On intertwining of maxima of sum of translates functions with nonsingular kernels

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Dedicated to the memory of
Yu. N. Subbotin and S. A. Telyakovskii,
excellent mathematicians and fine people

Abstract

In previous papers we investigated so-called sum of translates functions $F(\mathbf{x}, t) := J(t) + \sum_{j=1}^{n} \nu_j K(t - x_j)$, where $J : [0, 1] \to \underline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$ is a "sufficiently nondegenerate" and upper-bounded "field function", and $K : [-1, 1] \to \underline{\mathbb{R}}$ is a fixed "kernel function", concave both on $(-1, 0)$ and $(0, 1)$, and also satisfying the singularity condition $K(0) = \lim_{t \to 0} K(t) = -\infty$. For node systems $\mathbf{x} := (x_1, \ldots, x_n)$ with $x_0 := 0 \leq x_1 \leq \cdots \leq x_n \leq 1 =: x_{n+1}$, we analyzed the behavior of the local maxima vector $\mathbf{m} := (m_0, m_1, \ldots, m_n)$, where $m_j := m_j(\mathbf{x}) := \sup_{x_j \leq t \leq x_{j+1}} F(\mathbf{x}, t)$.

Among other results we proved a strong intertwining property: if the kernels are also decreasing on $(-1, 0)$ and increasing on $(0, 1)$, and the field function is upper semicontinuous, then for any two different node systems there are i, j such that $m_i(\mathbf{x}) < m_i(\mathbf{y})$ and $m_j(\mathbf{x}) > m_j(\mathbf{y})$.

Here we partially succeed to extend this even to nonsingular kernels.

Keywords: minimax problems, kernel function, sum of translates function, vector of local maxima, equioscillation, intertwining of interval maxima

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1 Introduction

In our papers [3, 4, 5] we analyzed interval maxima vectors of sum of translates functions. The very notion of the sum of translates functions originates from an ingenious paper of Fenton [7], who himself worked out results on them for use in his work proving a conjecture of Barry. About the origins and wide range of applications, of the approach, ranging from the strong polarization problem to Chebyshev constants and Bojanov theorems we refer the reader to the papers [4, 5] as well as to [1, 2].

A function $K : (-1, 0) \cup (0, 1) \to \mathbb{R}$ will be called a *kernel function* if it is concave on $(-1, 0)$ and on $(0, 1)$, and if it satisfies

$$\lim_{t \downarrow 0} K(t) = \lim_{t \uparrow 0} K(t).$$

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By the concavity assumption these limits exist, and a kernel function has one-sided limits also at -1 and 1. We set

$$K(0) := \lim_{t \to 0} K(t), \ K(-1) := \lim_{t \downarrow -1} K(t) \text{ and } K(1) := \lim_{t \uparrow 1} K(t).$$

Note explicitly that we thus obtain an extended continuous function $K : [-1, 1] \to \mathbb{R} \cup \{-\infty\} =: \overline{\mathbb{R}}$, and that we have $\sup K < \infty$. Also note that a kernel function is almost everywhere differentiable.

We say that the kernel function K is *strictly concave* if it is strictly concave on both of the intervals $(-1, 0)$ and $(0, 1)$.

Further, we call it *monotone*[1] if

$$K \text{ is nonincreasing on } (-1, 0) \text{ and nondecreasing on } (0, 1). \tag{M}$$

By concavity, under the monotonicity condition (M) the endpoint values $K(-1)$, $K(1)$ are also finite. If K is strictly concave, then (M) implies *strict monotonicity*:

$$K \text{ is strictly decreasing on } [-1, 0) \text{ and strictly increasing on } (0, 1], \tag{SM}$$

where we have extended the assertion to the finite endpoint values, too.

A kernel function K is called *singular* if

$$K(0) = -\infty. \tag{∞}$$

Let $n \in \mathbb{N} = \{1, 2, \dots\}$ be fixed. We will call a function $J : [0, 1] \to \overline{\mathbb{R}}$ an *external n-field function*[2], or—if the value of n is unambiguous from the context—simply a *field function*, if it is bounded above on $[0, 1]$, and it assumes finite values at more than n different points, where we count the points 0 and 1 with weight[3] $1/2$ only, while the points in $(0, 1)$ are accounted for with weight 1. Therefore, for a field function J the set $(0, 1) \setminus J^{-1}(\{-\infty\})$ has at least n elements, and if it has precisely n elements, then either $J(0)$ or $J(1)$ is finite, too.

Further, we consider the *open simplex*

$$S := S_n := \{\mathbf{y} = (y_1, \dots, y_n) \in (0, 1)^n, \ 0 < y_1 < \dots < y_n < 1\},$$

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[1] These conditions—and more, like C^2 smoothness and strictly negative second derivatives—were assumed on the kernel functions in the paper of Fenton [7].

[2] Again, the terminology of kernels and fields came to our mind by analogy, which in case of the logarithmic kernel $K(t) := \log|t|$ and an external field $J(t)$ arising from a weight $w(t) := \exp(J(t))$ are indeed discussed in logarithmic potential theory. However, in our analysis no further potential theoretic notions and tools will be applied. This is so in particular because our analysis is far more general, allowing different and almost arbitrary kernels and fields; yet the resemblance to the classical settings of logarithmic potential theory should not be denied.

[3] The weighted counting makes a difference only for the case when $J^{-1}(\{-\infty\})$ contains the two endpoints; with only $n-1$ further interior points in $(0, 1)$ the weights in this configuration add up to n only, hence the node system is considered inadmissible.

and its closure the *closed simplex*

$$\overline{S} := \{\mathbf{y} = (y_1, y_2, \ldots, y_n) \in [0,1]^n : 0 \leq y_1 \leq \cdots \leq y_n \leq 1\}.$$

For any given $n \in \mathbb{N}$, kernel function K, constants $\nu_j > 0$, $(j = 1, \ldots, n)$, and a given field function J we will consider the *pure sum of translates function*

$$f(\mathbf{y}, t) := \sum_{j=1}^{n} \nu_j K(t - y_j) \quad (\mathbf{y} \in \overline{S}, \ t \in [0,1]),$$

and also the *(weighted) sum of translates function*

$$F(\mathbf{y}, t) := J(t) + \sum_{j=1}^{n} \nu_j K(t - y_j) \quad (\mathbf{y} \in \overline{S}, \ t \in [0,1]).$$

Note that the functions J, K can take the value $-\infty$, but not $+\infty$, therefore both sum of translates functions can be defined meaningfully. Furthermore, $f : \overline{S} \times [0,1] \to \mathbb{R}$ is extended continuous, and $F(\mathbf{y}, \cdot)$ is not constant $-\infty$, hence $\sup_{t \in [0,1]} F(\mathbf{y}, t) > -\infty$ holds.[4]

We introduce the *singularity set* of the field function J as

$$X := X_J := \{t \in [0,1] : J(t) = -\infty\},$$

and note that the so-called *finiteness domain* of J, $X^c := [0,1] \backslash X$ has cardinality exceeding n (in the above described, weighted sense), in particular $X \neq [0,1]$. Similarly, the singularity set of $F(\mathbf{y}, \cdot)$ is

$$\widehat{X} := \widehat{X}(\mathbf{y}) := \{t \in [0,1] : F(\mathbf{y}, t) = -\infty\} \subsetneq [0,1].$$

Accordingly, an interval $I \subseteq [0,1]$ with $I \subseteq \widehat{X}(\mathbf{y})$ will be called *singular*.

Writing $y_0 := 0$ and $y_{n+1} := 1$ we also set for each $\mathbf{y} \in \overline{S}$ and $j \in \{0, 1, \ldots, n\}$

$$I_j(\mathbf{y}) := [y_j, y_{j+1}], \qquad m_j(\mathbf{y}) := \sup_{t \in I_j(\mathbf{y})} F(\mathbf{y}, t),$$

and

$$\overline{m}(\mathbf{y}) := \max_{j=0,\ldots,n} m_j(\mathbf{y}) = \sup_{t \in [0,1]} F(\mathbf{y}, t), \qquad \underline{m}(\mathbf{y}) := \min_{j=0,\ldots,n} m_j(\mathbf{y}).$$

Of interest are the simplex minimax and simplex maximin values which are defined as follows

$$M(S) := \inf_{\mathbf{y} \in S} \overline{m}(\mathbf{y}), \qquad m(S) := \sup_{\mathbf{y} \in S} \underline{m}(\mathbf{y}).$$

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[4] These require some careful considerations and the assumed degree of nonsingularity of J is in fact *the exact condition* to ensure $F \not\equiv -\infty$. For details see [4, 5].

As has been said above, for each $\mathbf{y} \in \overline{S}$ we have that $\overline{m}(\mathbf{y}) = \sup_{t \in [0,1]} F(\mathbf{y}, t) \in \mathbb{R}$ is finite. Observe that an interval $I \subseteq [0, 1]$ is contained in $\widehat{X}(\mathbf{y})$, i.e., I is singular, if and only if $F(\mathbf{y}, \cdot)|_I \equiv -\infty$. In particular $m_j(\mathbf{y}) = -\infty$ exactly when $I_j(\mathbf{y}) \subseteq \widehat{X}(\mathbf{y})$. A node system $\mathbf{y}$ is called *singular* if there is $j \in \{0, 1, \ldots, n\}$ with $I_j(\mathbf{y})$ singular, i.e., $m_j(\mathbf{y}) = -\infty$; and a node system $\mathbf{y} \in \partial S = \overline{S} \setminus S$ is called *degenerate*.

An essential role is played by the *regularity set* (set of regular node systems)

$$Y := Y_n := Y_n(X) := \{\mathbf{y} \in \overline{S} : \mathbf{y} \text{ is nonsingular}\}$$
$$= \{\mathbf{y} \in \overline{S} : I_j(\mathbf{y}) \nsubseteq \widehat{X}(\mathbf{y}) \text{ for } j = 0, 1, \ldots, n\}$$
$$= \{\mathbf{y} \in \overline{S} : m_j(\mathbf{y}) \neq -\infty \text{ for } j = 0, 1, \ldots, n\}.$$

An important observation is that the regularity set does not depend on the kernel function K, but on the set where K is $-\infty$ (which is subset of $\{-1, 0, 1\}$). Similarly, it only depends on the singularity set X_J of J, but not on the actual function J itself. If K is nonsingular and is finite valued at ± 1, too, then we necessarily have $\widehat{X} = X_J$, and the notion of singularity of intervals, hence of node systems, becomes totally independent of the kernel K itself. On the other hand if the kernel K is singular, then all degenerate node systems are outright singular, thus $Y \subset S$. Note also that we have $S \subset Y$ if and only if X (or equivalently $\widehat{X}$, which differs from it only by a finite number of points, if at all) has empty interior. In particular, if X has empty interior and K is singular, then $Y = S$.

We also introduce the *interval maxima vector function*

$$\mathbf{m}(\mathbf{w}) := (m_0(\mathbf{w}), m_1(\mathbf{w}), \ldots, m_n(\mathbf{w})) \in \mathbb{R}^{n+1} \quad (\mathbf{w} \in \overline{S}).$$

From the above it follows that for $\mathbf{w} \in \overline{S}$ we have $\mathbf{m}(\mathbf{w}) \neq (-\infty, \ldots, -\infty)$.

Definition 1.1 (Intertwining of maxima). Let $J : [0, 1] \to \mathbb{R}$ be an n-field function, K be a kernel function and $\nu_i > 0$, $i = 1, \ldots, n$ be given constants. We say that for this system *intertwining of maxima* holds, if for any two different regular node systems $\mathbf{x}, \mathbf{y} \in Y$ both $m_i(\mathbf{x}) > m_i(\mathbf{y})$ and $m_j(\mathbf{x}) < m_j(\mathbf{y})$ occur for some indices $i, j \in \{0, 1, \ldots, n\}$.

In our earlier papers we mentioned this same property under the terminology that "majorization does not occur", or simply "nonmajorization property". Majorization means for two node systems $\mathbf{x}, \mathbf{y}$ that $\mathbf{m}(\mathbf{x}) \geq \mathbf{m}(\mathbf{y})$, i.e. $m_i(\mathbf{x}) \geq m_i(\mathbf{y})$ for all $i = 0, 1, \ldots, n$.

For singular kernels, like $\log |t|$, the intertwining property was established under suitable assumptions—see the discussion in Section 2 below. In particular, exponentiating one of our results, we obtained the following—to the best of our knowledge, new—observation. The sequence of (local) absolute value maxima on $[0, 1]$ of the monic polynomials $P(\mathbf{x}, t) := \prod_{j=1}^{n} (t - x_j)$ for different node systems can never majorize each other.

In our earlier results on intertwining an essential role was played by a so-called "Homeomorphism Theorem", proved for singular kernels in [4]. If the

considered kernels are nonsingular, then this fundamental tool is no longer available. Also, other properties, like continuity of the $m_i(\mathbf{x})$, were obtained by heavy use of the singularity assumption (∞). Therefore, it was not clear what happens for general, not necessarily singular kernels. On the other hand Fenton formulated his results in [7] for possibly nonsingular kernels (even if under several other restrictive assumptions), so investigating the nonsingular case came as natural. Here is what we could prove.

Theorem 1.2 (Nonsingular intertwining). *Let $n \in \{1, 2, 3\}$, $\nu_1, \ldots, \nu_n > 0$, let K be a strictly concave and (strictly) monotone (SM) kernel function and let J be an upper semicontinuous n-field function.*

Then intertwining of maxima holds, and there is a unique equioscillation point i.e., a node system $\mathbf{w}$ for which $m_0(\mathbf{w}) = m_1(\mathbf{w}) = \cdots = m_n(\mathbf{w})$.

In other words, the first assertion of this theorem states that majorization— $m_i(\mathbf{x}) \geq m_i(\mathbf{y})$ for all $i = 0, 1, \ldots, n$—cannot hold, unless $\mathbf{x} = \mathbf{y}$. The existence of an equioscillation point follows from our earlier results, recalled as Theorem 2.3 below.

2 Some earlier results and a conjecture

Let us recall two of our earlier results on the behavior of $\mathbf{m}$ in case of singular kernels which put into context Theorem 1.2 and Conjecture 2.4 below. The first one is a combination of Corollary 3.2 and Corollary 4.2 of [5].

Theorem 2.1. *Let $n \in \mathbb{N}$, $\nu_1, \ldots, \nu_n > 0$, let K be a singular (∞) and monotone (M) kernel function, and let J be an upper semicontinuous n-field function.*

Then $M(S) = m(S)$ and there exists some node system $\mathbf{w} \in Y$ at which the simplex maximin and the simplex minimax are attained:

$$\underline{m}(\mathbf{w}) = m(S) = M(S) = \overline{m}(\mathbf{w}).$$

The node system $\mathbf{w}$ is an equioscillation point.

Moreover, there are no $\mathbf{x}, \mathbf{y} \in Y$ with $m_j(\mathbf{x}) > m_j(\mathbf{y})$ for every $j \in \{0, 1, \ldots, n\}$. For any equioscillation point $\mathbf{e}$ we have $\overline{m}(\mathbf{e}) = M(S)$.

Remark 2.2. (a) If there is an equioscillation point, then $M(S) \leq m(S)$. Indeed, if $\mathbf{e}$ is an equioscillation point, then $\mathbf{e} \in Y$ and $M(S) \leq \overline{m}(\mathbf{e})$, $\overline{m}(\mathbf{e}) = \underline{m}(\mathbf{e})$, $\underline{m}(\mathbf{e}) \leq m(S)$. Note that, for this we do not use any of the special properties of the kernel function or the field function.

(b) If there is intertwining on $\overline{S}$, then $M(S) \geq m(S)$. Indeed, if $M(S) < m(S)$, that is $\inf_{\mathbf{x}} \overline{m}(\mathbf{x}) < \sup_{\mathbf{y}} \underline{m}(\mathbf{y})$, then there are $\mathbf{x}, \mathbf{y} \in \overline{S}$ such that $\overline{m}(\mathbf{x}) < \underline{m}(\mathbf{y})$ which entails that $\mathbf{m}(\mathbf{y})$ strictly majorizes $\mathbf{m}(\mathbf{x})$. This observation again uses no properties of K and J. Let us mention here Example 5.4 from [5] where the kernel function K is strictly concave and singular but it is not monotone, J is constant and we have strict majorization there. Going beyond monotonicity gives rise to completely different phenomena.

The following, second theorem (Theorem 2.4 from [3]) can be viewed as the sharpest result in this direction.

Theorem 2.3. *Let $n \in \mathbb{N}$, $\nu_1, \ldots, \nu_n > 0$, let K be a* monotone (M) *kernel function and J be an arbitrary n-field function. Then $M(S) = m(S)$ and there exists some node system $\mathbf{w} \in \overline{S}$ at which the simplex minimax is attained:*

$$\overline{m}(\mathbf{w}) = M(S).$$

Moreover, there are no $\mathbf{x}, \mathbf{y} \in Y$ with $m_j(\mathbf{x}) > m_j(\mathbf{y})$ for every $j \in \{0, 1, \ldots, n\}$. For any equioscillation point $\mathbf{e}$ we have $\overline{m}(\mathbf{e}) = M(S)$, and if J is upper semicontinuous or K is singular, then, in fact, there exists an equioscillation point.

Based on these latter theorems and Theorem 1.2, we put forward the general case as a conjecture:

Conjecture 2.4 (Nonsingular Intertwining)**.** *Let $n \in \mathbb{N}$, $\nu_1, \ldots, \nu_n > 0$, let K be a strictly concave and (strictly) monotone (SM) kernel function and let J be an upper semicontinuous n-field function. The conclusion of Theorem 1.2 remains true even if $n > 3$, i.e., for $\mathbf{x}, \mathbf{y} \in Y$ the coordinatewise inequality $\mathbf{m}(\mathbf{x}) \leq \mathbf{m}(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$.*

In [8] Shi studied such intertwining type properties and the relation to minimax problems, via a "homeomorphism property", that is established under strong differentiability conditions. Such techniques are not applicable here (we lack good differentiability properties of the functions $m_0, \ldots, m_n$). We refer to [6] for a comparison with Shi's result (in the periodic case).

Remark 2.5. (a) In Conjecture 2.4 we need to restrict to $\mathbf{x}, \mathbf{y} \in Y$ as the following example shows. Let $K(t) := \log|t|$, $J(t) := -\infty$, if $t < 2/3$ and $J(t) := 0$ if $t \geq 2/3$, $n := 1$, $\mathbf{x} = (1/3)$, $\mathbf{y} = (2/3)$. Note that J is upper semicontinuous. Then $f(\mathbf{x}, t) > f(\mathbf{y}, t)$ for $t \in (2/3, 1]$. We have $m_0(\mathbf{x}) = m_0(\mathbf{y}) = -\infty$ and $m_1(\mathbf{y}) = \log|1/3| < m_1(\mathbf{x}) = \log|2/3|$, so $\mathbf{m}(\mathbf{x}) \geq \mathbf{m}(\mathbf{y})$ and $\mathbf{m}(\mathbf{x}) \neq \mathbf{m}(\mathbf{y})$, i.e., majorization occurs.

(b) In general, monotonicity of K is necessary to exclude majorization. Indeed, in Example 5.4 of [5] a non-monotone kernel K is given such that with $J \equiv 0$ strict majorization occurs.

It should be also clarified whether in Theorem 1.2 the upper semicontinuity of J is needed; our proof below uses this property. So we also ask the validity of the previous conjecture for not upper semicontinuous J.

3   Some technical lemmas

First, we recall a lemma from [3] (see Lemma 3.3 therein, but also Lemma 3.2 in [6]).

Lemma 3.1. *Let K be any kernel function. Let $0 < \alpha < a < b < \beta < 1$ and $p, q > 0$. Set*

$$\kappa := \frac{p(a - \alpha)}{q(\beta - b)}. \tag{1}$$

(a) *If K satisfies (M) and $\kappa \geq 1$, then for every $t \in [0, \alpha]$ we have*

$$pK(t - \alpha) + qK(t - \beta) \leq pK(t - a) + qK(t - b). \tag{2}$$

(b) *If K satisfies (M) and $\kappa \leq 1$, then (2) holds for every $t \in [\beta, 1]$.*

(c) *If $\kappa = 1$ (but K does not necessarily satisfy (M)), then (2) holds for every $t \in [0, \alpha] \cup [\beta, 1]$.*

(d) *In case of a strictly concave kernel function (a), (b) and (c) hold with strict inequality in (2).*

(e) *If K is a monotone kernel function (M), then for every $t \in [a, b]$*

$$pK(t - \alpha) + qK(t - \beta) \geq pK(t - a) + qK(t - b),$$

with strict inequality if K is strictly monotone (SM).

The following two lemmas settle particular cases of Theorem 1.2, but for general n.

Lemma 3.2. *Let $n \in \mathbb{N}$ be arbitrary, and assume that K is a strictly concave and monotone kernel function. Let J be an upper semicontinuous n-field function.*

If $\mathbf{x}, \mathbf{y} \in Y$ and $\mathbf{x} \leq \mathbf{y}$ in the sense that $x_i \leq y_i$, $i = 1, \ldots, n$, and $\mathbf{x} \neq \mathbf{y}$, then intertwining of maxima holds.

Proof. The monotonicity assumption (M) provides for all $i = 1, \ldots, n$ the inequalities $K(t - x_i) \leq K(t - y_i)$ for $0 \leq t \leq x_1$ and $K(t - x_i) \geq K(t - y_i)$ for $y_n \leq t \leq 1$; moreover, the inequalities are strict whenever $x_i < y_i$ is strict, for monotonicity has to be strict monotonicity in view of strict concavity.

Taking the positive linear combination of these inequalities and adding $J(t)$ to both sides, we get that $F(\mathbf{x}, t) \leq F(\mathbf{y}, t)$ for $0 \leq t \leq x_1$ and $F(\mathbf{x}, t) \geq F(\mathbf{y}, t)$ for $y_n \leq t \leq 1$. Moreover, as it is excluded that $x_i = y_i$ for all $i = 1, \ldots, n$, these inequalities have to be strict unless $J(t) = -\infty$. Picking points[5] $z \in [0, x_1]$ with $m_0(\mathbf{x}) = F(\mathbf{x}, z)$ and $w \in [y_n, 1]$ with $m_n(\mathbf{y}) = F(\mathbf{y}, w)$, the finiteness of $m_0(\mathbf{x}), m_n(\mathbf{y})$ (following from the assumption $\mathbf{x}, \mathbf{y} \in Y$) entails that $J(z), J(w) > -\infty$, hence we find

$$m_0(\mathbf{x}) = F(\mathbf{x}, z) < F(\mathbf{y}, z) \leq \max_{I_0(\mathbf{x})} F(\mathbf{y}, \cdot) \leq \max_{I_0(\mathbf{y})} F(\mathbf{y}, \cdot) = m_0(\mathbf{y})$$

and similarly $m_n(\mathbf{y}) = F(\mathbf{y}, w) < F(\mathbf{x}, w) \leq m_n(\mathbf{x})$. These altogether show intertwining of maxima for $\mathbf{x}$ and $\mathbf{y}$. □

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[5] Here and throughout we use that J, hence F is upper semicontinuous and thus on compact sets they have maximum points. Upper semicontinuity is thus an indispensable assumption for our argument.

Lemma 3.3. *Let $n \in \mathbb{N}$ be arbitrary, and assume that K is a strictly concave and monotone kernel function.*

Suppose that for any upper semicontinuous $n - 1$-field function J^* and for $n - 1$ nodes we know that intertwining holds.

Let J be an upper semicontinuous n-field function.

If $\mathbf{x} \neq \mathbf{y} \in Y$ consist of n nodes each, and there is i such that $x_i = y_i$, then intertwining holds for $\mathbf{x}$ and $\mathbf{y}$.

Proof. We apply the assertion to the new data $n^* := n - 1$, $J^*(x) := J(x) + \nu_i K(x - x_i)$—which is an upper semicontinuous n^*-field function—and $\nu_j^* := \nu_j$ for $j < i$ and $\nu_j^* := \nu_{j+1}$ for $i \leq j < n$. Put now $\mathbf{x}^* := (x_1^*, \ldots, x_{n-1}^*)$, where $x_j^* := x_j$ for $j < i$ and $x_j^* := x_{j+1}$ for $i \leq j < n$; and construct $\mathbf{y}^*$ similarly.

For these systems it is easy to see that

$$
m_j^*(\mathbf{x}^*) = \begin{cases} m_j(\mathbf{x}), & \text{if } j < i, \\ \max(m_i(\mathbf{x}), m_{i+1}(\mathbf{x})), & \text{if } j = i, \\ m_{j+1}(\mathbf{x}), & \text{if } i < j < n, \end{cases}
$$

and similarly for $\mathbf{y}^*$. So $\mathbf{x}^* \neq \mathbf{y}^*$ are two different node systems, and both are nonsingular. Then the assumption provides that we have both $m_j^*(\mathbf{x}^*) > m_j^*(\mathbf{y}^*)$ and also $m_k^*(\mathbf{x}^*) < m_k^*(\mathbf{y}^*)$ for some j, k. The same inequality immediately follows between the respective maxima for $\mathbf{x}$ and $\mathbf{y}$ unless $j = i$ or $k = i$. (Note that both cannot happen.) If e.g. $j = i$, then we only see $\max(m_i(\mathbf{x}), m_{i+1}(\mathbf{x})) > \max(m_i(\mathbf{y}), m_{i+1}(\mathbf{y}))$. But then if the maximum $m_i^*(\mathbf{x}^*) = m_i(\mathbf{x})$ then obviously we also have $m_i(\mathbf{x}) > m_i(\mathbf{y})$; and the same way if the maximum is $m_i^*(\mathbf{x}^*) = m_{i+1}(\mathbf{x})$ then $m_{i+1}(\mathbf{x}) > m_{i+1}(\mathbf{y})$. Similar argument works for the case $k = i$.

Therefore it follows that strong intertwining of maxima holds for $\mathbf{x}, \mathbf{y}$, too. $\square$

By the same argument, we can also obtain the following which will not be used, however.

Remark 3.4. *Let $n \in \mathbb{N}$ be arbitrary, and assume that K is a strictly concave and monotone kernel function.*

Suppose that for any $n - 1$-field function J^* and for $n - 1$ nodes we know that intertwining holds.

Let J be an n-field function.

If $\mathbf{x} \neq \mathbf{y} \in Y$ consist of n nodes each, and there is i such that $x_i = y_i$, then intertwining holds for $\mathbf{x}$ and $\mathbf{y}$.

4 Proof of Theorem 1.2

By Theorem 2.3, indeed, there exists an equioscillation point, since J was assumed to be upper semicontinuous.

Observe that the second assertion of the theorem is entailed by the first one: if $\mathbf{x} \neq \mathbf{y}$ are equioscillating, then either $m_i(\mathbf{x}) \leq m_i(\mathbf{y})$ for all $i = 0, 1, \ldots, n$ or $m_i(\mathbf{x}) > m_i(\mathbf{y})$ for all $i = 0, 1, \ldots, n$, whence majorization occurs, providing a contradiction.

Therefore the proof hinges upon excluding majorization i.e. proving intertwining of maxima. In what follows, we will use without further reference that an upper semicontinuous function attains its supremum on compact sets.

4.1 The case of only one node

If $n = 1$ and $\mathbf{x} := (x), \mathbf{y} := (y)$ are two nodes ("node systems"), then $\mathbf{x} \neq \mathbf{y}$ entails $x \neq y$, say $x < y$. This case is immediately solved by Lemma 3.2, $n = 1$.

4.2 The $n = 2$ case

If $n = 2$, and $\mathbf{x}, \mathbf{y} \in Y$ are two node systems, with say $x_1 \leq y_1$, then either also $x_2 \leq y_2$, and Lemma 3.2 applies, or we must have $x_2 > y_2$. Moreover, if $x_1 = y_1$, then we can refer back to the above Lemma 3.3 and the already settled case $n = 1$ to obtain intertwining of maxima for $\mathbf{x}, \mathbf{y}$. So, without loss of generality, we may assume that

$$0 < x_1 < y_1 < y_2 < x_2 < 1.$$

Taking $\alpha := x_1, a := y_1, b := y_2, \beta := x_2$ and $p := \nu_1, q := \nu_2$ in the above Lemma 3.1, according to (a), (b) and (d) we are led to

$$\nu_1 K(t - x_1) + \nu_2 K(t - x_2) < \nu_1 K(t - y_1) + \nu_2 K(t - y_2),$$

either on $I_0(\mathbf{x}) = [0, x_1]$ or on $I_2(\mathbf{x}) = [x_2, 1]$.

Consider the first case and pick some $z \in I_0(\mathbf{x})$ with $m_0(\mathbf{x}) = F(\mathbf{x}, z)$. In view of $\mathbf{x} \in Y$, $m_0(\mathbf{x}) > -\infty$, so also the value of J at z satisfies $J(z) > -\infty$. Therefore we have

$$\begin{aligned} m_0(\mathbf{x}) = F(\mathbf{x}, z) &= J(z) + \nu_1 K(z - x_1) + \nu_2 K(z - x_2) \\ &< J(z) + \nu_1 K(z - y_1) + \nu_2 K(z - y_2) = F(\mathbf{y}, z) \\ &\leq \max_{I_0(\mathbf{y})} F(\mathbf{y}, \cdot) = m_0(\mathbf{y}), \end{aligned}$$

using that $z \in I_0(\mathbf{x}) = [0, x_1] \subset [0, y_1] = I_0(\mathbf{y})$.

Similarly, in the other case we find $m_2(\mathbf{x}) < m_2(\mathbf{y})$.

On the other hand by strict monotonicity and part (e) of Lemma 3.1, we also have

$$\nu_1 K(t - x_1) + \nu_2 K(t - x_2) > \nu_1 K(t - y_1) + \nu_2 K(t - y_2) \text{ for } t \in [y_1, y_2]. \quad (3)$$

Let us take here a point $u \in I_1(\mathbf{y})$ with $m_1(\mathbf{y}) = F(\mathbf{y}, u)$; then by $\mathbf{y} \in Y$ we also have that $F(\mathbf{y}, u)$, hence also $J(u)$, are both finite. So adding $J(u)$ to (3) with $t = u$, we get $m_1(\mathbf{y}) = F(\mathbf{y}, u) < F(\mathbf{x}, u) \leq \max_{I_1(\mathbf{x})} F(\mathbf{x}, \cdot) = m_1(\mathbf{x})$.

Therefore, we have completed proving intertwining of maxima for $\mathbf{x}, \mathbf{y}$.

4.3 The $n = 3$ case

Assume now $n = 3$ and take two node systems $\mathbf{x}, \mathbf{y} \in Y$. Again, if there are equal ith coordinates, then induction settles the assertion according to Lemma 3.3 and Subsection 4.2. Further, if there is a coordinatewise ordering between the node systems, then Lemma 3.2 finishes the argument.

So we must have $x_i < y_i$ for some coordinates and also $x_i > y_i$ for all the other ones, each case occurring. By interchanging the role of the two nodes, we may settle with two $<$ and one $>$ inequality signs among the respective coordinates. Correspondingly, we will consider three cases according to occurrence of the inequality $x_i > y_i$ at $i = 1, 2$ or 3.

Case 1 $(i = 2)$. $x_1 < y_1 < y_2 < x_2 < x_3 < y_3$.

We will compare the values of the various $m_j(\mathbf{x}), m_j(\mathbf{y})$ through use of the intermediate node systems $\mathbf{z} := (y_1, x_2, x_3)$ and $\mathbf{w} := (y_1, y_2, x_3)$.

Observe that taking into account the condition (M) and $x_1 < y_1$ we get $m_0(\mathbf{z}) \geq m_0(\mathbf{x})$ and $m_j(\mathbf{z}) \leq m_j(\mathbf{x})$, $j = 1, 2, 3$.

Further, in view of Lemma 3.1 (a), (b) and (d) (with $[\alpha, \beta] = [y_2, y_3] \supset [a, b] = [x_2, x_3]$) we have either on $[0, y_2]$ or on $[y_3, 1]$ the inequality

$$\nu_2 K(t - y_2) + \nu_3 K(t - y_3) < \nu_2 K(t - x_2) + \nu_3 K(t - x_3).$$

Adding $J(t) + \nu_1 K(t - y_1)$ we find either on $[0, y_2]$ or on $[y_3, 1]$ the inequality

$$F(\mathbf{y}, t) \leq F(\mathbf{z}, t) \text{ with strict inequality if } J(t) > -\infty.$$

Now we consider the maxima of the sum of translate function $F(\mathbf{y}, \cdot)$ on various intervals: these maxima (and then also the respective values of J) have to be finite (for $\mathbf{y} \in Y$ with finite m_j values), whence at the points, where maxima are attained, one has even the strict inequality. It follows that either $m_0(\mathbf{y}) < m_0(\mathbf{z})$ and $m_1(\mathbf{y}) < \max_{[y_1, y_2]} F(\mathbf{z}, \cdot) \leq m_1(\mathbf{z})$ or $m_3(\mathbf{y}) < m_3(\mathbf{z})$.

In view of (M) and $x_1 < y_1$, we have $m_1(\mathbf{y}) < m_1(\mathbf{z}) \leq m_1(\mathbf{x})$ in the first case and $m_3(\mathbf{y}) < m_3(\mathbf{z}) \leq m_3(\mathbf{x})$ in the second case, so altogether $m_j(\mathbf{y}) < m_j(\mathbf{x})$ holds either for $j = 1$ or for $j = 3$.

We show similar comparison for $\mathbf{w}$ next. By the monotonicity assumption (M) and $x_3 < y_3$ we have $m_j(\mathbf{w}) \leq m_j(\mathbf{y})$, $j = 0, 1, 2$ (and $m_3(\mathbf{w}) \geq m_3(\mathbf{y})$).

According to Lemma 3.1 (a), (b) and (d) applied for $[\alpha, \beta] = [x_1, x_2] \supset [a, b] = [y_1, y_2]$, we have either on $[0, x_1]$ or on $[x_2, 1]$ the strict inequality

$$\nu_1 K(t - x_1) + \nu_2 K(t - x_2) < \nu_1 K(t - y_1) + \nu_2 K(t - y_2).$$

Adding $J(t) + \nu_3 K(t - x_3)$ we find either on $[0, x_1]$ or on $[x_2, 1]$ the inequality

$$F(\mathbf{x}, t) \leq F(\mathbf{w}, t) \text{ with strict inequality if } J(t) > -\infty.$$

Considering the maxima of the sum of translates function $F(\mathbf{x}, \cdot)$ on various intervals, and that the maxima $m_j(\mathbf{x})$ are finite (for $\mathbf{x} \in Y$), we conclude that at the points, where these maxima are attained, we have strict inequality. It

follows that either $m_0(\mathbf{x}) < m_0(\mathbf{w})$, or $m_2(\mathbf{x}) < m_2(\mathbf{w})$ and $m_3(\mathbf{x}) < m_3(\mathbf{w})$ simultaneously.

On combining the above we find that either $m_0(\mathbf{x}) < m_0(\mathbf{w}) \leq m_0(\mathbf{y})$ or $m_2(\mathbf{x}) < m_2(\mathbf{w}) \leq m_2(\mathbf{y})$. That is, either $m_0(\mathbf{x}) < m_0(\mathbf{y})$ or $m_2(\mathbf{x}) < m_2(\mathbf{y})$ must hold.

Therefore, intertwining of maxima holds for $\mathbf{x}, \mathbf{y} \in Y$.

Case 2 ($i = 3$). $x_1 < y_1, x_2 < y_2 < y_3 < x_3$.

One thing is immediate from (strict) monotonicity: we have $m_2(\mathbf{x}) > m_2(\mathbf{y})$.

Therefore, the proof hinges upon showing a reverse inequality for some j. We will distinguish two subcases in the proof of $m_j(\mathbf{x}) < m_j(\mathbf{y})$ for some j.

Case 2.1. If $\nu_1(y_1 - x_1) \geq \nu_3(x_3 - y_3)$.

We apply Lemma 3.1 (a) and (d) with $p := \nu_1, q := \nu_3, \alpha := x_1, a := y_1, b := y_3, \beta := x_3$. In this case κ, as defined in (1), will satisfy $\kappa \geq 1$, so we obtain

$$\nu_1 K(t - x_1) + \nu_3 K(t - x_3) < \nu_1 K(t - y_1) + \nu_3 K(t - y_3) \qquad (t \in [0, x_1]).$$

In view of monotonicity and $x_2 < y_2$, we also have here

$$\nu_2 K(t - x_2) < \nu_2 K(t - y_2) \qquad (t \in [0, x_1]).$$

Adding these inequalities and then $J(t)$ to both sides leads to

$$F(\mathbf{x}, t) \leq F(\mathbf{y}, t) \text{ for } t \in [0, x_1] \text{ with strict inequality if } J(t) > -\infty.$$

So pick now a point $v \in [0, x_1]$ with $m_0(\mathbf{x}) = F(\mathbf{x}, v)$: then with this v $F(\mathbf{x}, v)$, hence also $J(v)$ is finite, and the above entails

$$m_0(\mathbf{x}) = F(\mathbf{x}, v) < F(\mathbf{y}, v) \leq \max_{[0, x_1]} F(\mathbf{y}, \cdot) \leq m_0(\mathbf{y}),$$

providing us the required inequality with $j = 0$ in this case.

Case 2.2. If $\nu_1(y_1 - x_1) < \nu_3(x_3 - y_3)$.

In this subcase let us define $\mathbf{z} := (x_1, y_2, z_3)$ with $z_3 := \frac{\nu_1}{\nu_3}(y_1 - x_1) + y_3$. Note that then $x_1 < y_2 < y_3 < z_3 < x_3$ holds, so the nodes of $\mathbf{z}$ are listed in their natural order.

Applying Lemma 3.1 (c) and (d) with $p := \nu_1, q := \nu_3, \alpha := x_1, a := y_1, b := y_3, \beta := z_3$, we see $\kappa = 1$ and so the lemma provides inequalities *on both sides*:

$$\nu_1 K(t - x_1) + \nu_3 K(t - z_3) < \nu_1 K(t - y_1) + \nu_3 K(t - y_3) \qquad (t \in [0, x_1] \cup [z_3, 1]).$$

Let us add $\nu_2 K(t - y_2) + J(t)$ to both sides. Since y_2 is strictly between x_1 and z_3, we have $K(t - y_2) > -\infty$ everywhere in $[0, x_1] \cup [z_3, 1]$. So, we add $\nu_2 K(t - y_2) + J(t)$, a finite amount to the above inequality whenever $J(t) > -\infty$. This furnishes for all $t \in [0, x_1] \cup [z_3, 1]$

$$F(\mathbf{z}, t) \leq F(\mathbf{y}, t) \text{ with strict inequality whenever } J(t) > -\infty.$$

Take now two points $u \in [0, x_1]$ and $v \in [z_3, 1]$ where $m_0(\mathbf{z}) = F(\mathbf{z}, u)$ and $m_3(\mathbf{z}) = F(\mathbf{z}, v)$. Note that the interval $I_0(\mathbf{z}) = [0, x_1] = I_0(\mathbf{x})$ is nonsingular, because $\mathbf{x} \in Y$. So is the interval $I_3(\mathbf{z}) = [z_3, 1] \supset [x_3, 1] = I_3(\mathbf{x})$, because $I_3(\mathbf{x}) \not\subset \widehat{X}$ as $\mathbf{x}$ is nonsingular. It follows that the maxima $m_0(\mathbf{z}), m_3(\mathbf{z})$ are finite, thus so are $J(u), J(v)$, too. From these and $[0, x_1] \subset [0, y_1]$, $[z_3, 1] \subset [y_3, 1]$ we infer

$$m_0(\mathbf{z}) = F(\mathbf{z}, u) < F(\mathbf{y}, u) \le \max_{[0, y_1]} F(\mathbf{y}, \cdot) = m_0(\mathbf{y}),$$

$$m_3(\mathbf{z}) = F(\mathbf{z}, v) < F(\mathbf{y}, v) \le \max_{[y_3, 1]} F(\mathbf{y}, \cdot) = m_3(\mathbf{y}).$$

Therefore, we have the strict inequalities $m_i(\mathbf{z}) < m_i(\mathbf{y})$ for $i = 0, 3$.

Next, we apply Lemma 3.1 (a) and (b) with $p := \nu_2, q := \nu_3, \alpha := x_2, a := y_2, b := z_3, \beta := x_3$. We find that either on $[0, x_2]$ or on $[x_3, 1]$ the inequality

$$\nu_2 K(t - x_2) + \nu_3 K(t - x_3) \le \nu_2 K(t - y_2) + \nu_3 K(t - y_3)$$

must hold. Adding $J(t) + \nu_1 K(t - x_1)$ thus leads to

$$F(\mathbf{x}, t) \le F(\mathbf{z}, t)$$

either for all $t \in [0, x_2]$ or for all $t \in [x_3, 1]$. Taking maxima on $[0, x_1]$ or on $[x_3, 1]$ therefore furnishes either $m_0(\mathbf{x}) \le m_0(\mathbf{z})$ or $m_3(\mathbf{x}) \le \max_{[x_3, 1]} F(\mathbf{z}, \cdot) \le m_3(\mathbf{z})$.

Combining this with the above inequalities $m_i(\mathbf{z}) < m_i(\mathbf{y})$ $(i = 0, 3)$, we also obtain either $m_0(\mathbf{x}) < m_0(\mathbf{y})$ or $m_3(\mathbf{x}) < m_3(\mathbf{y})$. That is, we arrive at the desired inequality $m_j(\mathbf{x}) < m_j(\mathbf{y})$ either for $j = 0$ or for $j = 3$.

Case 3 ($i = 1$). $y_1 < x_1 < x_2 < y_2, x_3 < y_3$.

Consider the modified system $J^*(t) := J(1 - t)$, $\nu_j^* := \nu_{4-j}$ and $K^*(t) := K(-t)$ and the modified node systems $\mathbf{y}^* := (y_1^*, y_2^*, y_3^*) := (1 - x_3, 1 - x_2, 1 - x_1)$ and $\mathbf{x}^* := (x_1^*, x_2^*, x_3^*) := (1 - y_3, 1 - y_2, 1 - y_1)$.

Let us then compute $F^*(\mathbf{y}^*, s)$. We find

$$F^*(\mathbf{y}^*, s) = J(1 - s) + \nu_3 K(-(s - y_1^*)) + \nu_2 K(-(s - y_2^*)) + \nu_1 K(-(s - y_3^*))$$
$$= J(1 - s) + \nu_3 K(1 - s - x_3) + \nu_2 K(1 - s - x_2) + \nu_1 K(1 - s - x_1)$$
$$= F(\mathbf{x}, 1 - s).$$

Similarly, $F^*(\mathbf{x}^*, s) = F(\mathbf{y}, 1 - s)$. It follows that $m_j^*(\mathbf{x}^*) = m_{4-j}(\mathbf{y})$ and that $m_j^*(\mathbf{y}^*) = m_{4-j}(\mathbf{x})$, therefore the two function systems and nodes exhibit intertwining of maxima precisely in the corresponding cases.

However, for the modified system we have $y_1^* = 1 - x_3 > 1 - y_3 = x_1^*$, $y_2^* = 1 - x_2 > 1 - y_2 = x_2^*$ and $y_3^* = 1 - x_1 < 1 - y_1 = x_3^*$, so for these node systems we can return to Case 2. Therefore, $\mathbf{x}^*, \mathbf{y}^*$ have strict intertwining of maxima, hence so does $\mathbf{x}, \mathbf{y}$, too.

$\square$

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