem 3.1 implies that

$$
Q_{\alpha}^{*}(\rho \| \sigma)=\lim _{i}\left[Q_{\alpha}^{*}\left(e_{i} \rho e_{i} \| e_{i} \sigma e_{i}\right)+\rho\left(1-e_{i}\right)^{\alpha} \sigma\left(1-e_{i}\right)^{1-\alpha}\right]
$$

the assertion follows if we show the following:

- when $Q_{\alpha}^{*}(\rho \| \sigma)=+\infty, \lim _{i} Q_{\alpha}^{*}\left(e_{i} \rho e_{i} \| e_{i} \sigma e_{i}\right)=+\infty$,
- when $Q_{\alpha}^{*}(\rho \| \sigma)<+\infty, \lim _{i} \rho\left(1-e_{i}\right)^{\alpha} \sigma\left(1-e_{i}\right)^{1-\alpha}=0$.

These can indeed be proved in a similar way to the proof of [27, Theorem 4.5] by taking $\rho\left(1-e_{i}\right)^{\alpha} \sigma\left(1-e_{i}\right)^{1-\alpha}$ and $\rho\left(e_{j}-e_{i}\right)^{\alpha} \sigma\left(e_{j}-e_{i}\right)^{1-\alpha}$ in place of $\rho\left(1-e_{i}\right) f\left(\rho\left(1-e_{i}\right) / \sigma\left(1-e_{i}\right)\right)$ and $\rho\left(e_{j}-e_{i}\right) f\left(\rho\left(e_{j}-e_{i}\right) / \sigma\left(e_{j}-e_{i}\right)\right)$, respectively, there. The details are omitted here.

### 3.2 The strong converse exponent in injective von Neumann algebras

Let $\mathcal{M}$ be a von Neumann algebra and $\rho, \sigma \in \mathcal{M}_{*}^{+}$be non-zero. For each $n \in \mathbb{N}$ let $\mathcal{M}^{\otimes} n$ be the $n$-fold von Neumann algebra tensor product of $\mathcal{M}$, and $\rho_{n}:=\rho^{\otimes n}$ (resp., $\sigma_{n}:=\sigma^{\otimes n}$ ) be the $n$-fold tensor product of $\rho$ (resp., $\sigma$ ), which are elements of $\left(\mathcal{M}^{\bar{\otimes} n}\right)_{*}^{+}$; see [69, Chap. IV].

In this section we consider the simple hypothesis testing problem for the null hypothesis $H_{0}: \rho$ versus the alternative hypothesis $H_{1}: \sigma$, and extend the result [48] on the strong converse exponent to the injective von Neumann algebra case. For a test $T_{n} \in \mathcal{M}^{\bar{\otimes} n}$ with $0 \leq T_{n} \leq 1, \rho_{n}\left(1-T_{n}\right)$ and $\sigma_{n}\left(T_{n}\right)$ represent the type $I$ and the type II error probabilities, respectively (hence $\rho_{n}\left(T_{n}\right)$ is the type I success probability) when $\rho, \sigma$ are states. We begin by defining several forms of the strong converse exponents in the following:

Definition 3.5. For each type II error exponent $r \in \mathbb{R}$ we define the strong converse exponents of simple hypothesis testing for $H_{0}: \rho$ vs. $H_{1}: \sigma$ as follows:

$$
\begin{aligned}
& \underline{s c_{r}}(\rho \| \sigma):=\inf _{\left\{T_{n}\right\}}\left\{\liminf _{n \rightarrow \infty}-\frac{1}{n} \log \rho_{n}\left(T_{n}\right): \liminf _{n \rightarrow \infty}-\frac{1}{n} \log \sigma_{n}\left(T_{n}\right) \geq r\right\}, \\
& \overline{s c}_{r}(\rho \| \sigma):=\inf _{\left\{T_{n}\right\}}\left\{\limsup _{n \rightarrow \infty}-\frac{1}{n} \log \rho_{n}\left(T_{n}\right): \liminf _{n \rightarrow \infty}-\frac{1}{n} \log \sigma_{n}\left(T_{n}\right) \geq r\right\}, \\
& s c_{r}(\rho \| \sigma):=\inf _{\left\{T_{n}\right\}}\left\{\lim _{n \rightarrow \infty}-\frac{1}{n} \log \rho_{n}\left(T_{n}\right): \liminf _{n \rightarrow \infty}-\frac{1}{n} \log \sigma_{n}\left(T_{n}\right) \geq r\right\},
\end{aligned}
$$

where the infima are taken over all test sequences $\left\{T_{n}\right\}$ with $T_{n} \in \mathcal{M}^{\bar{\otimes} n}, 0 \leq T_{n} \leq 1(n \in$ $\mathbb{N}$ ) for which the indicated condition holds (and furthermore the limit exists for $s c_{r}(\rho \| \sigma)$ ). Also, we write $\underline{s c}_{r}^{0}(\rho \| \sigma), \overline{s c}_{r}^{0}(\rho \| \sigma)$ and $s c_{r}^{0}(\rho \| \sigma)$ for the above infima when the condition $\lim \inf _{n \rightarrow \infty}-\frac{1}{n} \log \sigma_{n}\left(T_{n}\right) \geq r$ is replaced with $\lim \inf _{n \rightarrow \infty}-\frac{1}{n} \log \sigma_{n}\left(T_{n}\right)>r$.

For any $r \in \mathbb{R}$ it is obvious that

$$
\begin{align*}
& \underline{s c}_{r}^{0}(\rho \| \sigma) \leq \overline{s c}_{r}^{0}(\rho \| \sigma) \leq s c_{r}^{0}(\rho \| \sigma) . \tag{3.5}
\end{align*}
$$

The following is the definition of the Hoeffding anti-divergence [48, 45] in the von Neumann algebra setting.

Definition 3.6. For any $\rho, \sigma \in \mathcal{M}_{*}^{+}$define

$$
\begin{aligned}
\psi^{*}(\rho \| \sigma \mid \alpha) & :=\log Q_{\alpha}^{*}(\rho \| \sigma)=(\alpha-1) D_{\alpha}^{*}(\rho \| \sigma), \quad \alpha \in(1,+\infty), \\
\tilde{\psi}^{*}(\rho \| \sigma \mid u) & :=(1-u) \psi^{*}\left(\rho \| \sigma \mid(1-u)^{-1}\right)=u D_{\frac{1}{1-u}}^{*}(\rho \| \sigma), \quad u \in(0,1) .
\end{aligned}
$$

The Hoeffding anti-divergence of $\rho$ and $\sigma$ is then defined for each $r \in \mathbb{R}$ by

$$
\begin{equation*}
H_{r}^{*}(\rho \| \sigma):=\sup _{\alpha>1} \frac{\alpha-1}{\alpha}\left\{r-D_{\alpha}^{*}(\rho \| \sigma)\right\}=\sup _{u \in(0,1)}\left\{u r-\tilde{\psi}^{*}(\rho \| \sigma \mid u)\right\} . \tag{3.6}
\end{equation*}
$$

The aim of this section is to find whether the strong converse exponents in (3.5) are all equal to $H_{r}^{*}(\rho \| \sigma)$ for given $r$, as in the finite-dimensional case. To do so, we may assume without loss of generality that $\rho, \sigma \in \mathcal{M}_{*}^{+}$are states. In fact, for any $\lambda, \mu>0$ it is immediate to check that if $\varepsilon_{r}(\rho \| \sigma)$ is any strong converse exponent in (3.5), then

$$
\varepsilon_{r}(\lambda \rho \| \mu \sigma)=\varepsilon_{r+\log \mu}(\rho \| \sigma)-\log \lambda .
$$

On the other hand, since $Q_{\alpha}^{*}(\lambda \rho \| \mu \sigma)=\lambda^{\alpha} \mu^{1-\alpha} Q_{\alpha}^{*}(\rho \| \sigma)$, one easily sees that

$$
\begin{equation*}
H_{r}^{*}(\lambda \rho \| \mu \sigma)=H_{r+\log \mu}^{*}(\rho \| \sigma)-\log \lambda . \tag{3.7}
\end{equation*}
$$

Thus, in the rest of the section, we will always assume that $\rho, \sigma \in \mathcal{M}_{*}^{+}$are states. Under this assumption, when $r<0$, any exponent in (3.5) is clearly equal to 0 by taking $T_{n}=1$ for all $n$, while $H_{r}^{*}(\rho \| \sigma)$ is also 0 whenever $D_{\alpha}^{*}(\rho \| \sigma)<+\infty$ for some $\alpha>1$, otherwise $H_{r}^{*}(\rho \| \sigma)=-\infty$ for all $r \in \mathbb{R}$. Therefore, when $r<0$, the conjecture holds in a trivial way, or otherwise it is not true. Furthermore, when $s(\rho) \not \leq s(\sigma)$, any exponent in (3.5) is 0 for all $r \in \mathbb{R}$ by taking $T_{n}=1-s\left(\sigma_{n}\right)=1-s(\sigma)^{\otimes n}$, while $H_{r}^{*}(\rho \| \sigma)=-\infty$ for all $r$ by definition. So the conjecture is not true for any $r$ in this case. Therefore, we may restrict our consideration to the case where $r \geq 0$ and $s(\rho) \leq s(\sigma)$.

When $\mathcal{M}$ is finite-dimensional and $s(\rho) \leq s(\sigma)$, it was proved in [48, Theorem 4.10] that $\underline{s c}_{r}(\rho \| \sigma)=H_{r}^{*}(\rho \| \sigma)$ for any $r \geq 0$. (Note that $\underline{s c}_{r}(\rho \| \sigma)$ here was denoted by $B_{e}^{*}(r)$ in [48].) The result has recently been extended in [45, Theorem IV.5] to the infinite-dimensional $\mathcal{B}(\mathcal{H})$ setting in such a way that $\underline{s c}_{r}(\rho \| \sigma)=\overline{s c}_{r}(\rho \| \sigma)=H_{r}^{*}(\rho \| \sigma)$ for any $r \in \mathbb{R}$ under the assumption that $D_{\alpha}^{*}(\rho \| \sigma)<+\infty$ for some $\alpha>1$. The main aim of this section is to further extend the result to the injective von Neumann algebra setting. For the convenience of the reader we recall the fundamental notions of injectivity and AFD for von Neumann algebras in Appendix E.

Theorem 3.7. Assume that $\mathcal{M}$ is injective. Let $\rho, \sigma \in \mathcal{M}_{*}^{+}$be states such that $D_{\alpha}^{*}(\rho \| \sigma)<$ $+\infty$ for some $\alpha>1$ (in particular, $s(\rho) \leq s(\sigma)$ ). Then for every $r \in \mathbb{R}$ we have

$$
\begin{equation*}
\underline{s c}_{r}(\rho \| \sigma)=\overline{s c}_{r}(\rho \| \sigma)=s c_{r}(\rho \| \sigma)=\underline{s c_{r}^{0}}(\rho \| \sigma)=\overline{s c}_{r}^{0}(\rho \| \sigma)=s c_{r}^{0}(\rho \| \sigma)=H_{r}^{*}(\rho \| \sigma) . \tag{3.8}
\end{equation*}
$$

Proof. To prove this, we may and do assume that $\mathcal{M}$ is $\sigma$-finite. Indeed, let $e:=s(\rho) \vee s(\sigma)$. Then $e \mathcal{M} e$ is injective and all the quantities in (3.8) for any $r \in \mathbb{R}$ as well as $D_{\alpha}^{*}(\rho \| \sigma)$ for any $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ are left unchanged when $\rho, \sigma$ are replaced with $\left.\rho\right|_{e \mathcal{M} e},\left.\sigma\right|_{e \mathcal{M} e}$.

When $r<0$, the quantities in (3.8) are all equal to 0 as explained in the paragraph after Definition 3.6. Hence we may assume that $r \geq 0$. By (3.5) it suffices to show the following two inequalities

$$
\begin{align*}
& {\underline{s c_{r}}}_{r}(\rho \| \sigma) \geq H_{r}^{*}(\rho \| \sigma),  \tag{3.9}\\
& s c_{r}^{0}(\rho \| \sigma) \leq H_{r}^{*}(\rho \| \sigma) . \tag{3.10}
\end{align*}
$$

The inequality (3.9) follows in the same way (using Nagaoka's method [53]) as the proof of [48, Lemma 4.7] in the finite-dimensional case, as mentioned in [9, Sec. 5]. The injectivity assumption is unnecessary for this part. As for the inequality (3.10), we need to first prove it in the finite-dimensional case, because the exponent $s c_{r}^{0}(\rho \| \sigma)$ was not treated in [48]. To make the proof of this part more understandable, we present it in Appendix F separately.

Now, by the injectivity assumption of $\mathcal{M}$ (see Appendix E), we can choose an increasing net $\left\{\mathcal{M}_{i}\right\}_{i \in \mathcal{I}}$ of finite-dimensional *-subalgebras containing the unit of $\mathcal{M}$ such that $\mathcal{M}=$ $\left(\bigcup_{i \in \mathcal{I}} \mathcal{M}_{i}\right)^{\prime \prime}$. For each $i \in \mathcal{I}$ let $\rho_{i}:=\left.\rho\right|_{\mathcal{M}_{i}}$ and $\sigma_{i}:=\left.\sigma\right|_{\mathcal{M}_{i}}$. According to [48, Corollary 3.11], $\alpha \mapsto \psi^{*}\left(\rho_{i} \| \sigma_{i} \mid \alpha\right)$ is a finite-valued convex function on $(1,+\infty)$, whence it is also continuous there. Thus, the function

$$
u \longmapsto \tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)=(1-u) \psi^{*}\left(\rho_{i} \| \sigma_{i} \mid(1-u)^{-1}\right)
$$

is also finite-valued, convex and continuous on $(0,1)$. In particular, it can be extended to a convex and continuous function on $[0,1]$ by

$$
\tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid 0\right):=\lim _{u \searrow 0} \tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)=0, \quad \tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid 1\right):=\lim _{u \nearrow 1} \tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)=D_{\max }\left(\rho_{i} \| \sigma_{i}\right),
$$

where the first equality is simple, and see (2.6) for the second one. We hence have

$$
H_{r}^{*}\left(\rho_{i} \| \sigma_{i}\right)=\sup _{u \in(0,1)}\left\{u r-\tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)\right\}=\max _{u \in[0,1]}\left\{u r-\tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)\right\} .
$$

Also, from the monotonicity of $D_{\alpha}^{*}$, note that $i \in \mathcal{I} \mapsto \psi^{*}\left(\rho_{i} \| \sigma_{i} \mid \alpha\right)$ is increasing for any $\alpha>1$, so that $i \in \mathcal{I} \mapsto u r-\tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)$ is decreasing for any $u \in[0,1]$. For every $r \geq 0$, since $s c_{r}^{0}(\rho \| \sigma) \leq s c_{r}^{0}\left(\rho_{i} \| \sigma_{i}\right)$ for all $i \in \mathcal{I}$ as immediately verified, we have

$$
\begin{align*}
s c_{r}^{0}(\rho \| \sigma) & \leq \inf _{i \in \mathcal{I}} s c_{r}^{0}\left(\rho_{i} \| \sigma_{i}\right)=\inf _{i \in \mathcal{I}} H_{r}^{*}\left(\rho_{i} \| \sigma_{i}\right) \\
& =\inf _{i \in \mathcal{I}} \max _{u \in[0,1]}\left\{u r-\tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)\right\} \\
& =\max _{u \in[0,1]} \inf _{i \in \mathcal{I}}\left\{u r-\tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)\right\} \\
& =\max _{u \in[0,1]}\left\{u r-\sup _{i} \tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)\right\}, \tag{3.11}
\end{align*}
$$

where the above first equality is due to Proposition F. 2 in Appendix F and the third equality is due to a minimax theorem in [45, Lemma II.3] (also see [46, Corollary A.2]).

Furthermore, Theorem 3.1 implies that

$$
\begin{equation*}
\sup _{i \in \mathcal{I}} \tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid u\right)=\sup _{i \in \mathcal{I}} \frac{\alpha-1}{\alpha} D_{\alpha}^{*}\left(\rho_{i} \| \sigma_{i}\right)=\frac{\alpha-1}{\alpha} D_{\alpha}^{*}(\rho \| \sigma)=\tilde{\psi}^{*}(\rho \| \sigma \mid u) \tag{3.12}
\end{equation*}
$$

for every $u \in(0,1)$ and $\alpha=(1-u)^{-1} \in(1,+\infty)$. By assumption, $\tilde{\psi}^{*}(\rho \| \sigma \mid u)<+\infty$ for some $u \in(0,1)$. Hence by Lemma G. 1 in Appendix G we confirm that

$$
\begin{equation*}
\sup _{i \in \mathcal{I}} \tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid 0\right)=\lim _{u \searrow 0} \tilde{\psi}^{*}(\rho \| \sigma \mid u), \quad \sup _{i \in \mathcal{I}} \tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid 1\right)=\lim _{u \nearrow 1} \tilde{\psi}^{*}(\rho \| \sigma \mid 1) \tag{3.13}
\end{equation*}
$$

From (3.11)-(3.13) we conclude that

$$
\begin{aligned}
s c_{r}^{0}(\rho \| \sigma) & \leq \sup _{u \in(0,1)}\left\{u r-\tilde{\psi}^{*}(\rho \| \sigma \mid u)\right\} \\
& =\sup _{\alpha>1} \frac{\alpha-1}{\alpha}\left\{r-D_{\alpha}^{*}(\rho \| \sigma)\right\}=H_{r}^{*}(\rho \| \sigma),
\end{aligned}
$$

proving (3.10).
Remark 3.8. According to [45, Example III.39] (also [27, Remark 5.4 (1)]), there exist commuting density operators $\rho, \sigma$ on $\mathcal{H}$ with $\operatorname{dim} \mathcal{H}=+\infty$ such that

$$
D(\rho \| \sigma)<+\infty, \quad D_{\alpha}^{*}(\rho \| \sigma)=+\infty \quad \text { for all } \alpha \in(1,+\infty)
$$

In this case, $H_{r}^{*}(\rho \| \sigma)=-\infty$ for all $r \in \mathbb{R}$, whereas any exponent in (3.5) is non-negative. See [45, Theorem IV.5] for a more precise result in this case. Therefore, the assumption of $D_{\alpha}^{*}(\rho \| \sigma)<+\infty$ for some $\alpha>1$ is essential in Theorem 3.7, as well as in the convergence (2.5).

For given states $\rho, \sigma \in \mathcal{M}_{*}^{+}$and any $r \geq 0$, the $n$th minimal type I error probability of Hoeffding type is defined as

$$
\begin{equation*}
\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right):=\min _{0 \leq T_{n} \leq 1}\left\{\rho_{n}\left(1-T_{n}\right): \sigma_{n}\left(T_{n}\right) \leq e^{-n r}\right\}, \tag{3.14}
\end{equation*}
$$

where the minimum is taken over all tests in $\mathcal{M}^{\bar{\otimes} n}$ with $\sigma_{n}\left(T_{n}\right) \leq e^{-n r}$. Note that the minimum exists since the set of such tests is weakly compact. The $n$th maximal type I success probability is then given as

$$
1-\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right)=\max _{0 \leq T_{n} \leq 1}\left\{\rho_{n}\left(T_{n}\right): \sigma_{n}\left(T_{n}\right) \leq e^{-n r}\right\}
$$

In terms of this we can reformulate the assertion of Theorem 3.7 on the strong converse exponents as follows:

Theorem 3.9. Under the same assumption as in Theorem 3.7, for every $r \geq 0$ we have

$$
\lim _{n \rightarrow \infty}-\frac{1}{n} \log \left\{1-\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right)\right\}=H_{r}^{*}(\rho \| \sigma) .
$$

Proof. For each $n$ one can choose a test $T_{n}$ in $\mathcal{M}^{\bar{\otimes} n}$ such that $\sigma_{n}\left(T_{n}\right) \leq e^{-n r}$ and $\rho_{n}\left(T_{n}\right)=$ $1-\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right)$. Then it follows that

$$
\liminf _{n \rightarrow \infty}-\frac{1}{n} \log \left\{1-\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right)\right\}=\liminf _{n \rightarrow \infty}-\frac{1}{n} \log \rho_{n}\left(T_{n}\right) \geq \underline{s c_{r}}(\rho \| \sigma)
$$

On the other hand, if a sequence of tests $\left\{T_{n}\right\}$ satisfies $\lim \inf _{n}-\frac{1}{n} \log \sigma_{n}\left(T_{n}\right)>r$, then we have $1-\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right) \geq \rho_{n}\left(T_{n}\right)$ for all sufficiently large $n$ and hence

$$
\limsup _{n \rightarrow \infty}-\frac{1}{n} \log \left\{1-\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right)\right\} \leq \limsup _{n \rightarrow \infty}-\frac{1}{n} \log \rho_{n}\left(T_{n}\right) .
$$

Therefore,

$$
\limsup _{n \rightarrow \infty}-\frac{1}{n} \log \left\{1-\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right)\right\} \leq \overline{s c}_{r}^{0}(\rho \| \sigma) .
$$

Since $\underline{s c}_{r}(\rho \| \sigma)=\overline{s c}_{r}^{0}(\rho \| \sigma)=H_{r}^{*}(\rho \| \sigma)$ by Theorem 3.7, we find that

$$
\lim _{n \rightarrow \infty}-\frac{1}{n} \log \left\{1-\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right)\right\}=H_{r}^{*}(\rho \| \sigma)
$$

as asserted.
Using Theorem 3.7 we can give a direct operational interpretation of the sandwiched Rényi divergences as generalized cutoff rates, following the idea of Csiszár [15].
Definition 3.10. Let $\rho, \sigma \in \mathcal{M}_{*}^{+}$be states, and $\kappa \in(0,1)$. The generalized $\kappa$-cutoff rate $C_{\kappa}(\rho \| \sigma)$ is defined to be the infimum of all $r_{0} \in \mathbb{R}$ such that $\underline{s c}_{r}(\rho \| \sigma) \geq \kappa\left(r-r_{0}\right)$ holds for every $r \in \mathbb{R}$.
Theorem 3.11. Assume that $\mathcal{M}$ is injective. Let $\rho, \sigma \in \mathcal{M}_{*}^{+}$be states such that $D_{\alpha_{0}}^{*}(\rho \| \sigma)<$ $+\infty$ for some $\alpha_{0}>1$. Then

$$
C_{\kappa}(\rho \| \sigma)=D_{\frac{1}{1-\kappa}}^{*}(\rho \| \sigma), \quad \text { or equivalently, } \quad D_{\alpha}^{*}(\rho \| \sigma)=C_{\frac{\alpha-1}{\alpha}}(\rho \| \sigma)
$$

for every $\alpha \in\left(1, \alpha_{0}\right)$ and corresponding $\kappa=(\alpha-1) / \alpha \in\left(0, \kappa_{0}\right)$, where $\kappa_{0}:=\left(\alpha_{0}-1\right) / \alpha_{0}$.
Proof. By Theorem 3.7 and (3.6),

$$
\begin{aligned}
\underline{s c_{r}}(\rho \| \sigma) & =H_{r}^{*}(\rho \| \sigma)=\sup _{u \in(0,1)}\left\{u r-\tilde{\psi}^{*}(\rho \| \sigma \mid u)\right\} \\
& \geq \kappa r-\tilde{\psi}^{*}(\rho \| \sigma \mid \kappa)=\kappa\left(r-D_{\frac{1}{1-\kappa}}^{*}(\rho \| \sigma)\right),
\end{aligned}
$$

showing $C_{\kappa}(\rho \| \sigma) \geq D_{\frac{1}{1-\kappa}}^{*}(\rho \| \sigma)$. As we have seen in the proof of Theorem 3.7, $\tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid \cdot\right)$ is convex and continuous on $(0,1)$ for every $i$, and hence $\tilde{\psi}^{*}(\rho \| \sigma \mid \cdot)=\sup _{i} \tilde{\psi}^{*}\left(\rho_{i} \| \sigma_{i} \mid \cdot\right)$ is convex and lower semi-continuous on $(0,1)$, where the equality is by Theorem 3.1. By assumption, and the monotonicity of $D_{\alpha}^{*}$ in $\alpha$ stated in Section 2 (see [38, Proposition 3.7]), $\tilde{\psi}^{*}(\rho \| \sigma \mid \cdot)$ is finite-valued on $\left(0, \kappa_{0}\right)$, and hence it has finite left- and right-derivatives at every $\kappa \in\left(0, \kappa_{0}\right)$. Thus, for any such $\kappa$ and $r \in\left[\partial^{-} \tilde{\psi}^{*}(\rho \| \sigma \mid \kappa), \partial^{+} \tilde{\psi}^{*}(\rho \| \sigma \mid \kappa)\right]$,

$$
\begin{aligned}
{\overline{s c_{r}}}_{r}(\rho \| \sigma) & =H_{r}^{*}(\rho \| \sigma)=\sup _{u \in(0,1)}\left\{u r-\tilde{\psi}^{*}(\rho \| \sigma \mid u)\right\} \\
& =\kappa r-\tilde{\psi}^{*}(\rho \| \sigma \mid \kappa)=\kappa\left(r-D_{\frac{1}{1-\kappa}}^{*}(\rho \| \sigma)\right),
\end{aligned}
$$

where the first equality is again due to Theorem 3.7. This shows $C_{\kappa}(\rho \| \sigma) \leq D_{\frac{1}{1-\kappa}}^{*}(\rho \| \sigma)$, completing the proof.

In the rest of this section we consider the (regularized) measured Rényi divergences. Let us first recall these notions in the present setting. A measurement (POVM) in $\mathcal{M}$ is given by a finite family $\mathfrak{M}=\left(M_{j}\right)_{1 \leq j \leq k}$ of $M_{j} \in \mathcal{M}_{+}(j=1, \ldots, k)$ such that $\sum_{j=1}^{k} M_{j}=1$. For each $\rho \in \mathcal{M}_{*}^{+}$let $\mathfrak{M}(\rho):=\left(\rho\left(M_{j}\right)\right)_{1 \leq j \leq k}$ denote the post-measurement probability distribution on $\{1, \ldots, k\}$.
Definition 3.12. The measured Rényi divergence $D_{\alpha}^{\text {meas }}(\rho \| \sigma)$ and its regularized version $\bar{D}_{\alpha}^{\text {meas }}(\rho \| \sigma)$ for $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ are defined as

$$
\begin{aligned}
& D_{\alpha}^{\text {meas }}(\rho \| \sigma):=\sup \left\{D_{\alpha}(\mathfrak{M}(\rho) \| \mathfrak{M}(\sigma)): \mathfrak{M} \text { a measurement in } M\right\}, \\
& \bar{D}_{\alpha}^{\text {meas }}(\rho \| \sigma):=\sup _{n \in \mathbb{N}} \frac{1}{n} D_{\alpha}^{\text {meas }}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right)=\lim _{n \rightarrow \infty} \frac{1}{n} D_{\alpha}^{\text {meas }}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right) .
\end{aligned}
$$

The last equality above holds since $n \mapsto D_{\alpha}^{\text {meas }}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right)$ is superadditive, as immediately seen from the definition of $D_{\alpha}^{\text {meas }}$.

One might also consider more general notions of measurements in the definition, but that does not change the value of the (regularized) measured Rényi divergence, as was shown in [29, Proposition 5.2]. In the opposite direction, one might restrict measurements $\mathfrak{M}$ to two-valued ones (tests), which leads to the notion of the test-measured Rényi divergence $D_{\alpha}^{\text {test }}(\rho \| \sigma)$ and its regularized version $\bar{D}_{\alpha}^{\text {test }}(\rho \| \sigma)$, defined as

$$
\begin{aligned}
& D_{\alpha}^{\text {test }}(\rho \| \sigma):=\sup _{T \in \mathcal{M}, 0 \leq T \leq 1} D_{\alpha}((\rho(T), \rho(1-T)) \|(\sigma(T), \sigma(1-T))), \\
& \bar{D}_{\alpha}^{\text {test }}(\rho \| \sigma):=\sup _{n \in \mathbb{N}} \frac{1}{n} D_{\alpha}^{\text {test }}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right) .
\end{aligned}
$$

It is obvious that

$$
\begin{align*}
& D_{\alpha}^{\text {test }}(\rho \| \sigma) \leq \bar{D}_{\alpha}^{\text {test }}(\rho \| \sigma)  \tag{3.15}\\
& \wedge \text { । } \\
& D_{\alpha}^{\text {meas }}(\rho \| \sigma) \leq \bar{D}_{\alpha}^{\text {meas }}(\rho \| \sigma)
\end{align*}
$$

for all $\alpha \in[1 / 2,+\infty) \backslash\{1\}$.
Let $\mathcal{M}$ be injective, and $\left\{\mathcal{M}_{i}\right\}_{i \in \mathcal{I}}$ be an increasing net of finite-dimensional *-subalgebras of $\mathcal{M}_{0}:=e \mathcal{M} e$ with $\mathcal{M}_{0}=\left(\bigcup_{i} \mathcal{M}_{i}\right)^{\prime \prime}$, where $e:=s(\rho) \vee s(\sigma)$ (see the proof of Theorem 3.7). Note that for each $n \in \mathbb{N}, \mathcal{M}_{0}^{\bar{\otimes} n}=\left(\bigcup_{i} \mathcal{M}_{i}^{\otimes n}\right)^{\prime \prime},\left(\left.\rho\right|_{\mathcal{M}_{0}}\right)^{\otimes n}=\left.\rho^{\otimes n}\right|_{\mathcal{M}_{0}^{区} n}$, and similarly for $\sigma$. Hence, for every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ and every $n \in \mathbb{N}$ one has by Theorem 3.1,

$$
\begin{align*}
D_{\alpha}^{*}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right) & =\lim _{i} D_{\alpha}^{*}\left(\left(\left.\rho\right|_{\mathcal{M}_{i}}\right)^{\otimes n} \|\left(\left.\sigma\right|_{\mathcal{M}_{i}}\right)^{\otimes n}\right) \\
& =\lim _{i} n D_{\alpha}^{*}\left(\left.\rho\right|_{\mathcal{M}_{i}} \|\left.\sigma\right|_{\mathcal{M}_{i}}\right)=n D_{\alpha}^{*}(\rho \| \sigma), \tag{3.16}
\end{align*}
$$

where the second equality above follows from the additivity of $D_{\alpha}^{*}$ under tensor product in the finite-dimensional case. Indeed, the additivity of $D_{\alpha}^{*}$ under tensor product holds true in the general von Neumann algebra case, as observed in [9] (see the footnote of p. 1860) in the approach of Araki and Masuda's $L^{p}$-norms.
Proposition 3.13. Assume that $\mathcal{M}$ is injective. Then for every $\rho, \sigma \in \mathcal{M}_{*}^{+}$and every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ we have

$$
\begin{equation*}
D_{\alpha}^{*}(\rho \| \sigma)=\bar{D}_{\alpha}^{\text {meas }}(\rho \| \sigma)=\lim _{n \rightarrow \infty} \frac{1}{n} D_{\alpha}^{\text {meas }}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right) \tag{3.17}
\end{equation*}
$$

and for every $\alpha>1$,

$$
\begin{equation*}
D_{\alpha}^{*}(\rho \| \sigma)=\bar{D}_{\alpha}^{\text {test }}(\rho \| \sigma)=\lim _{n \rightarrow \infty} \frac{1}{n} D_{\alpha}^{\text {test }}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right) \tag{3.18}
\end{equation*}
$$

Proof. For any $\alpha \in[1 / 2,+\infty) \backslash\{1\}$, by (3.16) and (3.15) note that

$$
D_{\alpha}^{*}(\rho \| \sigma) \geq \bar{D}_{\alpha}^{\text {meas }}(\rho \| \sigma) \geq \bar{D}_{\alpha}^{\text {test }}(\rho \| \sigma)
$$

To prove (3.17), it suffices to show that

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \frac{1}{n} D_{\alpha}^{\text {meas }}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right) \geq D_{\alpha}^{*}(\rho \| \sigma) . \tag{3.19}
\end{equation*}
$$

Let $e:=s(\rho) \vee s(\sigma)$. Let $v<D_{\alpha}^{*}(\rho \| \sigma)$ be arbitrary. By Theorem 3.1 there exists a finitedimensional ${ }^{*}$-subalgebra $\mathcal{N}_{0}$ of $e \mathcal{M}$ e such that $D_{\alpha}^{*}\left(\left.\rho\right|_{\mathcal{N}_{0}} \|\left.\sigma\right|_{\mathcal{N}_{0}}\right)>v$. Write $\rho_{0}:=\left.\rho\right|_{\mathcal{N}_{0}}$ and $\sigma_{0}:=\left.\sigma\right|_{\mathcal{N}_{0}}$. From [48, Theorem 3.7] and [24, Corollary 4] it follows that

$$
D_{\alpha}^{*}\left(\rho_{0} \| \sigma_{0}\right)=\lim _{n \rightarrow \infty} \frac{1}{n} D_{\alpha}^{\text {meas }}\left(\rho_{0}^{\otimes n} \| \sigma_{0}^{\otimes n}\right) .
$$

Hence there exists an $n_{0} \in \mathbb{N}$ such that $\frac{1}{n} D_{\alpha}^{\text {meas }}\left(\rho_{0}^{\otimes n} \| \sigma_{0}^{\otimes n}\right)>v$ for all $n \geq n_{0}$. Therefore,

$$
\liminf _{n \rightarrow \infty} \frac{1}{n} D_{\alpha}^{\text {meas }}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right) \geq \liminf _{n \rightarrow \infty} \frac{1}{n} D_{\alpha}^{\text {meas }}\left(\rho_{0}^{\otimes n} \| \sigma_{0}^{\otimes n}\right) \geq v
$$

which implies (3.19) by letting $v \nearrow D_{\alpha}^{*}(\rho \| \sigma)$.
When $\alpha>1$, the proof of (3.18) can proceed in the same way as above by appealing to [48, Corollary 4.6].

Remark 3.14. When $\alpha<1$, it may happen that $\bar{D}_{\alpha}^{\text {test }}(\rho \| \sigma)<\bar{D}_{\alpha}^{\text {meas }}(\rho \| \sigma)$, even for finitedimensional commuting states $\rho$ and $\sigma$; see [47].

## 4 The $C^{*}$-algebra case

### 4.1 Sandwiched and standard Rényi divergences in $C^{*}$-algebras

In this section we first extend the notion of the sandwiched and the standard Rényi divergences to positive linear functionals on a general unital $C^{*}$-algebra. In the rest of the section let $\mathcal{A}$ be a unital $C^{*}$-algebra. (The unitality assumption is not essential, since we can work with the unitization $\mathcal{A} \oplus \mathbb{C}$ if $\mathcal{A}$ is non-unital.) Let $\mathcal{A}_{+}^{*}$ be the set of positive linear functionals (automatically bounded) on $\mathcal{A}$. To define $D_{\alpha}^{*}$ and $D_{\alpha}$ for $\rho, \sigma \in \mathcal{A}_{+}^{*}$, let us consider the universal representation $\left\{\pi_{u}, \mathcal{H}_{u}\right\}$ of $\mathcal{A}$; then $\rho$ and $\sigma$ have the respective normal extensions $\bar{\rho}$ and $\bar{\sigma}$ to $\pi_{u}(\mathcal{A})^{\prime \prime} \cong \mathcal{A}^{* *}$ such that $\rho=\bar{\rho} \circ \pi$ and $\sigma=\bar{\sigma} \circ \pi$; see, e.g., [69, Definition III.2.3, Theorem III.2.4].
Definition 4.1. For every $\rho, \sigma \in \mathcal{A}_{+}^{*}$ let $\bar{\rho}, \bar{\sigma}$ be as stated above. For every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ we define the sandwiched Rényi $\alpha$-divergence of $\rho$ and $\sigma$ by

$$
\begin{equation*}
\widehat{D}_{\alpha}^{*}(\rho \| \sigma):=D_{\alpha}^{*}(\bar{\rho} \| \bar{\sigma}) . \tag{4.1}
\end{equation*}
$$

For $\alpha \in[0,+\infty) \backslash\{1\}$ define also the standard Rényi $\alpha$-divergence of $\rho$ and $\sigma$ by

$$
\begin{equation*}
\widehat{D}_{\alpha}(\rho \| \sigma):=D_{\alpha}(\bar{\rho} \| \bar{\sigma}) . \tag{4.2}
\end{equation*}
$$

One question arises immediately: when $\mathcal{A}$ itself is a von Neumann algebra and $\rho, \sigma$ are normal functionals, do definitions (4.1) and (4.2) give the same notions of sandwiched and standard Rényi divergences (see Section 2 for the definitions), i.e., do the equalities

$$
\begin{equation*}
\widehat{D}_{\alpha}^{*}(\rho \| \sigma)=D_{\alpha}^{*}(\rho \| \sigma), \quad \widehat{D}_{\alpha}(\rho \| \sigma)=D_{\alpha}(\rho \| \sigma) \tag{4.3}
\end{equation*}
$$

hold? Theorem 4.3 below shows that these are indeed the case. For stating it, let us introduce the following:

Definition 4.2. Let $\rho, \sigma \in \mathcal{A}_{+}^{*}$. We say that a representation $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is ( $\rho, \sigma$ )-normal, if there exist $\rho_{\pi}, \sigma_{\pi} \in\left(\pi(\mathcal{A})^{\prime \prime}\right)_{*}^{+}$such that $\rho=\rho_{\pi} \circ \pi, \sigma=\sigma_{\pi} \circ \pi$.

Theorem 4.3. Let $\rho, \sigma \in \mathcal{A}_{+}^{*}$ and $\pi$ be any $(\rho, \sigma)$-normal representation of $\mathcal{A}$. Then we have the following:
(i) For any $\alpha \in[1 / 2,+\infty) \backslash\{1\}$,

$$
\begin{equation*}
\widehat{D}_{\alpha}^{*}(\rho \| \sigma)=D_{\alpha}^{*}\left(\rho_{\pi} \| \sigma_{\pi}\right), \quad \text { i.e. }, \quad D_{\alpha}^{*}(\bar{\rho} \| \bar{\sigma})=D_{\alpha}^{*}\left(\rho_{\pi} \| \sigma_{\pi}\right) . \tag{4.4}
\end{equation*}
$$

(ii) For any $\alpha \in[0,+\infty) \backslash\{1\}$,

$$
\begin{equation*}
\widehat{D}_{\alpha}(\rho \| \sigma)=D_{\alpha}\left(\rho_{\pi} \| \sigma_{\pi}\right), \quad \text { i.e., } \quad D_{\alpha}(\bar{\rho} \| \bar{\sigma})=D_{\alpha}\left(\rho_{\pi} \| \sigma_{\pi}\right) . \tag{4.5}
\end{equation*}
$$

In particular, the equalities in (4.3) hold when $\mathcal{A}$ is a von Neumann algebra and $\rho, \sigma$ are normal functionals.

The proof of the theorem is somewhat technical based on Kosaki's interpolation $L^{p}$-spaces [42], so we defer it to Appendix H. Once (4.3) is confirmed by Theorem 4.3, we may and do rewrite $\widehat{D}_{\alpha}^{*}(\rho \| \sigma)$ and $\widehat{D}(\rho \| \sigma)$ for $\rho, \sigma \in \mathcal{A}_{+}^{*}$ as $D_{\alpha}^{*}(\rho \| \sigma)$ and $D_{\alpha}(\rho \| \sigma)$, respectively.

Remark 4.4. Recall that $D_{1 / 2}^{*}(\rho \| \sigma)=-2 \log F(\rho, \sigma)$ for $\rho, \sigma \in \mathcal{M}_{*}^{+}$, where $\mathcal{M}$ is a von Neumann algebra and $F(\rho, \sigma)$ is the fidelity of $\rho, \sigma$. Theorem 4.3 (for $\alpha=1 / 2$ ) shows that the fidelity of states $\rho, \sigma \in \mathcal{A}_{+}^{*}$ may be defined by $F\left(\rho_{\pi}, \sigma_{\pi}\right)$ via any $(\rho, \sigma)$-normal representation $\pi$ of $\mathcal{A}$. In view of (2.4), Theorem 4.3 also shows that the relative entropy $D(\rho \| \sigma)$ of $\rho, \sigma \in \mathcal{A}_{+}^{*}$ can be defined by $D\left(\rho_{\pi} \| \sigma_{\pi}\right)$ as above; see [3, Sec. 5] and [33, Lemma 3.1].

Based on Definition 4.1 and Theorem 4.3 we can easily extend properties of the sandwiched and the standard Rényi divergences from the von Neumann setting to the $C^{*}$-algebra setting. For instance, some important ones are given in the next proposition.

Proposition 4.5. Let $\mathcal{A}$ be a unital $C^{*}$-algebra and $\rho, \sigma \in \mathcal{A}_{+}^{*}$.
(1) Joint lower semi-continuity: The map $(\rho, \sigma) \in \mathcal{A}_{+}^{*} \times \mathcal{A}_{+}^{*} \mapsto D_{\alpha}^{*}(\rho \| \sigma)$ is jointly lower semi-continuous in the norm topology for every $\alpha \in(1,+\infty)$ and jointly continuous in the norm topology for every $\alpha \in[1 / 2,1)$. The map $(\rho, \sigma) \in \mathcal{A}_{+}^{*} \times \mathcal{A}_{+}^{*} \mapsto D_{\alpha}(\rho \| \sigma)$ is jointly lower semi-continuous in the weak (i.e., $\sigma\left(\mathcal{A}^{*}, \mathcal{A}^{* *}\right)$-) topology for every $\alpha \in(1,2]$ and jointly continuous in the norm topology for every $\alpha \in[0,1)$.
(2) Monotonicity (Data-processing inequality): Let $\Phi: \mathcal{B} \rightarrow \mathcal{A}$ be a unital positive linear map between unital $C^{*}$-algebras. Then for every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$,

$$
D_{\alpha}^{*}(\rho \circ \Phi \| \sigma \circ \Phi) \leq D_{\alpha}^{*}(\rho \| \sigma) .
$$

If $\Phi$ is, in addition, a Schwarz map, then for every $\alpha \in[0,2] \backslash\{1\}$,

$$
D_{\alpha}(\rho \circ \Phi \| \sigma \circ \Phi) \leq D_{\alpha}(\rho \| \sigma) .
$$

(3) Inequality between $D_{\alpha}^{*}$ and $D_{\alpha}$ : For every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$,

$$
D_{\alpha}^{*}(\rho \| \sigma) \leq D_{\alpha}(\rho \| \sigma)
$$

(4) Martingale convergence: Assume that $\left\{\mathcal{A}_{i}\right\}_{i \in \mathcal{I}}$ is an increasing net of $C^{*}$-subalgebras of $\mathcal{A}$ containing the unit of $\mathcal{A}$ such that $\bigcup_{i \in \mathcal{I}} \mathcal{A}_{i}$ is norm-dense in $\mathcal{A}$. Then for every $\alpha \in[0,2] \backslash\{1\}$,

$$
D_{\alpha}\left(\left.\rho\right|_{\mathcal{A}_{i}} \|\left.\sigma\right|_{\mathcal{A}_{i}}\right) \nearrow D_{\alpha}(\rho \| \sigma) .
$$

Moreover, if there exists a $(\rho, \sigma)$-normal representation $\pi$ of $\mathcal{A}$ such that $\pi(\mathcal{A})^{\prime \prime}$ is $\sigma$ finite (this is automatic if $\mathcal{A}$ is separable), then for every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$,

$$
D_{\alpha}^{*}\left(\left.\rho\right|_{\mathcal{A}_{i}} \|\left.\sigma\right|_{\mathcal{A}_{i}}\right) \nearrow D_{\alpha}^{*}(\rho \| \sigma) .
$$

Proof. (1) Let $\rho_{n}, \sigma_{n} \in \mathcal{A}_{+}^{*}, n \in \mathbb{N}$, be such that $\left\|\rho_{n}-\rho\right\| \rightarrow 0$ and $\left\|\sigma_{n}-\sigma\right\| \rightarrow 0$, with normal extensions $\bar{\rho}_{n}, \bar{\sigma}_{n}$ to $\pi_{u}(\mathcal{A})^{\prime \prime}$ as well as $\bar{\rho}, \bar{\sigma}$. By Kaplansky's density theorem, note that $\left\|\bar{\rho}_{n}-\bar{\rho}\right\|=\left\|\rho_{n}-\rho\right\|$ and $\left\|\bar{\sigma}_{n}-\bar{\sigma}\right\|=\left\|\sigma_{n}-\sigma\right\|$. Hence the assertions follow from the corresponding result in the von Neumann algebra case; see [38, Proposition 3.10] and [29, Theorem 3.16 (3)] for $D_{\alpha}^{*}$, and [27, Proposition 5.3 (6)] and [29, Corollary 3.8] for $D_{\alpha}$.
(2) As shown in the proof of [28, Proposition 7.4], $\Phi$ extends to a unital positive normal $\operatorname{map} \bar{\Phi}: \pi_{u}(\mathcal{B})^{\prime \prime} \rightarrow \pi_{u}(\mathcal{A})^{\prime \prime}$ in such a way that $\bar{\Phi} \circ \pi_{u}=\pi_{u} \circ \Phi$, where $\pi_{u}$ on the left-hand side is the universal representation of $\mathcal{B}$ and that of $\mathcal{A}$ is on the right-hand side. Then $\bar{\rho} \circ \bar{\Phi}$ and $\bar{\sigma} \circ \bar{\Phi}$ are the normal extensions of $\rho \circ \Phi$ and $\sigma \circ \Phi$ to $\pi_{u}(\mathcal{B})^{\prime \prime}$, as seen in the proof of [28, Proposition 7.4]. By [38, Theorem 3.14] and [39, Theorem 4.1] ${ }^{3}$ we have $D_{\alpha}^{*}(\rho \circ \Phi \| \sigma \circ \Phi) \leq D_{\alpha}^{*}(\rho \| \sigma)$ for all $\alpha \in[1 / 2,+\infty) \backslash\{1\}$. In addition, assume that $\Phi$ is a Schwarz map. Then it is easy to see by Kaplansky's density theorem that $\bar{\Phi}$ is a Schwarz map again. Hence [27, Proposition $5.3(9)]$ gives $D_{\alpha}(\rho \circ \Phi \| \sigma \circ \Phi) \leq D_{\alpha}(\rho \| \sigma)$ for all $\alpha \in[0,2] \backslash\{1\}$.
(3) By [9, Theorem 12] (also [38, Corollary 3.6]) we have

$$
D_{\alpha}^{*}(\rho \| \sigma)=D_{\alpha}^{*}(\bar{\rho} \| \bar{\sigma}) \leq D_{\alpha}(\bar{\rho} \| \bar{\sigma})=D_{\alpha}(\rho \| \sigma) .
$$

[^1](4) Let $\pi$ be as stated in the latter assertion (for $D_{\alpha}^{*}$ ). Then it is clear that $\left\{\pi\left(\mathcal{A}_{i}\right)^{\prime \prime}\right\}_{i \in \mathcal{I}}$ is an increasing net of von Neumann subalgebras of $\sigma$-finite $\pi(\mathcal{A})^{\prime \prime}$ containing the unit of $\pi(\mathcal{A})^{\prime \prime}$ such that $\left(\bigcup_{i} \pi\left(\mathcal{A}_{i}\right)^{\prime \prime}\right)^{\prime \prime}=\pi(\mathcal{A})^{\prime \prime}$. By Theorems 4.3 and 3.1 , for every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ we have
$$
D_{\alpha}^{*}\left(\left.\rho\right|_{\mathcal{A}_{i}} \|\left.\sigma\right|_{\mathcal{A}_{i}}\right)=D_{\alpha}^{*}\left(\left.\rho_{\pi}\right|_{\pi\left(\mathcal{A}_{i}\right)^{\prime \prime}} \|\left.\sigma_{\pi}\right|_{\pi\left(\mathcal{A}_{i}\right)^{\prime \prime}}\right) \nearrow D_{\alpha}^{*}\left(\rho_{\pi} \| \sigma_{\pi}\right)=D_{\alpha}^{*}(\rho \| \sigma)
$$

As for the first assertion, note that $Q_{\alpha}(\bar{\rho} \| \bar{\sigma})$ for $\alpha \in(1,2]$ is the standard $f$-divergence of $\bar{\rho}, \bar{\sigma}$ for $f(t)=t^{\alpha}$ and $-Q_{\alpha}(\bar{\rho} \| \bar{\sigma})$ for $\alpha \in[0,1)$ is that for $f(t)=-t^{\alpha}$. Since these functions $f_{\alpha}$ are operator convex on $(0,+\infty)$, the martingale convergence for $D_{\alpha}$ when $\alpha \in[0,2] \backslash\{1\}$ follows from [27, Theorem 4.1 (v)].

Example 4.6. Let $\mathcal{C}(\mathcal{H})$ be the compact operator ideal in $\mathcal{B}(\mathcal{H})$, and $\mathcal{C}_{1}(\mathcal{H})$ be the trace class on $\mathcal{H}$. Consider the unital $C^{*}$-algebra $\mathcal{A}:=\mathcal{C}(\mathcal{H})+\mathbb{C} 1$. It is well known that $\mathcal{A}^{*} \cong \mathcal{C}_{1}(\mathcal{H})$ and $\mathcal{C}_{1}(\mathcal{H})^{*} \cong \mathcal{B}(\mathcal{H})$. Thus we can identify $\mathcal{A}_{+}^{*}=\mathcal{C}_{1}(\mathcal{H})_{+}=\mathcal{B}(\mathcal{H})_{*}^{+}$. With this identification, for every $\rho, \sigma \in \mathcal{C}_{1}(\mathcal{H})_{+}$, the $C^{*}$-versions $D_{\alpha}^{*}(\rho \| \sigma)$ and $D_{\alpha}(\rho \| \sigma)$ of $\rho, \sigma$ considered as elements of $\mathcal{A}_{+}^{*}$ coincide with those of $\rho, \sigma$ as elements of $\mathcal{B}(\mathcal{H})_{*}^{+}$. Recently in [45], the $(\alpha, z)$-Rényi divergences, a more general notion than the sandwiched and the standard Rényi divergences, have been discussed for more general operators $\rho, \sigma$ in $\mathcal{B}(\mathcal{H})_{+}$, as well as their strong converse exponents.

Example 4.7. Let $(\mathcal{A}, G, \tau)$ be a $C^{*}$-dynamical system, where $G$ is a group and $\tau$ is an action of $G$ as ${ }^{*}$-automorphisms of $\mathcal{A}$. Let $\mathcal{S}_{G}(\mathcal{A})$ denote the set of $\tau$-invariant states of $\mathcal{A}$. Let $\varphi \in \mathcal{S}_{G}(\mathcal{A})$ and $\left\{\pi_{\varphi}, \mathcal{H}_{\varphi}, \xi_{\varphi}\right\}$ be the cyclic representation of $\mathcal{A}$ induced by $\varphi$. Then there is a unique unitary representation $U_{\varphi}$ of $G$ on $\mathcal{H}_{\varphi}$ such that $U_{\varphi}(g) \xi_{\varphi}=\xi_{\varphi}$ and $\pi_{\varphi}\left(\tau_{g}(a)\right)=$ $U_{\varphi}(g) \pi_{\varphi}(a) U_{\varphi}(g)^{*}$ for all $a \in \mathcal{A}, g \in G$. It is well known (see, e.g., [10, Corollary 4.3.11]) that $\mathcal{A}$ is $G$-abelian, i.e., $\pi_{\varphi}(\mathcal{A})^{\prime} \cap U_{\varphi}(G)^{\prime}$ is abelian for all $\varphi \in \mathcal{S}_{G}(\mathcal{A})$ if and only if $\mathcal{S}_{G}(\mathcal{A})$ is a (Choquet) simplex. In this case, each $\varphi \in \mathcal{S}_{G}(\mathcal{A})$ has a unique maximal representing measure $\mu_{\varphi}$ on $\mathcal{S}_{G}(\mathcal{A})$, which is the $\pi_{\varphi}(\mathcal{A})^{\prime} \cap U_{\varphi}(G)^{\prime}$-orthogonal measure of $\varphi$; see [10, Proposition 4.3.3]. Define a unital positive linear map $\Phi: \mathcal{A} \rightarrow C\left(\mathcal{S}_{G}(\mathcal{A})\right)$ (consisting of all continuous functions on $\left.\mathcal{S}_{G}(\mathcal{A})\right)$ by $(\Phi a)(\omega):=\omega(a)$ for $a \in \mathcal{A}, \omega \in \mathcal{S}_{G}(\mathcal{A})$. When $\mathcal{A}$ is $G$-abelian and $\rho, \sigma \in \mathcal{S}_{G}(\mathcal{A})$, since $\rho=\mu_{\rho} \circ \Phi$ and $\sigma=\mu_{\sigma} \circ \Phi$, Proposition 4.5 (2) gives

$$
\begin{array}{ll}
D_{\alpha}^{*}(\rho \| \sigma) \leq D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right), & \alpha \in[1 / 2,+\infty) \backslash\{1\}  \tag{4.6}\\
D_{\alpha}(\rho \| \sigma) \leq D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right), & \alpha \in[0,2] \backslash\{1\}
\end{array}
$$

where $D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right)$ is the classical Rényi relative entropy. Now assume that $\mathcal{A}$ is $G$-central, i.e., $\pi_{\varphi}(\mathcal{A})^{\prime} \cap U_{\varphi}(G)^{\prime} \subseteq \pi_{\varphi}(\mathcal{A})^{\prime \prime}\left(\right.$ hence $\pi_{\varphi}(\mathcal{A})^{\prime} \cap U_{\varphi}(G)^{\prime} \subseteq$ the center of $\left.\pi_{\varphi}(\mathcal{A})^{\prime \prime}\right)$ for all $\varphi \in \mathcal{S}_{G}(\mathcal{A})$, which is stronger than $G$-abeliannes and weaker than some other conditions of asymptotic abeliannes; see [10, Sec. 4.3], [17]. In this case, with $\varphi:=(\rho+\sigma) / 2$ there is an isomorphism $\theta_{\varphi}: L^{\infty}\left(\mathcal{S}_{G}(\mathcal{A}), \mu_{\varphi}\right) \rightarrow \pi_{\varphi}(\mathcal{A})^{\prime} \cap U_{\varphi}(G)^{\prime}$ such that

$$
\begin{equation*}
\left\langle\xi_{\varphi}, \theta_{\varphi}(f) \pi_{\varphi}(a) \xi_{\varphi}\right\rangle=\int f(\omega) \omega(a) d \mu_{\varphi}(\omega), \quad f \in L^{\infty}\left(\mathcal{S}_{G}(\mathcal{A}), \mu_{\varphi}\right) \tag{4.7}
\end{equation*}
$$

For this, see [10, Proposition 41.22]. Since $\rho, \sigma \leq 2 \varphi$, note [10, Corollary 4.1.17] that $\mu_{\rho}, \mu_{\sigma} \leq$ $2 \mu_{\varphi}$ so that one can take $g_{\rho}:=d \mu_{\rho} / d \mu_{\varphi}$ and $g_{\sigma}:=d \mu_{\sigma} / d \mu_{\varphi}$ in $L^{\infty}\left(\mathcal{S}_{G}(\mathcal{A}), \mu_{\varphi}\right)_{+}$. Then for every $a \in \mathcal{A}$ one has

$$
\rho(a)=\int g_{\rho}(\omega) \omega(a) d \mu_{\varphi}(\omega)=\left\langle\xi_{\varphi}, \theta_{\varphi}\left(g_{\rho}\right) \pi_{\varphi}(a) \xi_{\varphi}\right\rangle
$$

thanks to (4.7). Hence the normal extension of $\rho$ to $\pi_{\varphi}(\mathcal{A})^{\prime \prime}$ is given as

$$
\tilde{\rho}(x)=\left\langle\xi_{\varphi}, \theta_{\varphi}\left(g_{\rho}\right) x \xi_{\varphi}\right\rangle, \quad x \in \pi_{\varphi}(\mathcal{A})^{\prime \prime},
$$

which implies that

$$
\begin{aligned}
\tilde{\rho}\left(\theta_{\varphi}(f)\right) & =\left\langle\xi_{\varphi}, \theta_{\varphi}\left(g_{\rho} f\right) \xi_{\varphi}\right\rangle=\int g_{\rho}(\omega) f(\omega) d \mu_{\varphi}(\omega) \quad(\text { by }(4.7)) \\
& =\int f(\omega) d \mu_{\rho}(\omega), \quad f \in L^{\infty}\left(\mathcal{S}_{G}(\mathcal{A}), \mu_{\varphi}\right) .
\end{aligned}
$$

Hence $\tilde{\rho} \circ \theta_{\varphi}=\mu_{\rho}$ follows. Similarly, replacing $\rho, g_{\rho}$ with $\sigma, g_{\sigma}$ in the above argument, one has $\tilde{\sigma} \circ \theta_{\varphi}=\mu_{\sigma}$ for the normal extension $\tilde{\sigma}$ of $\sigma$ to $\pi_{\varphi}(\mathcal{A})^{\prime \prime}$. From Proposition $4.5(2)$ and Theorem 4.3 it follows that

$$
\begin{array}{ll}
D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right) \leq D_{\alpha}^{*}(\tilde{\rho} \| \tilde{\sigma})=D_{\alpha}^{*}(\rho \| \sigma), & \\
D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right) \leq D_{\alpha}(\tilde{\rho} \| \tilde{\sigma})=D_{\alpha}(\rho \| \sigma), & \alpha \in[0,2] \backslash\{1\} . \tag{4.8}
\end{array}
$$

Therefore, when $\mathcal{A}$ is $G$-central and $\rho, \sigma \in \mathcal{S}_{G}(\mathcal{A})$, it follows from (4.6) and (4.8) that $D_{\alpha}^{*}(\rho \| \sigma)=D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right)$ for all $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ and $D_{\alpha}(\rho \| \sigma)=D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right)$ for all $\alpha \in[0,2] \backslash\{1\}$, showing also $D(\rho \| \sigma)=D\left(\mu_{\rho} \| \mu_{\sigma}\right)$ as shown in [33, Theorem 3.2].

Example 4.8. Let $\mathcal{B}=\bigotimes_{1}^{\infty} \mathbb{M}_{d}$ be a UHF (or one-dimensional spin) $C^{*}$-algebra, and $\tau$ be the action of $S_{\infty}$ (the group of finite permutations on $\mathbb{N}$ ) given by $\tau\left(\otimes_{1}^{\infty} a_{n}\right)=\bigotimes_{1}^{\infty} a_{g(n)}, g \in S_{\infty}$. Let $K$ be a compact group and $u_{k}(k \in K)$ a continuous unitary representation of $K$ on $\mathbb{C}^{d}$, so a product action $\beta$ of $K$ on $\mathcal{B}$ is defined by $\beta_{k}:=\bigotimes_{1}^{\infty} \operatorname{Ad}\left(u_{k}\right)$ (where $\left.\operatorname{Ad}\left(u_{k}\right)=u_{k}(\cdot) u_{k}^{*}\right)$. The $\beta$-fixed point $C^{*}$-subalgebra $\mathcal{A}:=\mathcal{B}^{\beta}$ of $\mathcal{B}$ is called a gauge-invariant $C^{*}$-algebra. Let $\mathcal{B}_{n}:=\bigotimes_{1}^{n} \mathbb{M}_{d}$ and $\mathcal{A}_{n}:=\mathcal{B}_{n}^{\beta}=\mathcal{A} \cap \mathcal{B}_{n}$ for $n \geq 1$; then $\bigcup_{n=1}^{\infty} \mathcal{A}_{n}$ is norm-dense in $\mathcal{A}$ (so $\mathcal{A}$ is an AF $C^{*}$-algebra.) Then $\left(\mathcal{B}, S_{\infty}, \tau\right)$ and $\left(\mathcal{A}, S_{\infty},\left.\tau\right|_{\mathcal{A}}\right)$ are typical cases of Example 4.7, since $\mathcal{B}$ and $\mathcal{A}$ are asymptotically abelian with respect to $S_{\infty}$. Størmer's theorem [65] says that the extreme points of $\mathcal{S}_{S_{\infty}}(\mathcal{B})$ are the symmetric product states $\otimes_{1}^{\infty} \psi, \psi \in \mathcal{S}\left(\mathbb{M}_{d}\right)$. For every $\rho, \sigma \in \mathcal{S}_{S_{\infty}}(\mathcal{A})$, by Proposition 4.5 (4) and Example 4.7 one has

$$
\begin{array}{ll}
D_{\alpha}^{*}(\rho \| \sigma)=\lim _{n \rightarrow \infty} D_{\alpha}^{*}\left(\left.\rho\right|_{\mathcal{A}_{n}} \|\left.\sigma\right|_{\mathcal{A}_{n}}\right)=D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right), & \alpha \in[1 / 2,+\infty) \backslash\{1\}, \\
D_{\alpha}(\rho \| \sigma)=\lim _{n \rightarrow \infty} D_{\alpha}\left(\left.\rho\right|_{\mathcal{A}_{n}} \|\left.\sigma\right|_{\mathcal{A}_{n}}\right)=D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right), & \alpha \in[0,2] \backslash\{1\}, \tag{4.9}
\end{array}
$$

where $\mu_{\rho}, \mu_{\sigma}$ are the representing measures of $\rho, \sigma$ on the extreme boundary of $\mathcal{S}_{S_{\infty}}(\mathcal{A})$. In particular, it follows that $D_{\alpha}^{*}(\rho \| \sigma)=D_{\alpha}(\rho \| \sigma)$ for any $\rho, \sigma \in \mathcal{S}_{S_{\infty}}(\mathcal{A})$ and all $\alpha \in[1 / 2,2] \backslash$ $\{1\}$ even though $D_{\alpha}^{*}\left(\left.\rho\right|_{\mathcal{A}_{n}} \|\left.\sigma\right|_{\mathcal{A}_{n}}\right)<D_{\alpha}\left(\left.\rho\right|_{\mathcal{A}_{n}} \|\left.\sigma\right|_{\mathcal{A}_{n}}\right)$ unless $\left.\rho\right|_{\mathcal{A}_{n}}$ and $\left.\sigma\right|_{\mathcal{A}_{n}}$ commute. This phenomenon can naturally be understood by considering the asymptotic abeliannes of $\mathcal{A}$ with respect to $S_{\infty}$. Now let $\mathcal{T}(\mathcal{A})$ denote the set of tracial states on $\mathcal{A}$. Recall [61, Theorem 3.2] that any extreme point of $\mathcal{T}(\mathcal{A})$ is the restriction to $\mathcal{A}$ of a symmetric product state of $\mathcal{B}$, so that $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{S}_{S_{\infty}}(\mathcal{A})$. Furthermore, note (see [71, Sec. 4]) that $\mathcal{T}(\mathcal{A})$ is a face of $\mathcal{S}_{S_{\infty}}(\mathcal{A})$. For every $\rho, \sigma \in \mathcal{T}(\mathcal{A})$, since tracial states $\rho, \sigma$ commute, one has $D_{\alpha}^{*}(\rho \| \sigma)=D_{\alpha}(\rho \| \sigma)$ for all $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ by [29, Remark 3.18 (2)], and hence by (4.9),

$$
\begin{equation*}
D_{\alpha}(\rho \| \sigma)=\lim _{n \rightarrow \infty} D_{\alpha}\left(\left.\rho\right|_{\mathcal{A}_{n}} \|\left.\sigma\right|_{\mathcal{A}_{n}}\right)=D_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right), \quad \alpha \in[0,+\infty) \backslash\{1\}, \tag{4.10}
\end{equation*}
$$

where the representing measures $\mu_{\rho}, \mu_{\sigma}$ are supported on the extreme boundary of $\mathcal{T}(\mathcal{A})$. For example, let $d=2, K=\mathbb{T}=\{\zeta \in \mathbb{C}:|\zeta|=1\}$ and $u_{\zeta}:=\left[\begin{array}{ll}1 & 0 \\ 0 & \zeta\end{array}\right], \zeta \in \mathbb{T}$. Then $\mathcal{A}=\mathcal{B}^{\beta}$ is
the so-called GICAR algebra, and note that

$$
\mathcal{A}_{n}=\bigoplus_{k=0}^{n} \mathbb{M}_{\binom{n}{k}}, \quad n \geq 1
$$

It is well known that the extreme boundary of $\mathcal{T}(\mathcal{A})$ is parametrized by $\lambda \in[0,1]$; to be precise, it is $\left\{\omega_{\lambda}: \lambda \in[0,1]\right\}$, where $\omega_{\lambda}=\left.\left(\otimes_{1}^{\infty} \psi_{\lambda}\right)\right|_{\mathcal{A}}$ with $\psi_{\lambda}:=\operatorname{Tr}\left(\left[\begin{array}{cc}\lambda & 0 \\ 0 & 1-\lambda\end{array}\right].\right)$ on $\mathbb{M}_{2}$. Let $\rho, \sigma \in \mathcal{T}(\mathcal{A})$ and decompose them as $\rho=\int_{0}^{1} \omega_{\lambda} d \mu_{\rho}(\lambda), \sigma=\int_{0}^{1} \omega_{\lambda} d \mu_{\sigma}(\lambda)$ with unique probability measures $\mu_{\rho}, \mu_{\sigma}$ on $[0,1]$. Since

$$
\left.\omega_{\lambda}\right|_{\mathcal{A}_{n}}=\bigoplus_{k=0}^{n} \lambda^{k}(1-\lambda)^{n-k} \operatorname{Tr}_{\binom{n}{k}}
$$

(with convention $\lambda^{0}=1$ for $\lambda=0$ ), the convergence $\lim _{n \rightarrow \infty} Q_{\alpha}\left(\left.\rho\right|_{\mathcal{A}_{n}} \|\left.\sigma\right|_{\mathcal{A}_{n}}\right)=Q_{\alpha}\left(\mu_{\rho} \| \mu_{\sigma}\right)$ due to (4.10) can be rewritten in an explicit form as

$$
\begin{aligned}
\lim _{n \rightarrow \infty} & \sum_{k=0}^{n}\binom{n}{k}\left(\int_{0}^{1} \lambda^{k}(1-\lambda)^{n-k} d \mu_{1}(\lambda)\right)^{\alpha}\left(\int_{0}^{1} \lambda^{k}(1-\lambda)^{n-k} d \mu_{2}(\lambda)\right)^{1-\alpha} \\
& =\int_{0}^{1}\left(\frac{d \mu_{1}}{d\left(\mu_{1}+\mu_{2}\right)}\right)^{\alpha}\left(\frac{d \mu_{2}}{d\left(\mu_{1}+\mu_{2}\right)}\right)^{1-\alpha} d\left(\mu_{1}+\mu_{2}\right), \quad \alpha \in[0,+\infty)
\end{aligned}
$$

for any probability measures $\mu_{1}, \mu_{2}$ on $[0,1]$. (It does not seem easy to verify this convergence formula in a direct manner.)

### 4.2 The strong converse exponent in nuclear $C^{*}$-algebras

Let $\mathcal{A}$ and $\rho, \sigma \in \mathcal{A}_{+}^{*}$ be as in Section 4.1. For each $r \in \mathbb{R}$, we define the Hoeffding antidivergence $H_{r}^{*}(\rho \| \sigma)$ in the same way as in Definition 3.6 with use of $D_{\alpha}^{*}(\rho \| \sigma)$, i.e., $\widehat{D}_{\alpha}^{*}(\rho \| \sigma)$ in (4.1) or (4.4). For any $(\rho, \sigma)$-normal representation $\pi$ of $\mathcal{A}$, Theorem 4.3 gives

$$
\begin{equation*}
H_{r}^{*}(\rho \| \sigma)=H_{r}^{*}\left(\rho_{\pi} \| \sigma_{\pi}\right), \quad r \in \mathbb{R} \tag{4.11}
\end{equation*}
$$

We furthermore define strong converse exponents as in Definition 3.5 in the $C^{*}$-algebra setting. Here, recall (see, e.g., [69, Sec. IV.4]) that for $C^{*}$-algebras $\mathcal{A}$ and $\mathcal{B}$, a $C^{*}$-cross-norm on the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$ is not unique in general (though it exists always), and we have the smallest one $\|\cdot\|_{\min }$ and the largest one $\|\cdot\|_{\max }$. The minimal (or spatial) $C^{*}$ tensor product $\mathcal{A} \otimes_{\min } \mathcal{B}$ is the completion of $\mathcal{A} \odot \mathcal{B}$ with respect to $\|\cdot\|_{\text {min }}$, and the maximal $C^{*}$-tensor product $\mathcal{A} \otimes_{\max } \mathcal{B}$ is that with respect to $\|\cdot\|_{\max }$. To discuss the simple hypothesis testing on a $C^{*}$-algebra, the minimal $C^{*}$-tensor product is suitable for the following reasons: When $\mathcal{A}$ and $\mathcal{B}$ are realized in $\mathcal{B}\left(\mathcal{H}_{1}\right)$ and $\mathcal{B}\left(\mathcal{H}_{2}\right)$ respectively, $\mathcal{A} \otimes_{\min } \mathcal{B}$ is isomorphic to the $C^{*}$-algebra generated by $\mathcal{A} \odot \mathcal{B}$ in $\mathcal{B}\left(\mathcal{H}_{1} \otimes \mathcal{H}_{2}\right)$; see [69, Theorem IV.4.9 (iii)]. For any representations $\pi_{1}$ of $\mathcal{A}$ and $\pi_{2}$ of $\mathcal{B}$, the tensor product representation $\pi_{1} \otimes \pi_{2}$ of $\mathcal{A} \otimes_{\min } \mathcal{B}$ satisfies $\left(\pi_{1} \otimes \pi_{2}\right)\left(\mathcal{A} \otimes_{\min } \mathcal{B}\right)^{\prime \prime}=\pi_{1}(\mathcal{A})^{\prime \prime} \bar{\otimes} \pi_{2}(\mathcal{B})^{\prime \prime} ;$ see [69, Proposition IV.4.13].

Let $\rho, \sigma \in \mathcal{A}_{+}^{*}$. For each $n \in \mathbb{N}$ let $\mathcal{A}^{\otimes n}\left(=\mathcal{A}^{\otimes \min n}\right)$ be the $n$-fold minimal $C^{*}$-tensor product of $\mathcal{A}$, and $\rho_{n}:=\rho^{\otimes n}$ (resp., $\sigma_{n}:=\sigma^{\otimes n}$ ) be the $n$-fold tensor product of $\rho$ (resp., $\sigma$ ) on $\mathcal{A}^{\otimes n}$. The following are the strong converse exponents in the setting of the simple hypothesis testing for the null hypothesis $H_{0}: \rho$ versus the alternative hypothesis $H_{1}: \sigma$.

Definition 4.9. For each $r \in \mathbb{R}$ we define the strong converse exponents $\underline{s c}_{r}(\rho \| \sigma)$ etc. in the same expressions as in Definition 3.5, where $\left\{T_{n}\right\}$ in the present setting are taken as $T_{n} \in \mathcal{A}^{\otimes n}, 0 \leq T_{n} \leq 1, n \in \mathbb{N}$. Then the relation (3.5) is obvious as before.

Remark 4.10. The most general notion of a test in the $C^{*}$-algebra setting is not covered by tests taken in Definition 4.9. For instance, one could say that a measurement is represented by a self-adjoint element $X$ in $\mathcal{A}$, and the possible outcomes are the points of the spectrum of $X$. After measuring $X$, one could always make a classical post-processing, e.g., by dividing the spectrum of $X$ into two disjoint sets $B_{0}$ and $B_{1}$. If the measurement outcome falls into $B_{0}$, we accept $H_{0}$, otherwise we accept $H_{1}$. The error probabilities corresponding to such a test may not be described by an element $T \in \mathcal{A}$ with $0 \leq T \leq I$. However, the next lemma shows that considering such more general tests does not change the strong converse exponents.

Lemma 4.11. Let $\rho, \sigma \in \mathcal{A}_{+}^{*}$ and $\pi$ be a $(\rho, \sigma)$-normal representation of $\mathcal{A}$. For every $r \in \mathbb{R}$, let $\varepsilon_{r}(\rho \| \sigma)$ be any strong converse exponent given in Definition 4.9, and $\varepsilon_{r}\left(\rho_{\pi} \| \sigma_{\pi}\right)$ be the same strong converse exponent in Definition 3.5 for $\rho_{\pi}, \sigma_{\pi}$. Then we have

$$
\begin{equation*}
\varepsilon_{r}(\rho \| \sigma)=\varepsilon_{r}\left(\rho_{\pi} \| \sigma_{\pi}\right) \tag{4.12}
\end{equation*}
$$

Proof. Let $\mathcal{M}:=\pi(\mathcal{A})^{\prime \prime}$. For every $T_{n} \in \mathcal{A}^{\otimes n}, n \in \mathbb{N}$, with $0 \leq T_{n} \leq 1$, one has $\pi^{\otimes n}\left(T_{n}\right) \in$ $\mathcal{M}^{\bar{\otimes} n}, 0 \leq \pi^{\otimes n}\left(T_{n}\right) \leq 1, \rho_{\pi, n}\left(\pi^{\otimes n}\left(T_{n}\right)\right)=\rho_{n}\left(T_{n}\right)$ and $\sigma_{\pi, n}\left(\pi^{\otimes n}\left(T_{n}\right)\right)=\sigma_{n}\left(T_{n}\right)$, where $\rho_{\pi, n}:=$ $\left(\rho_{\pi}\right)^{\otimes n}$ and $\sigma_{\pi, n}:=\left(\sigma_{\pi}\right)^{\otimes n}$. Hence $\varepsilon_{r}\left(\rho_{\pi} \| \sigma_{\pi}\right) \leq \varepsilon_{r}(\rho \| \sigma)$ holds immediately. Conversely, for every $\widetilde{T}_{n} \in \mathcal{M}^{\otimes n}=\pi^{\otimes n}\left(\mathcal{A}^{\otimes n}\right)^{\prime \prime}, n \in \mathbb{N}$, with $0 \leq \widetilde{T}_{n} \leq 1$, by Kaplansky's density theorem and [35, Lemma IV.3.8] one can choose $T_{n} \in \mathcal{A}^{\otimes n}, n \in \mathbb{N}$, such that $0 \leq T_{n} \leq 1$ and

$$
\begin{aligned}
& \rho_{n}\left(T_{n}\right)=\rho_{\pi, n}\left(\pi^{\otimes n}\left(T_{n}\right)\right) \in \begin{cases}\left(e^{-1} \rho_{\pi, n}\left(\widetilde{T}_{n}\right), e \rho_{\pi, n}\left(\widetilde{T}_{n}\right)\right) & \text { if } \rho_{\pi, n}\left(\widetilde{T}_{n}\right)>0, \\
{\left[0,1 / n^{n}\right)} & \text { if } \rho_{\pi, n}\left(\widetilde{T}_{n}\right)=0,\end{cases} \\
& \sigma_{n}\left(T_{n}\right)=\sigma_{\pi, n}\left(\pi^{\otimes n}\left(T_{n}\right)\right) \in \begin{cases}\left(e^{-1} \sigma_{\pi, n}\left(\widetilde{T}_{n}\right), e \sigma_{\pi, n}\left(\widetilde{T}_{n}\right)\right) & \text { if } \sigma_{\pi, n}\left(\widetilde{T}_{n}\right)>0, \\
{\left[0,1 / n^{n}\right)} & \text { if } \sigma_{\pi, n}\left(\widetilde{T}_{n}\right)=0,\end{cases}
\end{aligned}
$$

which imply that

$$
\begin{aligned}
& -\frac{1}{n} \log \rho_{n}\left(T_{n}\right) \in \begin{cases}\left(-\frac{1}{n}-\frac{1}{n} \log \rho_{\pi, n}\left(\widetilde{T}_{n}\right), \frac{1}{n}-\frac{1}{n} \log \rho_{\pi, n}\left(\widetilde{T}_{n}\right)\right) & \text { if } \rho_{\pi, n}\left(\widetilde{T}_{n}\right)>0, \\
(\log n,+\infty] & \text { if } \rho_{\pi, n}\left(\widetilde{T}_{n}\right)=0,\end{cases} \\
& -\frac{1}{n} \log \sigma_{n}\left(T_{n}\right) \in \begin{cases}\left(-\frac{1}{n}-\frac{1}{n} \log \sigma_{\pi, n}\left(\widetilde{T}_{n}\right), \frac{1}{n}-\frac{1}{n} \log \sigma_{\pi, n}\left(\widetilde{T}_{n}\right)\right) & \text { if } \sigma_{\pi, n}\left(\widetilde{T}_{n}\right)>0, \\
(\log n,+\infty] & \text { if } \sigma_{\pi, n}\left(\widetilde{T}_{n}\right)=0 .\end{cases}
\end{aligned}
$$

From these one can see that $\varepsilon_{r}(\rho \| \sigma) \leq \varepsilon_{r}(\tilde{\rho} \| \tilde{\sigma})$. Hence (4.12) follows.
The next theorem is the $C^{*}$-algebra version of Theorem 3.7 under the assumption of the generated von Neumann algebra being injective, in particular, when $\mathcal{A}$ is nuclear. For the convenience of the reader, we recall the notion of nuclear $C^{*}$-algebras in Appendix E. Here we restrict $\rho, \sigma$ to states on $\mathcal{A}$ for the same reason as before.

Theorem 4.12. Let $\rho, \sigma \in \mathcal{A}_{+}^{*}$ be states such that $D_{\alpha}^{*}(\rho \| \sigma)<+\infty$ for some $\alpha>1$. If there exists a $(\rho, \sigma)$-normal representation $\pi$ of $\mathcal{A}$ such that $\pi(\mathcal{A})^{\prime \prime}$ is injective, then all the equalities in (3.8) hold for every $r \in \mathbb{R}$ in the present $C^{*}$-algebra situation too. In particular, this is the case if $\mathcal{A}$ is nuclear.

Proof. Let $\pi$ be as stated in the theorem. Then Theorem 3.7 says that the equalities in (3.8) for $\rho_{\pi}, \rho_{\sigma}$ in place of $\rho, \sigma$ hold for every $r \in \mathbb{R}$. Hence the assertion follows from Lemma 4.11 and (4.11).

We remark that the same formula as in Theorem 3.9 holds true under the assumption of Theorem 4.12, where $\alpha_{e^{-n r}}^{*}\left(\rho_{n} \| \sigma_{n}\right)$ is defined as in (3.14) with $T_{n} \in \mathcal{A}^{\otimes n}, 0 \leq T_{n} \leq 1$. The proof is the same as that of Theorem 3.9. Also, the generalized $\kappa$-cutoff rate $C_{\kappa}(\rho \| \sigma)$ in Definition 3.10 makes sense for states $\rho, \sigma \in \mathcal{A}_{+}^{*}$, and Theorem 3.11 is extended to the $C^{*}$-algebra setting with the same proof.

In the rest of the section let us discuss the (regularized) measured and the (regularized) test-measured Rényi divergences in the $C^{*}$-algebra setting. A measurement in $\mathcal{A}$ is given by $\mathfrak{M}=\left(M_{j}\right)_{1 \leq j \leq k}$ of $M_{j} \in \mathcal{A}_{+}$such that $\sum_{j=1}^{k} M_{j}=1$. Then $D_{\alpha}^{\text {meas }}(\rho \| \sigma)$ and $\bar{D}_{\alpha}^{\text {meas }}(\rho \| \sigma)$ for $\rho, \sigma \in \mathcal{A}_{+}^{*}$ are defined in the same way as in Definition 3.12 by taking measurements in $\mathcal{A}$ instead of $\mathcal{M}$. The test-measured versions $D_{\alpha}^{\text {test }}(\rho \| \sigma)$ and $\bar{D}_{\alpha}^{\text {test }}(\rho \| \sigma)$ for $\rho, \sigma \in \mathcal{A}_{+}^{*}$ are also defined in the same way as just after Definition 3.12 with tests $T \in \mathcal{A}, 0 \leq T \leq 1$.

Lemma 4.13. Let $\rho, \sigma \in \mathcal{A}_{+}^{*}$ and $\pi$ be any $(\rho, \sigma)$-normal representation of $\mathcal{A}$. Then for every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ we have

$$
\begin{align*}
D_{\alpha}^{\text {meas }}(\rho \| \sigma) & =D_{\alpha}^{\text {meas }}\left(\rho_{\pi} \| \sigma_{\pi}\right), & \bar{D}_{\alpha}^{\text {meas }}(\rho \| \sigma) & =\bar{D}_{\alpha}^{\text {meas }}\left(\rho_{\pi} \| \sigma_{\pi}\right),  \tag{4.13}\\
D_{\alpha}^{\text {test }}(\rho \| \sigma) & =D_{\alpha}^{\text {test }}\left(\rho_{\pi} \| \sigma_{\pi}\right), & \bar{D}_{\alpha}^{\text {test }}(\rho \| \sigma) & =\bar{D}_{\alpha}^{\text {test }}\left(\rho_{\pi} \| \sigma_{\pi}\right), \tag{4.14}
\end{align*}
$$

Proof. Let $\mathcal{M}:=\pi(\mathcal{A})^{\prime \prime}$. If $\left(M_{i}\right)_{1 \leq i \leq k}$ is a measurement in $\mathcal{A}$, then $\left(\pi\left(M_{i}\right)\right)_{1 \leq i \leq k}$ is a measurement in $\mathcal{M}$. Hence $D_{\alpha}^{\text {meas }}(\rho \| \sigma) \leq D_{\alpha}^{\text {meas }}\left(\rho_{\pi} \| \sigma_{\pi}\right)$ holds immediately. Conversely, let $\widetilde{\mathfrak{M}}=\left(\widetilde{M}_{i}\right)_{1 \leq i \leq k}$ be a measurement in $\mathcal{M}$. Consider the representation $\pi_{k}=\pi \otimes \operatorname{id}_{k}$ of $\mathbb{M}_{k}(\mathcal{A})=\mathcal{A} \otimes \mathbb{M}_{k}$ on $\mathcal{H} \otimes \mathbb{C}^{k}$ (where $\mathbb{M}_{k}$ is the $k \times k$ matrix algebra and $\mathcal{H}$ is the representing Hilbert space of $\pi$ ). Noting that $\pi_{k}\left(\mathbb{M}_{k}(\mathcal{A})\right)^{\prime \prime}=\mathbb{M}_{k}(\pi(\mathcal{A}))^{\prime \prime}=\mathbb{M}_{k}(\mathcal{M})$, we define $\tilde{A} \in \mathbb{M}_{k}(\mathcal{M})$ by

$$
\tilde{A}:=\left[\begin{array}{cc}
\widetilde{M}_{1}^{1 / 2} & \\
\vdots & 0 \\
\widetilde{M}_{k}^{1 / 2} &
\end{array}\right]
$$

Note that $\tilde{A}^{*} \tilde{A}$ is a projection with the $(1,1)$-block 1 and all other blocks 0 . Hence $\tilde{A}$ is a contraction. By Kaplansky's density theorem and [35, Lemma IV.3.8] there exists a net $\left\{A^{(\lambda)}\right\}$ of contractions in $\mathbb{M}_{k}(\mathcal{A})$ such that $\pi_{k}\left(A^{(\lambda)}\right) \rightarrow \tilde{A}$ in the strong* topology. Write $A^{(\lambda)}=\left[a_{i j}^{(\lambda)}\right]_{i, j=1}^{k}$ with $a_{i j}^{(\lambda)} \in \mathcal{A}$; then $\pi_{k}\left(A^{(\lambda)}\right)=\left[\pi\left(a_{i j}^{(\lambda)}\right)\right]_{i, j=1}^{k}$ so that $\pi\left(a_{i 1}^{(\lambda)}\right) \rightarrow \widetilde{M}_{i}^{1 / 2}$ in the strong* topology for $1 \leq i \leq k$. Set $M_{i}^{(\lambda)}:=\left(a_{i 1}^{(\lambda)}\right)^{*} a_{i 1}^{(\lambda)} \in \mathcal{A}_{+}, 1 \leq i \leq k$. Then $\pi\left(M_{i}^{(\lambda)}\right) \rightarrow \widetilde{M}_{i}$ strongly for $1 \leq i \leq k$, and $\sum_{i=1}^{k} M_{i}^{(\lambda)}=\sum_{i=1}^{k}\left(a_{i 1}^{(\lambda)}\right)^{*} a_{i 1}^{(\lambda)}$ is the (1,1)-block of $\left(A^{(\lambda)}\right)^{*} A^{(\lambda)}$ so that $\sum_{i=1}^{k} M_{i}^{(\lambda)} \leq 1$. Therefore, one can define a net $\left\{\mathfrak{M}_{\lambda}\right\}$ of measurements in $\mathcal{A}$ by $\mathfrak{M}_{\lambda}=\left(M_{i}^{(\lambda)}\right)_{1 \leq i \leq k+1}$, where $M_{k+1}^{(\lambda)}:=1-\sum_{i=1}^{k} M_{i}^{(\lambda)}$. Since $\pi\left(M_{k+1}^{(\lambda)}\right) \rightarrow 1-\sum_{i=1}^{k} \widetilde{M}_{i}=$ 0 , we have

$$
\mathfrak{M}_{\lambda}(\rho)=\left(\rho_{\pi} \circ \pi\left(M_{i}^{(\lambda)}\right)\right)_{1 \leq i \leq k+1} \longrightarrow \widetilde{\mathfrak{M}}\left(\rho_{\pi}\right) \oplus 0
$$

and similarly $\mathfrak{M}_{\lambda}(\sigma) \rightarrow \widetilde{\mathfrak{M}}\left(\sigma_{\pi}\right) \oplus 0$. From the lower semi-continuity of (classical) Rényi divergence, it follows that

$$
D_{\alpha}\left(\widetilde{\mathfrak{M}}\left(\rho_{\pi}\right) \| \widetilde{\mathfrak{M}}\left(\sigma_{\pi}\right)\right) \leq \underset{\lambda}{\liminf } D_{\alpha}\left(\mathfrak{M}_{\lambda}(\rho) \| \mathfrak{M}_{\lambda}(\sigma)\right) \leq D_{\alpha}^{\text {meas }}(\rho \| \sigma) .
$$

Hence $D_{\alpha}^{\text {meas }}\left(\rho_{\pi} \| \sigma_{\pi}\right) \leq D_{\alpha}^{\text {meas }}(\rho \| \sigma)$, implying the first equality of (4.13). Moreover, applying this to $\rho^{\otimes n}, \sigma^{\otimes n}$ and the representation $\pi^{\otimes n}$ of $\mathcal{A}^{\otimes n}$, we have for every $n \in \mathbb{N}$,

$$
D_{\alpha}^{\text {meas }}\left(\rho^{\otimes n} \| \sigma^{\otimes n}\right)=D_{\alpha}^{\text {meas }}\left(\rho_{\pi}^{\otimes n} \| \sigma_{\pi}^{\otimes n}\right)
$$

since $\rho_{\pi}^{\otimes n}, \sigma_{\pi}^{\otimes n}$ are the respective normal extensions of $\rho^{\otimes n}, \sigma^{\otimes n}$ to $\pi^{\otimes n}\left(\mathcal{A}^{\otimes n}\right)^{\prime \prime}=\mathcal{M}^{\otimes n}$. Thus, the second equality of (4.13) holds as well. The proof of (4.14) is similar (and easier) by approximating tests in $\mathcal{M}$ with those in $\mathcal{A}$ (similarly to the proof of Lemma 4.11).

Proposition 4.14. Assume that there exists a $(\rho, \sigma)$-normal representation $\pi$ of $\mathcal{A}$ such that $\pi(\mathcal{A})^{\prime \prime}$ is injective (in particular, this is the case if $\mathcal{A}$ is nuclear). Then (3.17) holds for every $\alpha \in[1 / 2,+\infty) \backslash\{1\}$ and (3.18) holds for every $\alpha>1$.

Proof. By Theorem 4.3 and Lemma 4.13, one can apply Proposition 3.13 to $\rho_{\pi}, \sigma_{\pi}$ to obtain the assertion.

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## A Relative modular operators

Let $\mathcal{M}$ be a general von Neumann algebra with the predual $\mathcal{M}_{*}$, and $\mathcal{M}_{*}^{+}$be the positive part of $\mathcal{M}_{*}$ consisting of normal positive linear functionals on $\mathcal{M}$. We consider $\mathcal{M}$ in its standard form $(\mathcal{M}, \mathcal{H}, J, \mathcal{P})[21]$, that is, $\mathcal{M}$ is represented on a Hilbert space $\mathcal{H}$ with the modular conjugation (a conjugate-linear involution) $J$ and the natural cone (a self-dual cone) $\mathcal{P}$, satisfying the following properties:
(1) $J M J=M^{\prime}\left(M^{\prime}\right.$ being the commutant of $\left.\mathcal{M}\right)$,
(2) $J x J=x^{*}, x \in \mathcal{M} \cap \mathcal{M}^{\prime}$ (the center of $\mathcal{M}$ ),
(3) $J \xi=\xi, \xi \in \mathcal{P}$,
(4) $x J x J \mathcal{P} \subseteq \mathcal{P}, x \in \mathcal{M}$.

Any von Neumann algebra has a unique (up to unitary conjugation) standard form; see [21, Theorem 2.3]. Any $\sigma \in \mathcal{M}_{*}^{+}$has a unique vector representative $\xi_{\sigma}$ in $\mathcal{P}$ so that $\sigma(x)=$ $\left\langle\xi_{\sigma}, x \xi_{\sigma}\right\rangle, x \in \mathcal{M}$. The support $s(\sigma)=s_{\mathcal{M}}(\sigma) \in \mathcal{M}$ of $\sigma$ is the orthogonal projection onto $\overline{\mathcal{M}^{\prime} \xi_{\sigma}}$, while the $\mathcal{M}^{\prime}$-support $s_{\mathcal{M}^{\prime}}(\sigma) \in \mathcal{M}^{\prime}$ is the orthogonal projection onto $\overline{\mathcal{M} \xi_{\sigma}}$ so that $s_{\mathcal{M}^{\prime}}(\sigma)=J s_{\mathcal{M}}(\sigma) J$.

For any $\rho, \sigma \in \mathcal{M}_{*}^{+}$, the closable conjugate-linear operators $S_{\rho, \sigma}$ and $F_{\rho, \sigma}$ are defined by

$$
\begin{aligned}
S_{\rho, \sigma}\left(x \xi_{\sigma}+\eta\right) & :=s_{\mathcal{M}}(\sigma) x^{*} \xi_{\sigma}, & & x \in \mathcal{M}, \eta \in\left(1-s_{\mathcal{M}^{\prime}}(\sigma)\right) \mathcal{H}, \\
F_{\rho, \sigma}\left(x^{\prime} \xi_{\sigma}+\zeta\right): & =s_{\mathcal{M}^{\prime}}(\sigma) x^{\prime *} \xi_{\sigma}, & & x^{\prime} \in \mathcal{M}^{\prime}, \zeta \in\left(1-s_{\mathcal{M}}(\sigma)\right) \mathcal{H},
\end{aligned}
$$

for which $S_{\rho, \sigma}^{*}=\bar{F}_{\rho, \sigma}$. The relative modular operator $\Delta_{\rho, \sigma}[3]$ is

$$
\Delta_{\rho, \sigma}:=S_{\rho, \sigma}^{*} \bar{S}_{\rho, \sigma}
$$

and the polar decomposition of $\bar{S}_{\rho, \sigma}$ is given as $\bar{S}_{\rho, \sigma}=J \Delta_{\rho, \sigma}^{1 / 2}$. When $\rho=\sigma, \Delta_{\sigma, \sigma}$ is the modular operator $\Delta_{\sigma}$.

When $\mathcal{M}=B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, consider the Hilbert-Schmidt class $\mathcal{C}_{2}(\mathcal{H})$ with the Hilbert-Schmidt inner product $\langle X, Y\rangle_{\mathrm{HS}}:=\operatorname{Tr}\left(X^{*} Y\right), X, Y \in \mathcal{C}_{2}(\mathcal{H})$; then the standard form of $B(\mathcal{H})$ is given as

$$
\left(B(\mathcal{H}), \mathcal{C}_{2}(\mathcal{H}), J={ }^{*}, \mathcal{C}_{2}(\mathcal{H})_{+}\right),
$$

where $B(\mathcal{H})$ is represented on $\mathcal{C}_{2}(\mathcal{H})$ by the left multiplications $L_{A} X:=A X$ for $A \in B(\mathcal{H})$, $X \in \mathcal{C}_{2}(\mathcal{H})$, and $\mathcal{C}_{2}(\mathcal{H})_{+}:=\left\{X \in \mathcal{C}_{2}(\mathcal{H}): X \geq 0\right\}$. Each $\rho \in B(\mathcal{H})_{*}^{+}$is identified with a trace-class operator $\hat{\rho} \geq 0$ so that $\rho(X)=\operatorname{Tr}(\hat{\rho} X)=\left\langle\hat{\rho}^{1 / 2}, X \hat{\rho}^{1 / 2}\right\rangle_{\mathrm{HS}}, X \in B(\mathcal{H})$, and $\hat{\rho}^{1 / 2} \in \mathcal{C}_{2}(\mathcal{H})_{+}$is the vector representative of $\rho$. For $\rho, \sigma \in B(\mathcal{H})_{*}^{+}$the relative modular operator $\Delta_{\rho, \sigma}$ is written as $\Delta_{\rho, \sigma}=L_{\hat{\rho}} R_{\hat{\sigma}^{-1}}$, where $\hat{\sigma}^{-1}$ is the generalized inverse (i.e., the inverse with restriction to the support $s(\sigma) \mathcal{H})$ of $\hat{\sigma}$ and $R_{\hat{\sigma}^{-1}}$ is the right multiplication by $\hat{\sigma}^{-1}$. Of course, when $\operatorname{dim} \mathcal{H}<+\infty$, we have $\mathcal{C}_{2}(\mathcal{H})=B(\mathcal{H})$.

## B Haagerup's $L^{p}$-spaces

Assume that $\mathcal{M}$ is $\sigma$-finite, i.e., there exists a faithful $\omega \in \mathcal{M}_{*}^{+}$. Let us denote by $\mathcal{N}$ the crossed product $\mathcal{M} \rtimes_{\omega} \mathbb{R}$ of $\mathcal{M}$ by the modular automorphism group $\sigma_{t}^{\omega}=\Delta_{\omega}^{i t}(\cdot) \Delta_{\omega}^{-i t}, t \in \mathbb{R}$. Le $\theta_{s}, s \in \mathbb{R}$, be the dual action of $\mathcal{N}$ so that $\tau \circ \theta_{s}=e^{-s} \tau, s \in \mathbb{R}$, where $\tau$ is the canonical trace on $\mathcal{N}$; the crossed product construction was developed in the structure theory of von Neumann algebras [68]. Let $\widetilde{\mathcal{N}}$ denote the space of $\tau$-measurable operators $[55,72]$ affiliated with $\mathcal{N}$. For each $p \in(0,+\infty]$, Haagerup's $L^{p}$-space $L^{p}(\mathcal{M})$ [72] is defined by

$$
L^{p}(\mathcal{M}):=\left\{x \in \widetilde{\mathcal{N}}: \theta_{s}(x)=e^{-s / p} x, s \in \mathbb{R}\right\}
$$

(in particular, $L^{\infty}(\mathcal{M})=\mathcal{M}$ ), whose positive part is $L^{p}(\mathcal{M})_{+}:=L^{p}(\mathcal{M}) \cap \widetilde{\mathcal{N}}_{+}$. There exists an order isomorphism $\mathcal{M}_{*} \cong L^{1}(\mathcal{M})$, given as $\psi \in \mathcal{M}_{*} \mapsto h_{\psi} \in L^{1}(\mathcal{M})$, so that $\operatorname{tr}\left(h_{\psi}\right):=\psi(\mathbf{1}), \psi \in \mathcal{M}_{*}$, defines a positive linear functional $\operatorname{tr}$ on $L^{1}(\mathcal{M})$. For $1 \leq p<+\infty$ the $L^{p}$-norm $\|a\|_{p}$ of $a \in L^{p}(\mathcal{M})$ is given by $\|a\|_{p}:=\operatorname{tr}\left(|a|^{p}\right)^{1 / p}$, and the $L^{\infty}$-norm $\|\cdot\|_{\infty}$ is the operator norm on $\mathcal{M}$. For $1 \leq p<+\infty, L^{p}(\mathcal{M})$ is a Banach space with the norm $\|\cdot\|_{p}$, whose dual Banach space is $L^{q}(\mathcal{M})$, where $1 / p+1 / q=1$, by the duality

$$
(a, b) \in L^{p}(\mathcal{M}) \times L^{q}(\mathcal{M}) \longmapsto \operatorname{tr}(a b)(=\operatorname{tr}(b a)) .
$$

In particular, $L^{2}(\mathcal{M})$ is a Hilbert space with the inner product $\langle a, b\rangle=\operatorname{tr}\left(a^{*} b\right)\left(=\operatorname{tr}\left(b a^{*}\right)\right)$. Then

$$
\left(\mathcal{M}, L^{2}(\mathcal{M}), J={ }^{*}, L^{2}(\mathcal{M})_{+}\right)
$$

becomes a standard form of $\mathcal{M}$, where $\mathcal{M}$ is represented on $L^{2}(\mathcal{M})$ by the left multiplication. Each $\rho \in \mathcal{M}_{*}^{+}$is represented as

$$
\rho(x)=\operatorname{tr}\left(x h_{\rho}\right)=\left\langle h_{\rho}^{1 / 2}, x h_{\rho}^{1 / 2}\right\rangle, \quad x \in \mathcal{M},
$$

with the vector representative $h_{\rho}^{1 / 2} \in L^{2}(\mathcal{M})_{+}$. Note that the support projection $s(\rho)(\in \mathcal{M})$ of the functional $\rho$ coincides with that of the operator $h_{\rho}$. For any projection $e \in \mathcal{M}$,

Haagerup's $L^{p}$-space $L^{p}(e \mathcal{M} e)$ is identified with $e L^{p}(\mathcal{M}) e$ and the standard form of $e \mathcal{M} e$ is given by $\left(e \mathcal{M} e, e L^{2}(\mathcal{M}) e, J={ }^{*}, e L^{2}(\mathcal{M})_{+} e\right)$.

Note that $L^{p}(\mathcal{M})$ is independent (up to isometric isomorphism) of the choice of $\omega$ (where $\omega$ can be a faithful normal semifinite weight unless $\mathcal{M}$ is $\sigma$-finite), and that when $\mathcal{M}$ is semifinite with a faithful normal semifinite trace $\tau_{0}, L^{p}(\mathcal{M})$ can be identified with the tracial $L^{p}$-space $L^{p}\left(\mathcal{M}, \tau_{0}\right)$ (see, e.g., [55]). In particular, when $\mathcal{M}=B(\mathcal{H})$ with $\omega=\operatorname{Tr}$ (and so $\Delta_{\omega}=\mathbf{1}$ ), note that $\mathcal{N}=\mathcal{M} \bar{\otimes} L^{\infty}(\mathbb{R})$ on $\mathcal{H} \otimes L^{2}(\mathbb{R})$ and the canonical trace on $\mathcal{N}$ is $\tau=\operatorname{Tr} \otimes \int_{\mathbb{R}}(\cdot) e^{t} d t$, so that $L^{p}(\mathcal{M})=\mathcal{C}_{p}(\mathcal{H}) \otimes e^{-t / p}$ with $\left\|X \otimes e^{-t / p}\right\|_{L^{p}(\mathcal{M})}=\|X\|_{p}$ for $X \in \mathcal{C}_{p}(\mathcal{H})$. Here, the symbol $e^{-t / p}$ is used to denote the multiplication operator on $L^{2}(\mathbb{R})$, and $\mathcal{C}_{p}(\mathcal{H})$ is the Schatten-von Neumann $p$-class with $\|X\|_{p}:=\left(\operatorname{Tr}|X|^{p}\right)^{1 / p}$. Therefore, $L^{p}(\mathcal{M})$ coincides with $\mathcal{C}_{p}(\mathcal{H})$ by just neglecting the superfluous tensor factor $e^{-t / p}$; see [30, Remark 8.16, Example 9.11] for more details on this matter.

It might be instructive to note that Haagerup's $L^{p}(\mathcal{M})$ is different from the tracial $L^{p}{ }_{-}$ space $L^{p}(\mathcal{N}, \tau)$ with the canonical trace $\tau$, even when $\mathcal{M}=B(\mathcal{H})$. In this case, $L^{p}(\mathcal{M})=$ $\mathcal{C}_{p}(\mathcal{H}) \otimes e^{-t / p}$ as stated above, and for every $X \in \mathcal{C}(\mathcal{H})$,

$$
\left\|X \otimes e^{-t / p}\right\|_{L^{p}(\mathcal{N}, \tau)}=\|X\|_{p}\left(\int_{\mathbb{R}}\left(e^{-t / p}\right)^{p} e^{t} d t\right)^{1 / p}=\|X\|_{p}\left(\int_{\mathbb{R}} d t\right)^{1 / p}=+\infty
$$

unless $X=0$. However, in the general case of $\mathcal{M}$, the exact relation of elements in $L^{p}(\mathcal{M})$ with the canonical trace $\tau$ on $\mathcal{N}$ is expressed as follows: for every $a \in L^{p}(\mathcal{M})$ and $p \in(0,+\infty)$,

$$
\mu_{t}(a)=t^{-1 / p}\|a\|_{p}, \quad t>0
$$

where $\mu_{t}(a)$ is the $t$ th generalized $s$-number of $a$ with respect to $\tau$; see [19, Lemma 4.8] and [30, Lemma 9.14]. The above expression is sometimes useful though it is not used in this paper.

## C Kosaki's interpolation $L^{p}$-spaces

Assume that $\mathcal{M}$ is $\sigma$-finite and let a faithful $\omega \in \mathcal{M}_{*}^{+}$be given with $h_{\omega} \in L^{1}(\mathcal{M})_{+}$. Consider an embedding $\mathcal{M}$ into $L^{1}(\mathcal{M})$ by $x \mapsto h_{\omega}^{1 / 2} x h_{\omega}^{1 / 2}$. Defining $\left\|h_{\omega}^{1 / 2} x h_{\omega}^{1 / 2}\right\|_{\infty}:=\|x\|_{\infty}$ on $h_{\omega}^{1 / 2} \mathcal{M} h_{\omega}^{1 / 2}$ we have a pair $\left(h_{\omega}^{1 / 2} \mathcal{M} h_{\omega}^{1 / 2}, L^{1}(\mathcal{M})\right)$ of compatible Banach spaces (see, e.g., [7]). For $1<p<+\infty$ Kosaki's (symmetric) $L^{p}$-space $L^{p}(\mathcal{M}, \omega)$ [42] with respect to $\omega$ is the complex interpolation Banach space

$$
C_{1 / p}\left(h_{\omega}^{1 / 2} \mathcal{M} h_{\omega}^{1 / 2}, L^{1}(\mathcal{M})\right)
$$

equipped with the interpolation norm $\|\cdot\|_{p, \omega}\left(=\|\cdot\|_{C_{1 / p}}\right)[7]$. Moreover, $L^{1}(\mathcal{M}, \omega):=L^{1}(\mathcal{M})$ with $\|\cdot\|_{1, \omega}=\|\cdot\|_{1}$ and $L^{\infty}(\mathcal{M}, \omega):=h_{\omega}^{1 / 2} \mathcal{M} h_{\omega}^{1 / 2}(\cong \mathcal{M})$ with $\|\cdot\|_{\infty, \omega}=\|\cdot\|_{\infty}$. Kosaki's theorem [42, Theorem 9.1] says that for every $p \in[1,+\infty]$ and $1 / p+1 / q=1$,

$$
\begin{align*}
& L^{p}(\mathcal{M}, \omega)=h_{\omega}^{\frac{1}{2 q}} L^{p}(\mathcal{M}) h_{\omega}^{\frac{1}{2 q}}\left(\subseteq L^{1}(\mathcal{M})\right)  \tag{C.1}\\
& \left\|h_{\omega}^{\frac{1}{2 q}} a h_{\omega}^{\frac{1}{2 q}}\right\|_{p, \omega}=\|a\|_{p}, \quad a \in L^{p}(\mathcal{M}) \tag{C.2}
\end{align*}
$$

that is, $L^{p}(\mathcal{M}) \cong L^{p}(\mathcal{M}, \omega)$ by the isometry $a \mapsto h_{\omega}^{\frac{1}{2 q}} a h_{\omega}^{\frac{1}{2 q}}$. Interpolation $L^{p}$-spaces were introduced in [42] in terms of more general embeddings $x \in \mathcal{M} \mapsto h_{\omega}^{\eta} x h_{\omega}^{1-\eta} \in L^{1}(\mathcal{M})$ with
$0 \leq \eta \leq 1$. (The $\eta=1 / 2$ case is the above symmetric $L^{1}(\mathcal{M}, \omega)$.) When $\mathcal{M}$ is general and $\omega \in \mathcal{M}_{*}^{+}$is not faithful with the support projection $e:=s(\omega) \in \mathcal{M}$, Kosaki's $L^{p}$-space $L^{p}(\mathcal{M}, \omega)$ with respect to $\omega$ is still defined over $e \mathcal{M e}$ so that (C.1) and (C.2) hold with $e L^{p}(\mathcal{M}) e$ in place of $L^{p}(\mathcal{M})$.

Consider now the special case $\mathcal{M}=B(\mathcal{H})$, and let $\omega \in B(\mathcal{H})_{*}^{+}$be given with $e:=s(\omega)$ and $\hat{\omega} \in \mathcal{C}_{1}(\mathcal{H})_{+}$representing $\omega$. When $1 \leq p \leq+\infty$ and $1 / p+1 / q=1$, Kosaki's $L^{p}$-space with respect to $\omega$ is $L^{p}(B(\mathcal{H}), \omega)=\hat{\omega}^{\frac{1}{2 q}} \mathcal{C}_{p}(\mathcal{H}) \hat{\omega}^{\frac{1}{2 q}}$ with $\left\|\hat{\omega}^{\frac{1}{2 q}} A \hat{\omega}^{\frac{1}{2 q}}\right\|_{p, \omega}=\|A\|_{p}$ for $A \in e \mathcal{C}_{p}(\mathcal{H}) e$ $\left(\right.$ where $\left.\mathcal{C}_{\infty}(\mathcal{H})=B(\mathcal{H})\right)$. In particular, when $\operatorname{dim} \mathcal{H}<+\infty, L^{p}(B(\mathcal{H}), \omega)=e B(\mathcal{H}) e=$ $B(e \mathcal{H})$ and the interpolation $L^{p}$-norm is $\|A\|_{p, \omega}=\left\|\hat{\omega}^{-\frac{1}{2 q}} A \hat{\omega}^{-\frac{1}{2 q}}\right\|_{p}$ for any $A \in B(e \mathcal{H})$. The interpolation norm in the finite-dimensional case was used in [6] for instance.

## D Generalized conditional expectations

Let $\mathcal{M}$ and $\mathcal{N}$ be ( $\sigma$-finite) von Neumann algebras, with standard forms ( $\mathcal{M}, \mathcal{H}, J, \mathcal{P}$ ) and $\left(\mathcal{N}, \mathcal{H}_{0}, J_{0}, \mathcal{P}_{0}\right)$, respectively (see Appendix A). Let $\Phi: \mathcal{N} \rightarrow \mathcal{M}$ be a unital positive map. Let a faithful $\omega \in \mathcal{M}_{*}^{+}$be given, and assume that $\omega \circ \Phi$ is normal and faithful on $\mathcal{N}$. In this case, $\Phi$ is automatically normal and faithful (i.e., $\Phi\left(x^{*} x\right)=0 \Longrightarrow x=0$ ). Then it was shown in [1] that there exists a unique unital normal positive map $\Phi_{\omega}^{*}: \mathcal{M} \rightarrow \mathcal{N}$ such that

$$
\begin{equation*}
\langle J x \Omega, \Phi(y) \Omega\rangle=\left\langle J_{0} \Phi_{\omega}^{*}(x) \Omega_{0}, y \Omega_{0}\right\rangle, \quad x \in \mathcal{M}, y \in \mathcal{N}, \tag{D.1}
\end{equation*}
$$

where $\Omega \in \mathcal{P}$ and $\Omega_{0} \in \mathcal{P}_{0}$ are the vector representatives of $\omega$ and $\omega \circ \Phi$, respectively. The $\operatorname{map} \Phi_{\omega}^{*}$ is also faithful. Moreover, we have

$$
\begin{equation*}
\omega \circ \Phi \circ \Phi_{\omega}^{*}=\omega, \tag{D.2}
\end{equation*}
$$

and $\Phi_{\omega}^{*}$ is completely positive if and only if so is $\Phi$. This map $\Phi_{\omega}^{*}$ is called the $\omega$-dual map of $\Phi$, or the Petz recovery map (see [59]), whose definition by (D.1) is independent of the choice of the standard forms of $\mathcal{M}, \mathcal{N}$. In terms of Haagerup's $L^{1}$-elements $h_{\omega}$ and $h_{\omega \circ \Phi}$ (see Appendix B), we note [29, Lemma 8.3] that the map $\Phi_{\omega}^{*}$ is determined by

$$
\begin{equation*}
\Phi_{*}\left(h_{\omega}^{1 / 2} x h_{\omega}^{1 / 2}\right)=h_{\omega \circ \Phi}^{1 / 2} \Phi_{\omega}^{*}(x) h_{\omega \circ \Phi}^{1 / 2}, \quad x \in \mathcal{M} \tag{D.3}
\end{equation*}
$$

where $\Phi_{*}: L^{1}(\mathcal{M}) \rightarrow L^{1}(\mathcal{N})$ is the predual map of $\Phi$ via $\mathcal{M}_{*} \cong L^{1}(\mathcal{M})$ and $\mathcal{N}_{*} \cong L^{1}(\mathcal{N})$, i.e., $\Phi_{*}\left(h_{\psi}\right)=h_{\psi \circ \Phi}, \psi \in \mathcal{M}_{*}$. Note that the construction of $\Phi_{\omega}^{*}$ is possible even when $\omega$ and/or $\omega \circ \Phi$ are not faithful (see [29, Theorem 6.1 and Lemma 8.3]), though the above setting is sufficient for our present purpose.

In particular, let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ containing the unit of $\mathcal{M}$, and $\omega \in \mathcal{M}_{*}^{+}$be faithful. The $\omega$-dual map $\Phi_{\omega}^{*}$ of the injection $\Phi: \mathcal{N} \hookrightarrow \mathcal{M}$ is called the generalized conditional expectation with respect to $\omega$ [1], which we denote by $\mathcal{E}_{\mathcal{N}, \omega}: \mathcal{M} \rightarrow \mathcal{N}$. The map $\mathcal{E}_{\mathcal{N}, \omega}$ is unital, normal, completely positive, and faithful. Property (D.2) becomes

$$
\omega \circ \mathcal{E}_{\mathcal{N}, \omega}=\omega .
$$

In the present case, the standard Hilbert space $\mathcal{H}_{0}$ for $\mathcal{N}$ is taken as $\mathcal{H}_{0}=\overline{\mathcal{N} \Omega}$, where the vector representative $\Omega_{0}$ of $\omega \circ \Phi=\left.\omega\right|_{\mathcal{N}}$ is equal to $\Omega$. Let $P$ be the orthogonal projection from $\mathcal{H}=\overline{\mathcal{M} \Omega}$ onto $\mathcal{H}_{0}=\overline{\mathcal{N} \Omega}$. In this situation, note [1] that $\mathcal{E}_{\mathcal{N}, \omega}=\Phi_{\omega}^{*}$ given in (D.1) and (D.3) can be written more explicitly as

$$
\begin{equation*}
\mathcal{E}_{\mathcal{N}, \omega}(x)=J_{0} P J x J P J_{0}=J_{0} P J x J J_{0}, \quad x \in \mathcal{M} \tag{D.4}
\end{equation*}
$$

which is also determined by $\mathcal{E}_{\mathcal{N}, \omega}(x) \Omega=J_{0} P J x \Omega, x \in \mathcal{M}$. As is well known [67], there exists a (genuine) conditional expectation (i.e., a norm-one projection) $E: \mathcal{M} \rightarrow \mathcal{N}$ such that $\omega \circ E=\omega$ on $\mathcal{M}$, if and only if $\mathcal{N}$ is globally invariant under the modular automorphism group $\sigma_{t}^{\omega}$ (see Appendix B) of $\mathcal{M}$ with respect to $\omega$, i.e., $\sigma_{t}^{\omega}(\mathcal{N})=\mathcal{N}, t \in \mathbb{R}$. If this is the case, $J_{0}=\left.J\right|_{\mathcal{H}_{0}}$ and $J P=P J$ hold so that $\mathcal{E}_{\mathcal{N}, \omega}=E$. An important property of $E$ is the bimodule property $E(a x b)=a E(x) b$ for $a, b \in \mathcal{N}$ and $x \in \mathcal{M}$, which $\mathcal{E}_{\mathcal{N}, \omega}$ does not satisfy in general. A merit of $\mathcal{E}_{\mathcal{N}, \omega}$ is that it always exists, while the existence of $E$ is very restrictive as stated above.

## E Injective von Neumann algebras and nuclear $C^{*}$-algebras

A von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ is injective if and only if there exists a (not necessarily normal) conditional expectation (i.e., a projection of norm one [73]) from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$; see, e.g., [70, Corollary XV.1.3]. A fundamental result of Connes [14] (see also [70, Theorem XVI.1.9]) says that a von Neumann algebra $\mathcal{M}$ of separable predual is injective if and only if $\mathcal{M}$ is $A F D$ (approximately finite dimensional), i.e., there exists an increasing sequence $\left\{\mathcal{M}_{j}\right\}_{j=1}^{\infty}$ of finite-dimensional *-subalgebras of $\mathcal{M}$ such that $\mathcal{M}=\left(\bigcup_{j=1}^{\infty} \mathcal{M}_{j}\right)^{\prime \prime}$. In [18] the result was furthermore extended in such a way that a (general) von Neumann algebra $\mathcal{M}$ is injective if and only if there is an increasing net $\left\{\mathcal{M}_{i}\right\}_{i \in \mathcal{I}}$ of finite-dimensional ${ }^{\text {-subalgebras of } \mathcal{M}}$ with $\mathcal{M}=\left(\bigcup_{i \in \mathcal{I}} \mathcal{M}_{i}\right)^{\prime \prime}$. (Here, $\mathcal{A}^{\prime \prime}$ denotes the double commutant, i.e., the commutant of $\mathcal{A}^{\prime}$, for any $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$.)

Next, a $C^{*}$-algebra $\mathcal{A}$ is said to be nuclear if, for every $C^{*}$-algebra $\mathcal{B}$, there is a unique $C^{*}$-cross-norm on $\mathcal{A} \odot \mathcal{B}$, i.e., $\mathcal{A} \otimes_{\min } \mathcal{B}=\mathcal{A} \otimes_{\max } \mathcal{B}$; see, e.g., [70, Chap. XV]. Concerning nuclear $C^{*}$-algebras, among many others, the most fundamental result is that $\mathcal{A}$ is nuclear if and only if $\mathcal{A}^{* *}$ is injective. Here, $\mathcal{A}^{*}$ denotes the Banach space dual of $\mathcal{A}$, and $\mathcal{A}^{* *}$ the second Banach space dual of $\mathcal{A}$. Note that $\mathcal{A}^{* *}$ is isometrically isomorphic to the universally enveloping von Neumann algebra of $\mathcal{A}$, and so it is customary to use $\mathcal{A}^{* *}$ to denote the latter as well; see [69, Sec. III.2]. Therefore, if $\mathcal{A}$ is nuclear, then $\pi(\mathcal{A})^{\prime \prime}$ is injective for every representation $\pi$ of $\mathcal{A}$. Typical examples of nuclear $C^{*}$-algebras are AF $C^{*}$-algebras, in particular, the compact operator ideal $\mathcal{C}(\mathcal{H})$ (or rather $\mathcal{C}(\mathcal{H})+\mathbb{C} 1$ in our present setting; see Example 4.6). Here, recall that a $C^{*}$-algebra $\mathcal{A}$ is AF if there exists an increasing sequence $\left\{\mathcal{A}_{k}\right\}_{k=1}^{\infty}$ of finite-dimensional *-subalgebras of $\mathcal{A}$ such that $\bigcup_{k=1}^{\infty} \mathcal{A}_{k}$ is norm-dense in $\mathcal{A}$. More intricate examples are provided by groups. For a discrete group $G$, the $C^{*}$-algebra generated by the left regular representation on $\ell^{2}(G)$ is the (reduced) group $C^{*}$-algebra $C_{r}^{*}(G)$, while the generated von Neumann algebra is the group von Neumann algebra $W^{*}(G)$. Then $G$ is amenable $\Longleftrightarrow C_{r}^{*}(G)$ is nuclear $\Longleftrightarrow W^{*}(G)$ is injective.

## F Strong converse exponent in the finite-dimensional case

In this appendix we assume that a von Neumann algebra $\mathcal{M}$ is finite-dimensional, so $\mathcal{M} \subseteq$ $\mathcal{B}(\mathcal{H})$ with a finite-dimensional Hilbert space $\mathcal{H}$. Note that $\mathcal{M}$ is isomorphic to $\bigoplus_{i=1}^{m} \mathcal{B}\left(\mathcal{H}_{i}\right)$, a finite direct sum of finite-dimensional $\mathcal{B}\left(\mathcal{H}_{i}\right)$, so it is clear that all the arguments in [48] are valid with $\mathcal{M}$ in place of $\mathcal{B}(\mathcal{H})$. Let $\operatorname{Tr}$ be the usual trace on $\mathcal{M}$ (such that $\operatorname{Tr}(e)=1$ of all minimal projections $e \in \mathcal{M}$ ). Below, to designate states of $\mathcal{M}$, we use density operators $\rho, \sigma$ with respect to Tr rather than positive functionals. Recall that both of the relative entropy $D(\rho \| \sigma)$ and the max-relative entropy $D_{\max }(\rho \| \sigma)$ showing up in (2.4)-(2.6) play an important
role to describe $\psi(s):=\psi^{*}(\rho \| \sigma \mid s+1)$ and $H_{r}^{*}(\rho \| \sigma)$ in [48, Sec. 4].
The aim of this appendix is to give Proposition F.2, which is used in Section 3.2. The main assertion is that for finite-dimensional density operators we have $s c_{r}^{0}(\rho \| \sigma) \leq H_{r}^{*}(\rho \| \sigma)$, $r \geq 0$, which in turn can be obtained easily from the weaker inequalities $s c_{r}(\rho \| \sigma) \leq H_{r}^{*}(\rho \| \sigma)$, $r \geq 0$. The latter was proved in [48]; however, the proof contains a gap, as it is implicitly assumed there that $D(\rho \| \sigma)<D_{\max }(\rho \| \sigma)$. Our main contribution in Proposition F. 2 is filling this gap; for this we give a characterization of the case $D(\rho \| \sigma)=D_{\max }(\rho \| \sigma)$, which may be of independent interest.

Lemma F.1. For density operators $\rho, \sigma$ in $\mathcal{M}$ with $s(\rho) \leq s(\sigma)$, the following conditions are equivalent:
(a) $s \mapsto \psi(s):=\psi^{*}(\rho \| \sigma \mid s+1)$ is affine on $(0,+\infty)$;
(b) $D(\rho \| \sigma)=D_{\max }(\rho \| \sigma)$;
(c) $\rho$ and $\sigma$ commute, and $\rho \sigma^{-1}=\gamma s(\rho)$ for some constant $\gamma>0$;
(d) $s(\rho) \sigma=\sigma s(\rho)$ and $\rho=\gamma \sigma s(\rho)$ for some constant $\gamma>0$.

Moreover, if the above hold, then we have $\gamma \geq 1, D(\rho \| \sigma)=\log \gamma$ and

$$
\begin{equation*}
H_{r}^{*}(\rho \| \sigma)=(r-D(\rho \| \sigma))_{+}, \quad r \geq 0 . \tag{F.1}
\end{equation*}
$$

Proof. (a) $\Longleftrightarrow(\mathrm{b})$. Since $\psi(s)$ is a differentiable convex function on $[0,+\infty)$, this is clear from [48, Lemma 4.2].
(b) $\Longrightarrow$ (c). Consider $D_{2}(\rho \| \sigma):=\log \operatorname{Tr} \rho^{2} \sigma^{-1}$, the standard (or Petz-type) Rényi 2divergence of $\rho, \sigma$. Note that

$$
D(\rho \| \sigma) \leq D_{2}^{*}(\rho \| \sigma) \leq D_{2}(\rho \| \sigma) \leq D_{\max }(\rho \| \sigma)
$$

where the first inequality is seen from the properties noted in Section 2, the second is due to the Araki-Lieb-Thirring inequality, and the last was shown in [11, Lemma 7]. Hence (b) implies that $D_{2}^{*}(\rho \| \sigma)=D_{2}(\rho \| \sigma)$, i.e., $\operatorname{Tr}\left(\sigma^{-1 / 4} \rho \sigma^{-1 / 4}\right)^{2}=\operatorname{Tr} \sigma^{-1 / 2} \rho^{2} \sigma^{-1 / 2}$. Using [26, Theorem 2.1] we find that $\rho, \sigma$ commute. Hence, (a) says that $\psi(s)=\log \operatorname{Tr} \rho^{s+1} \sigma^{-s}$ is affine on $(0,+\infty)$. From [32, Lemma 3.2] for the commutative case, it follows that $\rho \sigma^{-1}=\gamma s(\rho)$, implying (c).
$(\mathrm{c}) \Longrightarrow(\mathrm{d})$ is obvious.
$(\mathrm{d}) \Longrightarrow(\mathrm{b})$. From condition (c) it easily follows that $\gamma \geq 1$ and $D_{\max }(\rho \| \sigma)=\log \gamma$. Moreover, since

$$
\begin{aligned}
D(\rho \| \sigma) & =\operatorname{Tr}(\rho \log \rho-\rho \log (s(\rho) \sigma)) \\
& =\operatorname{Tr}\left(\rho \log \rho-\rho \log \left(\gamma^{-1} \rho\right)\right)=\log \gamma
\end{aligned}
$$

(b) follows.

Finally, if (b) and hence (a) hold, then $\psi(s)=D(\rho \| \sigma) s$ for all $s>0$, from which (F.1) follows immediately.

The next proposition is used in the proof of Theorem 3.7, while the former is a specialized case of the latter.

Proposition F.2. For every density operators $\rho, \sigma$ in $\mathcal{M}$ with $s(\rho) \leq s(\sigma)$ and any $r \geq 0$ we have

$$
\underline{s c}_{r}(\rho \| \sigma)=s c_{r}^{0}(\rho \| \sigma)=H_{r}^{*}(\rho \| \sigma) .
$$

Proof. Since $s c_{r}^{0}(\rho \| \sigma) \geq \underline{s c}_{r}(\rho \| \sigma) \geq H_{r}^{*}(\rho \| \sigma)$, where the second inequality is by [48, Lemma 4.7], it suffices to prove that $s c_{r}^{0}(\rho \| \sigma) \leq H_{r}^{*}(\rho \| \sigma), r \geq 0$. Moreover, the last inequality follows if we can prove that $s c_{r}(\rho \| \sigma) \leq H_{r}^{*}(\rho \| \sigma), r \geq 0$, since then

$$
s c_{r}^{0}(\rho \| \sigma) \leq \inf _{r^{\prime}>r} s c_{r}(\rho \| \sigma) \leq \inf _{r^{\prime}>r} H_{r^{\prime}}^{*}(\rho \| \sigma)=H_{r}^{*}(\rho \| \sigma),
$$

where the first inequality is obvious by definition, the second inequality is to be proved below, and the equality follows from the fact that $r \mapsto H_{r}^{*}(\rho \| \sigma)$ is a monotone increasing finite-valued convex function on $\mathbb{R}$, whence it is also continuous.

Let us therefore prove $s c_{r}(\rho \| \sigma) \leq H_{r}^{*}(\rho \| \sigma), r \geq 0$. The proof of [48, Theorem 4.10] gives this when $D(\rho \| \sigma)<D_{\max }(\rho \| \sigma)$. Assume thus that $D(\rho \| \sigma)=D_{\max }(\rho \| \sigma)=: D$. For any $r \geq 0$, the test sequence $T_{n, r}:=e^{-n(r-D)+} s(\rho)^{\otimes n}, n \in \mathbb{N}$, yields

$$
-\frac{1}{n} \log \operatorname{Tr} \rho_{n} T_{n, r}=(r-D)_{+}=H_{r}^{*}(\rho \| \sigma),
$$

where the last equality is by (F.1), and

$$
-\frac{1}{n} \log \operatorname{Tr} \sigma_{n} T_{n, r}=-\frac{1}{n} \log \left(e^{-n(r-D)_{+}} \operatorname{Tr}(\sigma s(\rho))^{\otimes n}\right)=D+(r-D)_{+} \geq r,
$$

where we have used that $\sigma s(\rho)=e^{-D} \rho$ by Lemma F.1. This proves $s c_{r}(\rho \| \sigma) \leq H_{r}^{*}(\rho \| \sigma)$.

## G Boundary values of convex functions on $(0,1)$

Let $\left\{\phi_{i}\right\}_{i \in \mathcal{I}}$ be a set of convex functions on $(0,1)$ with values in $(-\infty,+\infty]$. Define

$$
\phi(u):=\sup _{i \in \mathcal{I}} \phi_{i}(u), \quad u \in(0,1),
$$

which is obviously convex on $(0,1)$ with values in $(-\infty,+\infty]$. We extend $\phi_{i}$ and $\phi$ to $[0,1]$ by continuity as

$$
\begin{aligned}
\phi_{i}(u) & :=\lim _{u \searrow 0} \phi_{i}(u), & \phi_{i}(1) & :=\lim _{u \nearrow 1} \phi_{i}(u), \\
\phi(0) & :=\lim _{u \searrow 0} \phi(u), & \phi(1) & :=\lim _{u \nearrow 1} \phi(u) .
\end{aligned}
$$

We then give the next lemma to use it in the proof of Theorem 3.7.
Lemma G.1. In the situation stated above, if $\phi(u)<+\infty$ for some $u \in(0,1)$, then

$$
\phi(0)=\sup _{i \in \mathcal{I}} \phi_{i}(0), \quad \phi(1)=\sup _{i \in \mathcal{I}} \phi_{i}(1) .
$$

Proof. By assumption we have a $u_{0} \in(0,1)$ with $\phi\left(u_{0}\right)<+\infty$. Obviously, $\phi_{i}(0) \leq \phi(0)$ and $\phi_{i}(1) \leq \phi(1)$ for all $i \in \mathcal{I}$. Hence it suffices to show that $\phi(0) \leq \sup _{i} \phi_{i}(0)$ and $\phi(1) \leq$ $\sup _{i} \phi_{i}(1)$. Let us prove the first inequality (the proof of the latter is similar). Set $\xi:=$
$\sup _{i} \phi_{i}(0)$. If $\xi=+\infty$, the assertion holds trivially. So assume $\xi<+\infty$. By convexity, for every $i \in \mathcal{I}$ we have

$$
\phi_{i}(u) \leq \frac{u_{0}-u}{u_{0}} \xi+\frac{u}{u_{0}} \phi\left(u_{0}\right), \quad u \in\left(0, u_{0}\right),
$$

so that

$$
\phi(u) \leq \frac{u_{0}-u}{u_{0}} \xi+\frac{u}{u_{0}} \phi\left(u_{0}\right), \quad u \in\left(0, u_{0}\right) .
$$

This implies that $\phi(0) \leq \xi=\sup _{i} \phi_{i}(0)$.

## H Proof of Theorem 4.3

Let $\rho, \sigma \in \mathcal{A}_{+}^{*}$ and $\pi$ be any $(\rho, \sigma)$-normal representation of $\mathcal{A}$ with $\tilde{\rho}=\rho_{\pi}$ and $\tilde{\sigma}=\sigma_{\pi}$, the normal extensions to $\mathcal{M}:=\pi(\mathcal{A})^{\prime \prime}$. Also, let $\bar{\rho}, \bar{\sigma}$ be the normal extensions of $\rho, \sigma$ to the enveloping von Neumann algebra $\mathcal{A}^{* *}$ and $\bar{\pi}: \mathcal{A}^{* *} \rightarrow \mathcal{M}$ be the normal extension of $\pi$ to $\mathcal{A}^{* *}$ (see [69, p. 121]). Let $s(\bar{\pi})$ be the support projection of $\bar{\pi}$. Concerning the support projections $s(\tilde{\rho})$ and $s(\bar{\rho})$ we have $s(\tilde{\rho})=\bar{\pi}(s(\bar{\rho}))$ with $s(\bar{\rho}) \leq s(\bar{\pi})$. Therefore, $s(\tilde{\rho}) \leq s(\tilde{\sigma})$ is equivalent to $s(\bar{\rho}) \leq s(\bar{\sigma})$. This means that the condition $s(\tilde{\rho}) \leq s(\tilde{\sigma})$ is independent of the choice of a representation $\pi$ as above. (The condition is called the absolute continuity of $\rho$ with respect to $\sigma$ [25].)

Now let $\hat{\pi}$ be another $(\rho, \sigma)$-normal representation of $\mathcal{A}$ with $\hat{\rho}:=\rho_{\hat{\pi}}$ and $\hat{\sigma}:=\sigma_{\hat{\pi}}$, the normal extensions to $\hat{\mathcal{M}}:=\hat{\pi}(\mathcal{A})^{\prime \prime}$. The next lemma is a main ingredient of the proof of Theorem 4.3.

Lemma H.1. In the situation stated above, assume that $s(\tilde{\rho}) \leq s(\tilde{\sigma})$ (hence $s(\hat{\rho}) \leq s(\hat{\sigma})$ as well). Let $z_{0}, \hat{z}_{0}$ denote the central supports of $s(\tilde{\sigma}), s(\hat{\sigma})$, respectively. Then there exists an isomorphism $\Lambda: \mathcal{M} z_{0} \rightarrow \hat{\mathcal{M}} \hat{z}_{0}$ for which we have

$$
\begin{equation*}
\tilde{\rho}(x)=\hat{\rho} \circ \Lambda(x), \quad \tilde{\sigma}(x)=\hat{\sigma} \circ \Lambda(x), \quad x \in \mathcal{M} z_{0}, \tag{H.1}
\end{equation*}
$$

and for every $p \in[1,+\infty)$,

$$
\begin{equation*}
\operatorname{tr}\left(h_{\tilde{\sigma}}^{1 / 2 p} x h_{\tilde{\sigma}}^{1 / 2 p}\right)^{p}=\operatorname{tr}\left(h_{\tilde{\sigma}}^{1 / 2 p} \Lambda(x) h_{\hat{\sigma}}^{1 / 2 p}\right)^{p}, \quad x \in \mathcal{M} z_{0} \tag{H.2}
\end{equation*}
$$

where $h_{\tilde{\sigma}} \in L^{1}(\mathcal{M})_{+}$and $h_{\hat{\sigma}} \in L^{1}(\hat{\mathcal{M}})_{+}$are Haagerup's $L^{1}$-elements corresponding to $\tilde{\sigma} \in$ $\mathcal{M}_{*}^{+}$and $\hat{\sigma} \in \hat{\mathcal{M}}_{*}^{+}$, respectively.

Proof. We will work in the standard forms

$$
\left(\mathcal{M}, L^{2}(\mathcal{M}), J={ }^{*}, L^{2}(\mathcal{M})_{+}\right), \quad\left(\hat{\mathcal{M}}, L^{2}(\hat{\mathcal{M}}), \hat{J}={ }^{*}, L^{2}(\hat{\mathcal{M}})_{+}\right) .
$$

For brevity we write

$$
\left\{\begin{array} { l } 
{ h _ { 0 } : = h _ { \tilde { \rho } } \in L ^ { 1 } ( \mathcal { M } ) _ { + } , } \\
{ k _ { 0 } : = h _ { \tilde { \sigma } } \in L ^ { 1 } ( \mathcal { M } ) _ { + } , } \\
{ e _ { 0 } : = s ( \tilde { \sigma } ) = s ( k _ { 0 } ) \in \mathcal { M } , } \\
{ e _ { 0 } ^ { \prime } : = J e _ { 0 } J \in \mathcal { M } ^ { \prime } , }
\end{array} \quad \left\{\begin{array}{l}
\hat{h}_{0}:=h_{\hat{\rho}} \in L^{1}(\hat{\mathcal{M}})_{+}, \\
\hat{k}_{0}:=h_{\hat{\sigma}} \in L^{1}(\hat{\mathcal{M}})_{+}, \\
\hat{e}_{0}:=s(\hat{\sigma})=s\left(\hat{k}_{0}\right) \in \hat{\mathcal{M}}, \\
\hat{e}_{0}^{\prime}:=\hat{J} \hat{e}_{0} \hat{J} \in \hat{\mathcal{M}}^{\prime} .
\end{array}\right.\right.
$$

Below the proof is divided into several steps.

Step 1. Note that

$$
\overline{\pi(\mathcal{A}) k_{0}^{1 / 2}}=\overline{\mathcal{M} k_{0}^{1 / 2}}=L^{2}(\mathcal{M}) e_{0}=e_{0}^{\prime} L^{2}(\mathcal{M})
$$

and for every $a \in \mathcal{A}$,

$$
\left\langle k_{0}^{1 / 2}, \pi(a) e_{0}^{\prime} k_{0}^{1 / 2}\right\rangle=\left\langle k_{0}^{1 / 2}, \pi(a) k_{0}^{1 / 2}\right\rangle=\tilde{\sigma} \circ \pi(a)=\sigma(a)
$$

Hence $\left\{\pi(\cdot) e_{0}^{\prime}, e_{0}^{\prime} L^{2}(\mathcal{M}), k_{0}^{1 / 2}\right\}$ is the cyclic representation of $\mathcal{A}$ with respect to $\sigma$, and similarly $\left\{\hat{\pi}(\cdot) \hat{e}_{0}^{\prime}, \hat{e}_{0}^{\prime} L^{2}(\hat{\mathcal{M}}), \hat{k}_{0}^{1 / 2}\right\}$ is the same. By the uniqueness (up to unitary conjugation) of the cyclic representation, there exists a unitary $V: L^{2}(\mathcal{M}) e_{0} \rightarrow L^{2}(\hat{\mathcal{M}}) \hat{e}_{0}$ such that

$$
\begin{equation*}
V k_{0}^{1 / 2}=\hat{k}_{0}^{1 / 2}, \quad V\left(\pi(a) e_{0}^{\prime}\right) V^{*}=\hat{\pi}(a) \hat{e}_{0}^{\prime}, \quad a \in \mathcal{A} . \tag{H.3}
\end{equation*}
$$

We hence have an isomorphism $V \cdot V^{*}: \mathcal{M} e_{0}^{\prime} \rightarrow \hat{\mathcal{M}} \hat{e}_{0}^{\prime}$.
Step 2. Since $z_{0}$ is the central support of $e_{0}^{\prime}$, note that $x z_{0} \in \mathcal{M} z_{0} \mapsto x e_{0}^{\prime} \in \mathcal{M} e_{0}^{\prime}(x \in \mathcal{M})$ is an isomorphism, and similarly so is $\hat{x} \hat{z}_{0} \in \hat{\mathcal{M}} \hat{z}_{0} \mapsto \hat{x} \hat{e}_{0}^{\prime} \in \hat{\mathcal{M}} \hat{e}_{0}^{\prime}(\hat{x} \in \hat{\mathcal{M}})$. Hence one can define an isomorphism $\Lambda: \mathcal{M} z_{0} \rightarrow \hat{\mathcal{M}} \hat{z}_{0}$ as follows:

$$
\begin{equation*}
\Lambda: \mathcal{M} z_{0} \cong \mathcal{M} e_{0}^{\prime} \cong \hat{\mathcal{M}} \hat{e}_{0}^{\prime} \cong \hat{\mathcal{M}} \hat{z}_{0}, \quad x z_{0} \mapsto x e_{0}^{\prime} \mapsto V\left(x e_{0}^{\prime}\right) V^{*}=\hat{x} \hat{e}_{0}^{\prime} \mapsto \hat{x} \hat{z}_{0} \tag{H.4}
\end{equation*}
$$

Note [21, Lemma 2.6] that the standard forms of $\mathcal{M} z_{0}$ and $\hat{\mathcal{M}} \hat{z}_{0}$ are respectively given by

$$
\begin{aligned}
& \left(\mathcal{M} z_{0}, z_{0} L^{2}(\mathcal{M}) z_{0}=L^{2}(\mathcal{M}) z_{0}, J={ }^{*}, z_{0} L^{2}(\mathcal{M})_{+} z_{0}=L^{2}(\mathcal{M})_{+} z_{0}\right) \\
& \left(\hat{\mathcal{M}} \hat{z}_{0}, \hat{z}_{0} L^{2}(\hat{\mathcal{M}}) \hat{z}_{0}=L^{2}(\hat{\mathcal{M}}) \hat{z}_{0}, \hat{J}={ }^{*}, \hat{z}_{0} L^{2}(\hat{\mathcal{M}})_{+} \hat{z}_{0}=L^{2}(\hat{\mathcal{M}})_{+} \hat{z}_{0}\right) .
\end{aligned}
$$

By the uniqueness (up to unitary conjugation) of the standard form (under isomorphism) [21, Theorem 2.3], there exists a unitary $U: L^{2}(\mathcal{M}) z_{0} \rightarrow L^{2}(\hat{\mathcal{M}}) \hat{z}_{0}$ such that

$$
\begin{align*}
& \Lambda(x)=U x U^{*}, \quad x \in \mathcal{M} z_{0}  \tag{H.5}\\
& (U \xi)^{*}=U\left(\xi^{*}\right), \quad \xi \in z_{0} L^{2}(\mathcal{M}) z_{0}  \tag{H.6}\\
& U\left(L^{2}(\mathcal{M})_{+} z_{0}\right)=L^{2}(\hat{\mathcal{M}})_{+} \hat{z}_{0} \tag{H.7}
\end{align*}
$$

Step 3. Since $s\left(h_{0}\right) \leq e_{0} \leq z_{0}$ by assumption, one has $h_{0}^{1 / 2}, k_{0}^{1 / 2} \in L^{2}(\mathcal{M}) z_{0}$, and similarly $\hat{h}_{0}^{1 / 2}, \hat{k}_{0}^{1 / 2} \in L^{2}(\hat{\mathcal{M}}) \hat{z}_{0}$. By (H.7) one has $U h_{0}^{1 / 2}, U k_{0}^{1 / 2} \in L^{2}(\hat{\mathcal{M}})_{+} \hat{z}_{0}$. Here we confirm that

$$
\begin{equation*}
U h_{0}^{1 / 2}=\hat{h}_{0}^{1 / 2}, \quad U k_{0}^{1 / 2}=\hat{k}_{0}^{1 / 2} \tag{H.8}
\end{equation*}
$$

To show this, for every $a \in \mathcal{A}$ we find that

$$
\begin{aligned}
\left\langle U h_{0}^{1 / 2},\left(\hat{\pi}(a) \hat{z}_{0}\right) U h_{0}^{1 / 2}\right\rangle & =\left\langle h_{0}^{1 / 2}, \Lambda^{-1}\left(\hat{\pi}(a) \hat{z}_{0}\right) h_{0}^{1 / 2}\right\rangle \quad(\text { by }(\text { H.5) }) \\
& =\left\langle h_{0}^{1 / 2},\left(\pi(a) z_{0}\right) h_{0}^{1 / 2}\right\rangle \quad(\text { by }(\text { H.3 }) \text { and (H.4)) } \\
& =\tilde{\rho} \circ \pi(a)=\rho(a)=\hat{\rho} \circ \hat{\pi}(a) \\
& =\left\langle\hat{h}_{0}^{1 / 2},\left(\hat{\pi}(a) \hat{z}_{0}\right) \hat{h}_{0}^{1 / 2}\right\rangle
\end{aligned}
$$

which implies that $U h_{0}^{1 / 2}=\hat{h}_{0}^{1 / 2}$. The proof of $U k_{0}^{1 / 2}=\hat{k}_{0}^{1 / 2}$ is similar. By (H.5) and (H.8) we have also

$$
\begin{equation*}
\Lambda(x) \hat{h}_{0}^{1 / 2}=U\left(x h_{0}^{1 / 2}\right), \quad \Lambda(x) \hat{k}_{0}^{1 / 2}=U\left(x k_{0}^{1 / 2}\right), \quad x \in \mathcal{M} z_{0} \tag{H.9}
\end{equation*}
$$

These imply (H.1). Furthermore, by (H.8) and (H.9) we have $\Lambda\left(e_{0}\right) \hat{k}_{0}^{1 / 2}=U k_{0}^{1 / 2}=\hat{k}_{0}^{1 / 2}$, from which $\Lambda\left(e_{0}\right) \geq \hat{e}_{0}$ follows. Applying the same argument to $\Lambda^{-1}(\hat{x})=U^{*} \hat{x} U\left(\hat{x} \in \hat{\mathcal{M}} \hat{z}_{0}\right)$ with $k_{0}^{1 / 2}, \hat{k}_{0}^{1 / 2}$ exchanged gives $\Lambda^{-1}\left(\hat{e}_{0}\right) \geq e_{0}$ as well. Therefore,

$$
\begin{equation*}
\Lambda\left(e_{0}\right)=\hat{e}_{0} . \tag{H.10}
\end{equation*}
$$

Step 4. We define

$$
\left(\Lambda^{-1}\right)_{*}: L^{1}\left(\mathcal{M} z_{0}\right)=L^{1}(\mathcal{M}) z_{0} \rightarrow L^{1}\left(\hat{\mathcal{M}} \hat{z}_{0}\right)=L^{1}(\hat{\mathcal{M}}) \hat{z}_{0}
$$

by transforming $\psi \in\left(\mathcal{M} z_{0}\right)_{*} \mapsto \psi \circ \Lambda^{-1} \in\left(\hat{\mathcal{M}} \hat{z}_{0}\right)_{*}$ via $L^{1}\left(\mathcal{M} z_{0}\right) \cong\left(\mathcal{M} z_{0}\right)_{*}$ and $L^{1}\left(\hat{\mathcal{M}} \hat{z}_{0}\right) \cong$ $\left(\hat{\mathcal{M}} \hat{z}_{0}\right)_{*}$, that is, $\left(\Lambda^{-1}\right)_{*}: h_{\psi} \in L^{1}\left(\mathcal{M} z_{0}\right) \mapsto h_{\psi \circ \Lambda^{-1}} \in L^{1}\left(\hat{\mathcal{M}} \hat{z}_{0}\right)$ for $\psi \in\left(\mathcal{M} z_{0}\right)_{*}$. Of course, $\left(\Lambda^{-1}\right)_{*}$ is an isometry with respect to $\|\cdot\|_{1}$. Now, Kosaki's (symmetric) interpolation $L^{p}$-spaces enter into our discussions. Here we confirm that

$$
\begin{equation*}
\left(\Lambda^{-1}\right)_{*}\left(k_{0}^{1 / 2} x k_{0}^{1 / 2}\right)=\hat{k}_{0}^{1 / 2} \Lambda(x) \hat{k}_{0}^{1 / 2}, \quad x \in \mathcal{M} z_{0} \tag{H.11}
\end{equation*}
$$

Indeed, for every $x, y \in \mathcal{M} z_{0}$ we find that

$$
\begin{aligned}
\operatorname{tr}\left(\Lambda(y)\left(\Lambda^{-1}\right)_{*}\left(k_{0}^{1 / 2} x k_{0}^{1 / 2}\right)\right) & =\operatorname{tr}\left(y k_{0}^{1 / 2} x k_{0}^{1 / 2}\right)=\left\langle\left(y k_{0}^{1 / 2}\right)^{*}, x k_{0}^{1 / 2}\right\rangle \\
& =\left\langle U\left(\left(y k_{0}^{1 / 2}\right)^{*}\right), U\left(x k_{0}^{1 / 2}\right)\right\rangle \\
& =\left\langle\left(\Lambda(y) \hat{k}_{0}^{1 / 2}\right)^{*}, \Lambda(x) \hat{k}_{0}^{1 / 2}\right\rangle \quad(\text { by }(\text { H.6) and (H.9) }) \\
& =\operatorname{tr}\left(\Lambda(y) \hat{k}_{0}^{1 / 2} \Lambda(x) \hat{k}_{0}^{1 / 2}\right),
\end{aligned}
$$

which yields (H.11).
Step 5. Thanks to (H.11) we see that the isometry $\left(\Lambda^{-1}\right)_{*}: L^{1}\left(\mathcal{M} z_{0}\right) \rightarrow L^{1}\left(\hat{\mathcal{M}} \hat{z}_{0}\right)$ with respect to $\|\cdot\|_{1}$ is restricted to an isometry from $k_{0}^{1 / 2}\left(\mathcal{M} z_{0}\right) k_{0}^{1 / 2}$ (embedded into $\left.L^{1}\left(\mathcal{M} z_{0}\right)\right)$ onto $\hat{k}_{0}^{1 / 2}\left(\hat{\mathcal{M}} \hat{z}_{0}\right) \hat{k}_{0}^{1 / 2}\left(\right.$ embedded into $\left.L^{1}\left(\hat{\mathcal{M}} \hat{z}_{0}\right)\right)$ with respect to $\|\cdot\|_{\infty}$, i.e.,

$$
\left\|k_{0}^{1 / 2} x k_{0}^{1 / 2}\right\|_{\infty}=\|x\|=\|\Lambda(x)\|=\left\|\hat{k}_{0}^{1 / 2} \Lambda(x) \hat{k}_{0}^{1 / 2}\right\|_{\infty}, \quad x \in \mathcal{M} z_{0}
$$

By Kosaki's construction in [42] (or the Riesz-Thorin theorem) it follows that $\left(\Lambda^{-1}\right)_{*}$ gives rise to an isometry

$$
\begin{aligned}
& \left(\Lambda^{-1}\right)_{*}: L^{p}\left(\mathcal{M} z_{0}, \tilde{\sigma}\right)=C_{1 / p}\left(\left(k_{0}^{1 / 2}\left(\mathcal{M} z_{0}\right) k_{0}^{1 / 2}, L^{1}\left(\mathcal{M} z_{0}\right)\right)\right. \\
& \quad \rightarrow L^{p}\left(\hat{\mathcal{M}} \hat{z}_{0}, \hat{\sigma}\right)=C_{1 / p}\left(\hat{k}_{0}^{1 / 2}\left(\hat{\mathcal{M}} \hat{z}_{0}\right) \hat{k}_{0}^{1 / 2}, L^{1}\left(\hat{\mathcal{M}} \hat{z}_{0}\right)\right)
\end{aligned}
$$

with respect to the interpolation norms $\|\cdot\|_{p, \tilde{\sigma}}$ and $\|\cdot\|_{p, \hat{\sigma}}$ for any $p \in[1,+\infty)$. For every $x \in \mathcal{M} z_{0}$, applying this to $k_{0}^{1 / 2} x k_{0}^{1 / 2}$ with (H.11) gives

$$
\left\|k_{0}^{1 / 2} x k_{0}^{1 / 2}\right\|_{p, \tilde{\sigma}}=\left\|\hat{k}_{0}^{1 / 2} \Lambda(x) \hat{k}_{0}^{1 / 2}\right\|_{p, \hat{\sigma}}
$$

By [42, Theorem 9.1], for every $p \in[1,+\infty)$ the above equality is rephrased as Haagerup's $L^{p}$-norm equality

$$
\left\|k_{0}^{1 / 2 p} x k_{0}^{1 / 2 p}\right\|_{p}=\left\|\hat{k}_{0}^{1 / 2 p} \Lambda(x) \hat{k}_{0}^{1 / 2 p}\right\|_{p},
$$

which is (H.2), as asserted.
We are now in a position to prove Theorem 4.3.

Proof of (i). We use the variational expressions in Proposition 2.3 based on Lemma H.1. Assume first that $\alpha>1$. If $s(\tilde{\rho}) \not \leq s(\tilde{\sigma})$, then $s(\hat{\rho}) \not \leq s(\hat{\sigma})$ (as mentioned at the beginning of this appendix) so that both of $D_{\alpha}^{*}(\tilde{\rho} \| \tilde{\sigma})$ and $D_{\alpha}^{*}(\hat{\rho} \| \hat{\sigma})$ are $+\infty$. Hence we assume that $s(\tilde{\rho}) \leq s(\tilde{\sigma})$ (and $s(\hat{\rho}) \leq s(\hat{\sigma})$ ). Using (2.7) we have

$$
\begin{aligned}
Q_{\alpha}^{*}(\tilde{\rho} \| \tilde{\sigma}) & =\sup _{x \in \mathcal{M}}\left[\alpha \tilde{\rho}(x)-(\alpha-1) \operatorname{tr}\left(h_{\tilde{\sigma}}^{\frac{\alpha-1}{2 \alpha}} x h_{\tilde{\sigma}}^{\frac{\alpha-1}{2 \alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right] \\
& =\sup _{x \in\left(\mathcal{M} z_{0}\right)_{+}}\left[\alpha \tilde{\rho}(x)-(\alpha-1) \operatorname{tr}\left(h_{\tilde{\sigma}}^{\frac{\alpha-1}{2 \alpha}} x h_{\tilde{\sigma}}^{\frac{\alpha-1}{2 \alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right] \quad\left(\text { since } s(\tilde{\rho}) \leq s(\tilde{\sigma}) \leq z_{0}\right) \\
& =\sup _{x \in\left(\mathcal{M} z_{0}\right)_{+}}\left[\alpha \hat{\rho}(\Lambda(x))-(\alpha-1) \operatorname{tr}\left(h_{\tilde{\sigma}}^{\frac{\alpha-1}{2 \alpha}} \Lambda(x) h_{\tilde{\sigma}}^{\frac{\alpha-1}{2 \alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right] \quad(\text { by Lemma H.1) } \\
& =\sup _{\hat{x} \in\left(\hat{\mathcal{M}} \hat{z}_{0}\right)_{+}}\left[\alpha \hat{\rho}(\hat{x})-(\alpha-1) \operatorname{tr}\left(h_{\tilde{\sigma}}^{\frac{\alpha-1}{2 \alpha}} \hat{x} h_{\tilde{\sigma}}^{\frac{\alpha-1}{2 \alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right] \\
& =Q_{\alpha}^{*}(\hat{\rho} \| \hat{\sigma}) .
\end{aligned}
$$

Next, assume that $1 / 2 \leq \alpha<1$. When $s(\tilde{\rho}) \leq s(\tilde{\sigma})$, we use (2.8) as above to have

$$
\begin{aligned}
Q_{\alpha}^{*}(\tilde{\rho} \| \tilde{\sigma}) & =\inf _{x \in\left(\mathcal{M} z_{0}\right)_{++}}\left[\alpha \tilde{\rho}(x)+(1-\alpha) \operatorname{tr}\left(h_{\tilde{\sigma}}^{\frac{1-\alpha}{2 \alpha}} x^{-1} h_{\tilde{\sigma}}^{\frac{1-\alpha}{2 \alpha}}\right)^{\frac{\alpha}{1-\alpha}}\right] \\
& =\inf _{x \in\left(\mathcal{M} z_{0}\right)_{++}}\left[\alpha \hat{\rho}(\Lambda(x))+(1-\alpha) \operatorname{tr}\left(h_{\hat{\sigma}^{\frac{1-\alpha}{2 \alpha}}}^{\text {2 }} \Lambda\left(x^{-1}\right) h^{\frac{1-\alpha}{2 \alpha}}\right)^{\frac{\alpha}{1-\alpha}}\right] \quad \text { (by Lemma H.1) } \\
& =\inf _{\hat{x} \in\left(\hat{\mathcal{M}} \hat{z}_{0}\right)_{++}}\left[\alpha \hat{\rho}(\hat{x})+(1-\alpha) \operatorname{tr}\left(h_{\tilde{\sigma}}^{\frac{1-\alpha}{2 \alpha}} \hat{x}^{-1} h_{\hat{\sigma}}^{\frac{1-\alpha}{2 \alpha}}\right)^{\frac{\alpha}{1-\alpha}}\right] \quad\left(\text { since } \Lambda\left(x^{-1}\right)=\Lambda(x)^{-1}\right) \\
& =Q_{\alpha}^{*}(\hat{\rho} \| \hat{\sigma}) .
\end{aligned}
$$

For general $\rho, \sigma \in \mathcal{A}_{+}^{*}$, let $\sigma_{\varepsilon}:=\sigma+\varepsilon \rho$ for any $\varepsilon>0$. Then $\sigma_{\varepsilon}$ has the normal extensions $\tilde{\sigma}_{\varepsilon}=\tilde{\sigma}+\varepsilon \tilde{\rho}$ to $\mathcal{M}$ and $\hat{\sigma}_{\varepsilon}=\hat{\sigma}+\varepsilon \hat{\rho}$ to $\hat{\mathcal{M}}$. The above case yields $Q_{\alpha}^{*}\left(\tilde{\rho} \| \tilde{\sigma}_{\varepsilon}\right)=Q_{\alpha}^{*}\left(\hat{\rho} \| \hat{\sigma}_{\varepsilon}\right)$ for all $\varepsilon>0$. From the continuity of $Q_{\alpha}^{*}$ on $\mathcal{M}_{*}^{+} \times \mathcal{M}_{*}^{+}$in the norm topology when $1 / 2 \leq \alpha<1$ (see [29, Theorem $3.16(3)])$, letting $\varepsilon \searrow 0$ gives $Q_{\alpha}^{*}(\tilde{\rho} \| \tilde{\sigma})=Q_{\alpha}^{*}(\hat{\rho} \| \hat{\sigma})$, implying (4.4).

Proof of (ii). Assume first that $s(\tilde{\rho}) \leq s(\tilde{\sigma})$ (hence $s(\hat{\rho}) \leq s(\hat{\sigma})$ ). Below let us use the same symbols as in the proof of Lemma H.1. Recall [3] that the relative modular operator $\Delta_{\tilde{\rho}, \tilde{\sigma}}$ is defined as $\Delta_{\tilde{\rho}, \tilde{\sigma}}:=S_{\tilde{\rho}, \tilde{\tilde{\sigma}}}^{*} \overline{S_{\tilde{\rho}, \tilde{\sigma}}}$, where $S_{\tilde{\rho}, \tilde{\sigma}}$ is a closable conjugate linear operator defined by

$$
S_{\tilde{\rho}, \tilde{\sigma}}\left(x k_{0}^{1 / 2}+\zeta\right):=e_{0} x^{*} h_{0}^{1 / 2}, \quad x \in \mathcal{M}, \zeta \in\left(L^{2}(\mathcal{M}) e_{0}\right)^{\perp}
$$

Similarly, $\Delta_{\hat{\rho}, \hat{\sigma}}:=S_{\hat{\rho}, \hat{\sigma}}^{*} \overline{S_{\hat{\rho}, \hat{\sigma}}}$ is given, where

$$
S_{\hat{\rho}, \hat{\sigma}}\left(\hat{x} \hat{k}_{0}^{1 / 2}+\hat{\zeta}\right):=\hat{e}_{0} \hat{x}^{*} \hat{h}_{0}^{1 / 2}, \quad \hat{x} \in \hat{\mathcal{M}}, \hat{\zeta} \in\left(L^{2}(\hat{\mathcal{M}}) \hat{e}_{0}\right)^{\perp} .
$$

Since $s\left(h_{0}\right) \leq e_{0} \leq z_{0}$, we can consider $S_{\tilde{\rho}, \tilde{\sigma}}$ and $\Delta_{\tilde{\rho}, \tilde{\sigma}}$ as operators on $L^{2}(\mathcal{M}) z_{0}$ (they are zero operators on $\left.\left(L^{2}(\mathcal{M}) z_{0}\right)^{\perp}\right)$. Similarly, $S_{\hat{\rho}, \hat{\sigma}}$ and $\Delta_{\hat{\rho}, \hat{\sigma}}$ are considered on $L^{2}(\hat{\mathcal{M}}) \hat{z}_{0}$. Let us use an isomorphism $\Lambda: \mathcal{M} z_{0} \rightarrow \hat{\mathcal{M}} \hat{z}_{0}$ and a unitary $U: L^{2}(\mathcal{M}) z_{0} \rightarrow L^{2}(\hat{\mathcal{M}}) \hat{z}_{0}$. Since $\overline{\mathcal{M} k_{0}^{1 / 2}}=$ $L^{2}(\mathcal{M}) e_{0}$ and $\overline{\hat{\mathcal{M}} \hat{k}_{0}^{1 / 2}}=L^{2}(\hat{\mathcal{M}}) \hat{e}_{0}$, it follows from (H.9) that $U\left(L^{2}(\mathcal{M}) e_{0}\right)=L^{2}(\hat{\mathcal{M}}) \hat{e}_{0}$ and hence $U\left(\left(L^{2}(\mathcal{M}) e_{0}\right)^{\perp}\right)=\left(L^{2}(\hat{\mathcal{M}}) \hat{e}_{0}\right)^{\perp}$. For every $\hat{x}=\Lambda(x) \in \hat{\mathcal{M}} \hat{z}_{0}$ (with $x \in \mathcal{M} z_{0}$ ) and
$\hat{\zeta} \in\left(L^{2}(\hat{\mathcal{M}}) \hat{e}_{0}\right)^{\perp}$, we find that

$$
\begin{aligned}
S_{\hat{\rho}, \hat{\sigma}}\left(\hat{x} \hat{k}_{0}^{1 / 2}+\hat{\zeta}\right) & =U e_{0} U^{*} U x^{*} U^{*} U h_{0}^{1 / 2} \quad(\text { by }(\text { H.10 }),(\text { H.5) and (H.8)) } \\
& =U e_{0} x^{*} h_{0}^{1 / 2}=U S_{\tilde{\rho}, \tilde{\sigma}}\left(x k_{0}^{1 / 2}+U^{*} \hat{\zeta}\right) \\
& =U S_{\tilde{\rho}, \tilde{\sigma}}\left(U^{*} \hat{x} \hat{k}_{0}^{1 / 2}+U^{*} \hat{\zeta}\right) \quad(\text { by }(\mathrm{H} .9)) \\
& =U S_{\tilde{\rho}, \tilde{\sigma}} U^{*}\left(\hat{x} \hat{k}_{0}^{1 / 2}+\hat{\zeta}\right)
\end{aligned}
$$

This implies that $S_{\hat{\rho}, \hat{\sigma}}=U S_{\tilde{\rho}, \tilde{\sigma}} U^{*}$ and hence $\Delta_{\hat{\rho}, \hat{\sigma}}=U \Delta_{\tilde{\rho}, \tilde{\sigma}} U^{*}$. Therefore, for every $\alpha \in$ $[0,+\infty), \hat{k}_{0}^{1 / 2}$ is in $\mathcal{D}\left(\Delta_{\hat{\rho}, \hat{\sigma}}^{\alpha / 2}\right)$ if and only if $k_{0}^{1 / 2}=U^{*} \hat{k}_{0}^{1 / 2}$ is in $\mathcal{D}\left(\Delta_{\tilde{\rho}, \tilde{\sigma}}^{\alpha / 2}\right)$, and in this case,

$$
Q_{\alpha}(\tilde{\rho} \| \tilde{\sigma})=\left\|\Delta_{\tilde{\rho}, \tilde{\sigma}}^{\alpha / 2} k_{0}^{1 / 2}\right\|^{2}=\left\|\Delta_{\hat{\rho}, \tilde{\sigma}}^{\alpha / 2} \hat{k}_{0}^{1 / 2}\right\|^{2}=Q_{\alpha}(\hat{\rho} \| \hat{\sigma}) .
$$

Otherwise, $Q_{\alpha}(\tilde{\rho} \| \tilde{\sigma})=Q_{\alpha}(\hat{\rho} \| \hat{\sigma})=+\infty$.
Next assume that $s(\tilde{\rho}) \not \leq s(\tilde{\sigma})$, equivalently $s(\hat{\rho}) \not \leq s(\hat{\sigma})$. Then $Q_{\alpha}(\tilde{\rho} \| \tilde{\sigma})=Q_{\alpha}(\hat{\rho} \| \hat{\sigma})=+\infty$ for $\alpha>1$. When $0 \leq \alpha<1$, let $\sigma_{\varepsilon}:=\sigma+\varepsilon \rho$ for every $\varepsilon>0$. From the continuity of $Q_{\alpha}$ on $\mathcal{M}_{*}^{+} \times \mathcal{M}_{*}^{+}$[29, Corollary 3.8], the above case yields

$$
Q_{\alpha}(\tilde{\rho} \| \tilde{\sigma})=\lim _{\varepsilon \searrow 0} Q_{\alpha}\left(\tilde{\rho} \| \tilde{\sigma}_{\varepsilon}\right)=\lim _{\varepsilon \searrow 0} Q_{\alpha}\left(\hat{\rho} \| \hat{\sigma}_{\varepsilon}\right)=Q_{\alpha}(\hat{\rho} \| \hat{\sigma}),
$$

implying (4.5).
Remark H.2. The notion of standard $f$-divergences $S_{f}(\rho \| \sigma)$ with a parametrization of operator convex functions $f$ on $(0,+\infty)$ has been studied in [27] in the von Neumann algebra setting. From the above proof of (ii) we observe that $S_{f}(\rho \| \sigma)$ can be extended to $\rho, \sigma \in \mathcal{A}_{+}^{*}$ as

$$
S_{f}(\rho \| \sigma):=S_{f}\left(\rho_{\pi} \| \sigma_{\pi}\right)
$$

independently of the choice of a $(\rho, \sigma)$-normal representation $\pi$ of $\mathcal{A}$. Then we can easily extend properties of $S_{f}(\rho \| \sigma)$ given in [27] to the $C^{*}$-algebra setting (like Proposition 4.5 for the sandwiched and the standard Rényi divergences).

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[^1]:    ${ }^{3}$ In [38, 39] (also [29]) it was implicitly assumed that $\mathcal{M}$ is $\sigma$-finite. But this assumption can be removed. Indeed, let $\Phi: \mathcal{N} \rightarrow \mathcal{M}$ be a unital positive normal map between von Neumann algebras and $\rho, \sigma \in \mathcal{M}_{*}^{+}$. Let $e:=s(\rho) \vee s(\sigma)$ and $e_{0}:=s(\rho \circ \Phi) \vee s(\sigma \circ \Phi)$. Since $s(\rho)\left(1-\Phi\left(e_{0}\right)\right) s(\rho)=0, s(\rho)=s(\rho) \Phi\left(e_{0}\right)=\Phi\left(e_{0}\right) s(\rho)$ and $s(\rho) \leq \Phi\left(e_{0}\right)$. Similarly, $s(\sigma)=s(\sigma) \Phi\left(e_{0}\right)=\Phi\left(e_{0}\right) s(\sigma)$ and $s(\sigma) \leq \Phi\left(e_{0}\right)$. Let $P:=1-s(\rho), Q:=1-s(\sigma)$ and $A:=1-\Phi\left(e_{0}\right) \geq 0$; hence $P \geq A$ and $Q \geq A$. Note that $P Q P \geq P A P=A, Q P Q P Q \geq Q A Q=A$, and so on. Hence $(P Q)^{n} P \geq A$ for all $n \geq 1$. Since $(P Q)^{n} P \rightarrow P \wedge Q$ strongly, we have $P \wedge Q \geq A$, which means that $e \leq \Phi\left(e_{0}\right)$. We thus find that $\tilde{\Phi}:=\left.e \Phi(\cdot) e\right|_{e_{0} \mathcal{N} e_{0}}$ is a unital positive normal map from $e_{0} \mathcal{N} e_{0}$ to $e \mathcal{M} e$. Note that $e_{0} \mathcal{N} e_{0}$ and $e \mathcal{M} e$ are $\sigma$-finite. Moreover, $D_{\alpha}^{*}(\rho \| \sigma)=D_{\alpha}^{*}\left(\left.\rho\right|_{e \mathcal{M} e} \|\left.\sigma\right|_{e \mathcal{M} e}\right)$ and

    $$
    D_{\alpha}^{*}(\rho \circ \Phi \| \sigma \circ \Phi)=D_{\alpha}^{*}\left(\left.\rho \circ \Phi\right|_{e_{0} \mathcal{N} e_{0}} \|\left.\sigma \circ \Phi\right|_{e_{0} \mathcal{N} e_{0}}\right)=D_{\alpha}^{*}\left(\left(\left.\rho\right|_{e \mathcal{M} e}\right) \circ \tilde{\Phi} \|\left(\left.\sigma\right|_{e \mathcal{M} e}\right) \circ \tilde{\Phi}\right) .
    $$

