New boundary monodromy matrices for classical sigma models

Tamás Gombor

Lendület Holographic QFT Group, Wigner Research Centre for Physics, Konkoly-Thege Miklós u. 29-33, 1121 Budapest , Hungary

Abstract

The 2d principal models without boundaries have $G\times G$ symmetry. The already known integrable boundaries have either $H\times H$ or G_D symmetries, where H is such a subgroup of G for which G/H is a symmetric space while G_D is the diagonal subgroup of $G\times G$. These boundary conditions have a common feature: they do not contain free parameters. We have found new integrable boundary conditions for which the remaining symmetry groups are either $G\times H$ or $H\times G$ and they contain one free parameter. The related boundary monodromy matrices are also described.

Keywords: principal chiral model, non-linear sigma model, boundary conditions, double row monodromy matrix, classical boundary Yang-Baxter equation

Contents

Email address: gombor.tamas@wigner.mta.hu (Tamás Gombor)

1. Introduction

In this paper we investigate $1 + 1$ dimensional $\mathrm{O}(N)$ sigma and principal chiral models (PCMs). These are integrable at the quantum level i.e. infinite many conserved charges survive the quantization [1, 2]. The scattering matrices (S-matrices) are factorized and they can be constructed from the two particle S-matrices which satisfy the Yang-Baxter equation (YBE). Thus, integrable theories at infinite volume can be defined by the solutions of the YBE. For example, it has been verified that the minimum solution of the $\mathrm{O}(N)$ symmetric YBE is the S-matrix of the $\mathrm{O}(N)$ sigma model [3].

In this paper we are interested in boundary conditions for these systems. There are three interesting type of boundary conditions which are:

I Classically conformal - which means that the boundary condition does not break the classical conformal symmetry, which guaranties infinitely many conserved charges

II Boundary conditions with zero curvature representation which means that there exists a κ-matrix (or classical reflection matrix) from which double row monodromy matrices can be constructed

III Quantum integrable, which means that there exist a higher spin conserved charge even on the half line.

The basic examples of boundary conditions of $\mathrm{O}(N)$ sigma models are:

1. Restricted boundary conditions when we restrict the field to a lower dimensional sphere
   (a) with arbitrary radius
   (b) with maximal radius
2. Using boundary Lagrangian $L_b = \mathbf{n}^T M \dot{\mathbf{n}}$ with $M \in \mathfrak{so}(N)$ (See notations in Section 3)
   (a) where M is arbitrary
   (b) where $M^2 = c1$
   (c) where $M^2 = \mathrm{diag}(c, c, 0, \dots, 0)$. .

The basic examples of boundary conditions of PCM of group G are as follows:

i Restricted boundary condition when we restrict the field to a subgroup H.
   (a) where H is arbitrary
   (b) where G/H is a symmetric space.

ii Using boundary Lagrangian $L_b = \mathrm{Tr}\,(M J_0)$ with $M \in \mathfrak{g}$ (See notations in Section 2)
   (a) where M is arbitrary
   (b) where the G/H is a symmetric space for $H := \left\{h \in G | h M h^{-1} = M\right\}$. .

These boundary conditions were investigated in [4, 5, 6, 7, 8, 9] and was shown that all of them are conformal. What can we say about the quantum integrability of these boundary conditions? In some of these cases, one can also use the Goldschmidt-Witten argument [5, 7] which is a sufficient condition for quantum integrability. With this argument it can be shown that boundary conditions 1b and ib are integrable at the quantum level.

There is also a necessary condition for quantum integrability which comes from the boundary bootstrap. As we know, quantum integrable theories with boundary can be defined with the

	I	II	III
1a	✓	×	×
1b	✓	✓	✓
2a	✓	?	×
2b	✓	?	?
2c	✓	?	?
ia	✓	×	×
ib	✓	✓	✓
iia	✓	?	×
iib	✓	?	?

Table 1: Properties of boundary conditions

bulk S-matrix and the boundary scattering matrix (or reflection matrix, R-matrix). Reflection matrices are solutions of the boundary Yang-Baxter equation (bYBE). They are classified for the $\mathrm{O}(N)$ sigma model [7, 10]. There are two classes which have symmetries either $\mathrm{O}(k)\times\mathrm{O}(N-k)$ or $\mathrm{U}(n)$ if $N=2n$. There is a free parameter in the reflection matrix when the remaining symmetries are $\mathrm{O}(2)\times\mathrm{O}(N-2)$ and $\mathrm{U}(n)$. Thus we can infer that if the center of the residual symmetry algebra is $\mathfrak{u}(1)$ then the reflection matrix contains a free parameter [11].

We can also classify the residual symmetries of PCMs. The bulk theory has $G_L\times G_R$ symmetry and the particles transform with respect to some representations of this symmetry. If the reflection matrix has a factorized form ($R=R_L\otimes R_R$), then the bYBE can be separated into equation for left and right reflection matrices. Thus, in principle, arbitrarily combined solutions R_L and R_R can be used to construct the full reflection matrix R. This implies that the remaining left and right symmetries can be different.

From the classification of the quantum reflection matrices [7, 6, 10, 11, 12] we can extract the possible residual symmetries therefore we can conclude that 1a, 2a, ia and iia can not be quantum integrable because their residual symmetries are different.

The zero curvature description is also known for some boundary conditions [4, 8]. Their classical reflection matrices are constant matrices without any parameters.

The state of the art about boundary conditions and their integrability can be summarized in Table 1. With question marks we indicated the open questions. For example, 1b is quantum integrable (Goldschmidt-Witten argument) and it has $\mathrm{O}(k)\times\mathrm{O}(N-k)$ symmetry so it can be matched to the reflection matrix (coming from the bootstrap) with the same symmetry. Contrary, we have a $\mathrm{U}(N/2)$ symmetric reflection matrix with a free parameter and one can ask which boundary condition belongs to it. The boundary condition 2b is a natural candidate because it has a free parameter and the same symmetry. Indeed, in this paper we show that it has a zero curvature representation which may indicate the quantum integrability in view of the fact that a restricted boundary condition preserved the integrability at the quantum level if and only if there exists a zero curvature representation (see the table above).

In the PCM the remaining symmetries for the known classical integrable boundary conditions are $H_L\times H_R$ where $H_L\cong H_R$ which means $R_L\cong R_R$ (or the residual symmetry is G_D which is the diagonal subgroup of $G_L\times G_R$ but in this case the reflection matrix is not factorized) [5, 6]. This paper also provides a zero curvature representation for boundary condition iib where only the left or the right symmetries are broken therefore these can be candidates for reflection matrices where $R_L\ncong R_R$.

We also derive that the traces of these new monodromy matrices Poisson commute therefore there are infinitely many conserved charges in involution. This Poisson algebra of the one and double row monodromy matrices are consistent if the r-matrix and classical reflection matrix

(κ-matrix) satisfy the classical Yang-Baxter (cYBE) and the classical boundary Yang-Baxter equations (cbYBE). In [4] and [8] the Poisson algebra was investigated for non-ultralocal theories with constant κ-matrix. In [13] this was done for ultralocal theories with dynamical κ-matrix when the Poisson bracket of the κ-matrix and the Lax-connection vanished. In this paper we derive the Poisson algebra of non-ultralocal theories with κ-matrix whose Poisson-bracket with the Lax-connection does not vanish. However, the possible solutions of this equation have only been examined in a few cases. In this paper we classify the solutions of the field independent cbYBE and check that the new field dependent κ-matrix is satisfies the cbYBE for O(N) sigma models.

The paper is structured as follows. In the next section, we start with the Lax formalism of the PCMs where we construct classical reflection matrices and use them to build double row transfer matrices. The conservation of these matrices (which is equivalent to the existence of infinite many conserved charges) provides the boundary conditions of the theories which belong to these boundary Lax representations. Using these results, we derive new double row monodromy matrices for the O($2n$) sigma models and the corresponding boundary conditions will be determined too. In Section 4 we derive the Poisson algebra of the double row monodromy matrices and the cbYBE which is satisfied for the new κ-matrices.

2. Principal Chiral Models on the half line

In this section the new boundary monodromy matrix will be introduced. In the first subsection we will overview the Lax formalism of PCMs. After that the new reflection matrix and the related boundary condition will be derived. Finally we will show the corresponding Lagrangian descriptions and the unbroken symmetries of these models.

2.1. Lax formalism for PCMs

Let $\mathfrak{g}$ be a semi-simple Lie algebra and $G = \exp(\mathfrak{g})$. We use only matrix Lie-algebra and we work in the defining representation. The field variable is a map $g : \Sigma \to G$ where the space-time $\Sigma = \mathbb{R} \times (-\infty, 0]$ is parameterized with $(x^0, x^1) = (t, x)$. We can define two currents $\mathrm{J}^R = g^{-1}\mathrm{d}g$ and $\mathrm{J}^L = g\mathrm{d}g^{-1}$ where $\mathrm{J}^{L/R} = J_0^{L/R}\mathrm{d}x^0 + J_1^{L/R}\mathrm{d}x^1 \left(= J_t^{L/R}\mathrm{d}t + J_x^{L/R}\mathrm{d}x\right)$[1]. These two currents satisfy the flatness condition (by definition):

$$\mathrm{dJ}^{L/R} + \mathrm{J}^{L/R} \wedge \mathrm{J}^{L/R} = 0$$

The bulk equation of motion (E.O.M) is

$$\mathrm{d} * \mathrm{J}^{L/R} = 0.$$

The E.O.M and the flatness condition is equivalent to the flatness condition of the Lax connection:

$$\mathrm{dL}^{L/R}(\lambda) + \mathrm{L}^{L/R}(\lambda) \wedge \mathrm{L}^{L/R}(\lambda) = 0 \tag{1}$$

where

$$\mathrm{L}^{L/R}(\lambda) = \frac{1}{1-\lambda^2}\mathrm{J}^{L/R} + \frac{\lambda}{1-\lambda^2} * \mathrm{J}^{L/R}.$$

We will also use the following notations

$$\mathcal{M}^{L/R}(\lambda) = L_0^{L/R}(\lambda) \qquad\qquad \mathcal{L}^{L/R}(\lambda) = L_1^{L/R}(\lambda)$$

[1] The ordinary letters denote forms and the italic letters denote the local coordinate functions of these.

Using these, the zero curvature condition can be written as

$$\partial_t \mathcal{L}(\lambda) - \partial_x \mathcal{M}(\lambda) + [\mathcal{M}(\lambda), \mathcal{L}(\lambda)] = 0.$$

The usefulness of the Lax connection lies in the fact that one can generate from it an infinite family of conserved charges. At first we define the one row monodromy matrix

$$T_{L/R}(\lambda) = \mathcal{P}\overleftarrow{\exp}\left(-\int_{-\infty}^{0} \mathcal{L}^{L/R}(\lambda)\mathrm{d}x\right). \tag{2}$$

These monodromy matrices have an inversion property

$$T_R(\lambda) = g^{-1}(0) T_L(1/\lambda) g(-\infty). \tag{3}$$

The monodromy matrix in the boundary case takes a double row type form

$$\Omega_{L/R}(\lambda) = T_{L/R}(-\lambda)^{-1} \kappa_{L/R}(\lambda) T_{L/R}(\lambda), \tag{4}$$

where the $\kappa_L(\lambda), \kappa_R(\lambda) \in G$ are the reflection matrices which will be specified later. In the following we use the right currents therefore we introduce the following notation $\mathrm{J}(\lambda) = \mathrm{J}^R(\lambda)$, $\mathrm{L}(\lambda) = \mathrm{L}^R(\lambda)$, $T(\lambda) = T_R(\lambda)$, $\Omega(\lambda) = \Omega_R(\lambda)$, $\kappa(\lambda) = \kappa_R(\lambda)$, $\mathcal{M}(\lambda) = \mathcal{M}^R(\lambda)$ and $\mathcal{L}(\lambda) = \mathcal{L}^R(\lambda)$

The existence of infinitely many conserved quantities requires that the time derivative of the monodromy matrix has to vanish $\dot{\Omega}(\lambda) = 0$, which is equivalent to:

$$\kappa(\lambda)\mathcal{M}(\lambda)\Big|_{x=0} - \mathcal{M}(-\lambda)\Big|_{x=0}\kappa(\lambda) = \dot{\kappa}(\lambda), \tag{5}$$

where we assumed that the currents vanish at $-\infty$. This is the *boundary flatness condition.*

This equation can be translated to boundary conditions for the J^R current. The consistency of the theory requires that the number of boundary conditions have to be equal to $\dim(\mathfrak{g})$. Based on these, we call $\kappa(\lambda)$ a consistent solution of (5) if it leads to exactly $\dim(\mathfrak{g})$ boundary conditions.

The consistency of the definitions of double row monodromy matrices Ω_L and Ω_R (the boundary flatness condition implies the same boundary conditions with Ω_L and Ω_R) implies that

$$\kappa_R(\lambda) = g^{-1}(0)\kappa_L(1/\lambda) g(0). \tag{6}$$

Using this equation, the double row monodromy matrices also have an inversion property:

$$\Omega_R(\lambda) = g^{-1}(-\infty)\Omega_L(1/\lambda) g(-\infty). \tag{7}$$

Hereinafter, we look for consistent solutions for the equation (5). The most obvious ansatz for the reflection matrix is $\kappa(\lambda) = U$ where $U \in G$ is a constant matrix. Using this ansatz, the equation (5) is equivalent to the following two equations:

$$\begin{aligned} J_0 &= U J_0 U^{-1}, \\ -J_1 &= U J_1 U^{-1}. \end{aligned}$$

Clearly, J_0 and J_1 are elements of the eigenspaces of the linear transformation $\mathrm{Ad}_U : \mathfrak{g} \to \mathfrak{g}$ with $+1$ and -1 eigenvalues. These are equivalent to $\dim(\mathfrak{g})$ boundary conditions if and only if U^2 is proportional to 1. Thus, there is a $\mathbb{Z}_2$ graded decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}$ where $\mathfrak{h}$ and $\mathfrak{f}$ are the $+1$ and -1 eigenspaces of the Ad_U automorphism of $\mathfrak{g}$. Therefore the boundary conditions imply $J_0 \in \mathfrak{h}$ and $J_1 \in \mathfrak{f}$. These are well known integrable boundary conditions [6]. In the next subsection, we will try to find new consistent solutions with non-trivial spectral parameter dependency.

Before that, we note that there is another possibility for the definition of the double row monodromy matrix, namely:

$$\Omega(\lambda) = T_L(-\lambda)^{-1} U T_R(\lambda),$$

This leads to the following boundary conditions

$$J_0^L = U J_0^R U^{-1}, \tag{8}$$

$$-J_1^L = U J_1^R U^{-1}. \tag{9}$$

Let us calculate the number of boundary conditions. For this, let us use the relation between the left and right currents.

$$-g J_0^R g^{-1} = U J_0^R U^{-1}, \quad \Rightarrow \quad -J_0^R = \left(U^{-1} g\right)^{-1} J_0^R U^{-1} g,$$

$$+g J_1^R g^{-1} = U J_1^R U^{-1}, \quad \Rightarrow \quad +J_1^R = \left(U^{-1} g\right)^{-1} J_1^R U^{-1} g.$$

We saw previously that this type of boundary condition is consistent if the operator $\mathrm{Ad}_{U^{-1}g}$ is an involution on $\mathfrak{g}$ which is equivalent to

$$U^{-1} g U^{-1} g = e. \tag{10}$$

Clearly this restricted boundary condition is invariant under the transformation $g \to U g_0^{-1} U^{-1} g g_0$ therefore it has the diagonal symmetry G_D.

Finally, let us note that there is an other representation of this boundary condition. Using the inversion property (3) we can obtain an equivalent double row monodromy matrix:

$$\Omega(\lambda) = T_R(-1/\lambda)^{-1} \left(g(0)^{-1} U\right) T_R(\lambda),$$

The conservation of this double row monodromy matrix requires that the following boundary flatness condition has to vanish.

$$g^{-1} U \left(J_0^R - \lambda J_1^R\right) + \left(\lambda^2 J_0^R + \lambda J_1^R\right) g^{-1} U = (1-\lambda^2)\partial_0 \left(g^{-1} U\right) = (\lambda^2 - 1) J_0^R g^{-1} U$$

Multiplying this by g from the right, we obtain

$$U \left(J_0^R - \lambda J_1^R\right) + \lambda g J_1^R g^{-1} U = -g J_0^R g^{-1} U$$

which leads to the equations (8) and (9).

2.2. *Spectral parameter dependent κ-matrices*

In the previous subsection we summarized the spectral parameter independent κ-matrices. In this subsection, we try to find new *spectral parameter dependent* κs.

2.2.1. *Solution of the boundary flatness equation*

Let us use the following ansatz:

$$\kappa(\lambda) = k(\lambda)(1 + \lambda M + \lambda^2 N), \tag{11}$$

where $k(z)$ is a scalar and $M \in \mathfrak{g}$. Using this ansatz the equation (5) takes the following form:

$$\left(1 + \lambda M + \lambda^2 N\right)(J_0 - \lambda J_1) - (J_0 + \lambda J_1)\left(1 + \lambda M + \lambda^2 N\right) = 0.$$

Which leads to the following system of equations:

$$\lambda^1 : \qquad [M, J_0] - 2J_1 = 0 \tag{12}$$
$$\lambda^2 : \qquad [N, J_0] - [M, J_1]_+ = 0 \tag{13}$$
$$\lambda^3 : \qquad [N, J_1]_+ = 0 \tag{14}$$

where $[,]_+$ is the anti-commutator i.e. $[X, Y]_+ = XY + YX$. Since equation (12) provides already $\dim(\mathfrak{g})$ boundary conditions, the consistency requires that the equations (13) and (14) should follow from (12). In the following, we look for constraints on M and N which ensure this.

Taking the anti-commutator of equation (12) with M gives

$$[M, J_1]_+ = \frac{1}{2}[M, [M, J_0]]_+ = \frac{1}{2}\left[M^2, J_0\right].$$

The r.h.s is equal to $[N, J_0]$ if

$$N - \frac{1}{2}M^2 = c1, \tag{15}$$

where c is a constant. From this we can see that M commutes with N. Using this and the equation (14) we can obtain:

$$\left[N, [M, J_1]_+\right]_+ = 0.$$

Therefore, by taking the anti-commutator of equation (13) with N, we get

$$\left[N^2, J_0\right] = 0.$$

Since J_0 spans the whole defining representation of $\mathfrak{g}$ therefore N^2 has to be proportional to 1 so the automorphism Ad_N has $+1$ and -1 eigenvalues and we denote the corresponding eigenspaces by $\mathfrak{h}$ and $\mathfrak{f}$. Therefore N defines a $\mathbb{Z}_2$ graded decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}$.

Equation (14) means that $J_1 \in \mathfrak{f}$ i.e $\Pi_{\mathfrak{h}}(J_1) = 0$ where $\Pi_{\mathfrak{h}}$ is the projection operator of $\mathfrak{h}$ subspace. Putting this into (13):

$$\Pi_{\mathfrak{h}}(J_1) = \frac{1}{2}\Pi_{\mathfrak{h}}([M, J_0]) = \frac{1}{2}[M, \Pi_{\mathfrak{h}}(J_0)]$$

where we used that $[M, N] = 0$ which implies $M \in \mathfrak{h}$. We can see from the last equation that equation (14) follows from (12) if M commutes with $\mathfrak{h}$.

Summarizing, consistency of the solutions requires the following conditions

$$2N - M^2 \sim 1 \qquad \text{and} \qquad N^2 \sim 1. \tag{16}$$

These implies that Ad_N generates a $\mathbb{Z}_2$ graded decomposition and M is an element of $\mathfrak{h}$ and also commutes with $\mathfrak{h}$. Therefore $\mathfrak{h}$ has a non-trivial center which is generated by M. It follows that every $\mathbb{Z}_2$ graded decomposition where $\mathfrak{h}$s are not semi-simple belong to these type of reflection matrices and boundary conditions.

There are two classes of these κ matrices. The first is $N \neq 0$. The second case is $N = 0$, which implies that $M^2 \sim 1$. In this case M defines the $\mathbb{Z}_2$ graded decomposition. The projection operators to the $\mathfrak{h}$ and $\mathfrak{f}$ are:

$$\Pi_{\mathfrak{h}}(X) = \frac{1}{2}\left(X + UXU^{-1}\right),$$
$$\Pi_{\mathfrak{f}}(X) = \frac{1}{2}\left(X - UXU^{-1}\right),$$

where $U = N$ when $N \neq 0$ otherwise $U = M$. The classification of these κ-matrices for classical Lie-algebras are shown in the following.

2.2.2. Examples

We saw that the integrable boundary conditions described above belongs to a $(\mathfrak{g}, \mathfrak{h})$ symmetric pair for which G/H is a symmetric spaces ($G = \exp(\mathfrak{g})$, $H = \exp(\mathfrak{h})$). The symmetric spaces are classified [14]. The spectral parameter dependent solutions belongs to not semi-simple $\mathfrak{h}$ therefore there are three types of spectral parameter dependent κ-matrices.

1. $\mathfrak{g} = \mathfrak{su}(n)$ and $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{su}(m) \oplus \mathfrak{su}(n-m)$. The $\mathfrak{u}(1) \subset \mathfrak{h}$ sub-algebra is generated by the matrix M and condition (16) leads to the following N:

$$M = i\frac{2a}{k-m}\begin{pmatrix} -k1_{m\times m} & 0_{m\times k} \\ 0_{k\times m} & m1_{k\times k}\end{pmatrix}, \qquad N = a^2\frac{n}{k-m}\begin{pmatrix} -1_{m\times m} & 0_{m\times k} \\ 0_{k\times m} & 1_{k\times k}\end{pmatrix},$$

where $k = n - m$. One can choose a function $k(\lambda)$ for which $\kappa(z) \in \mathrm{U}(n)$ when $z \in \mathbb{R}$:

$$\kappa_1(\lambda|a) = \begin{pmatrix} \frac{1+ia\lambda}{1-ia\lambda}1_{m\times m} & 0_{m\times k} \\ 0_{k\times m} & 1_{k\times k}\end{pmatrix}.$$

2. $\mathfrak{g} = \mathfrak{so}(n)$ and $\mathfrak{h} = \mathfrak{so}(2) \oplus \mathfrak{so}(n-2)$. The M, N and the $\kappa(\lambda) \in \mathrm{SO}(n)$ can be written as:

$$M = 2a\begin{pmatrix} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad N = a^2\begin{pmatrix} -1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\kappa_2(\lambda|a) = \begin{pmatrix} A(\lambda|a) & -B(\lambda|a) & 0 & 0 & \cdots \\ B(\lambda|a) & A(\lambda|a) & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$A(\lambda|a) = \frac{1-\lambda^2a^2}{1+\lambda^2a^2}, \qquad B(\lambda|a) = \frac{2\lambda a}{1+\lambda^2a^2}.$$

3. $\mathfrak{g} = \mathfrak{so}(2n)$ or $\mathfrak{g} = \mathfrak{sp}(n)$ and $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{su}(n)$ For this case

$$M = a\begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n}\end{pmatrix}.$$

Since $M^2 = -a^2 1$ then $N = 0$. The κ-matrix is the following:

$$\kappa_3(\lambda|a) = \frac{1}{\sqrt{1+\lambda^2a^2}}\begin{pmatrix} 1_{n\times n} & -\lambda a1_{n\times n} \\ \lambda a1_{n\times n} & 1_{n\times n}\end{pmatrix}.$$

We can check that $\kappa_3(\lambda) \in \mathrm{SO}(2n)$ and $\kappa_3(\lambda) \in \mathrm{Sp}(n)$ too.

These matrices are the classical counterparts of the $\mathfrak{h} = \mathfrak{u}(1)\oplus\mathfrak{su}(m)\oplus\mathfrak{su}(n-m)$, $\mathfrak{h} = \mathfrak{u}(1)\oplus\mathfrak{su}(n)$ and $\mathfrak{h} = \mathfrak{so}(2)\oplus\mathfrak{so}(n-2)$ symmetric solutions of the quantum boundary Yang-Baxter equation [7][6][9]. The quantum reflection matrices are

$$R_1(\theta|c) = \nu_1(\theta|c)\begin{pmatrix} \frac{c-\theta}{c+\theta}1_{m\times m} & 0_{m\times k} \\ 0_{k\times m} & 1_{k\times k}\end{pmatrix},$$
$$R_2(\theta|c) = \nu_2(\theta|c)\begin{pmatrix} \tilde{A}(\theta|c) & -\tilde{B}(\theta|c) & 0 & 0 & \cdots \\ \tilde{B}(\theta|c) & \tilde{A}(\theta|c) & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
$$R_3(\theta|c) = \nu_3(\theta|c)\begin{pmatrix} c1_{n\times n} & -i\theta 1_{n\times n} \\ i\theta 1_{n\times n} & c1_{n\times n}\end{pmatrix},$$

where $\nu_i(\theta)$ are some dressing phases and

$$\tilde{A}(\theta|c) = \frac{1}{2}\left(\frac{c-k-\theta}{c-k+\theta} + \frac{-c-k-\theta}{-c-k+\theta}\right),$$
$$\tilde{B}(\theta|c) = \frac{1}{2}\left(\frac{c-k-\theta}{c-k+\theta} - \frac{-c-k-\theta}{-c-k+\theta}\right),$$
$$k = -i\frac{\pi}{2}\frac{n-4}{n-2}.$$

For the classical limit we define a scaling variable h for which

$$\theta = \lambda/h, \qquad\qquad c = i/(ha).$$

The classical limit is $h\to 0$. In this limit the R-matrices are proportional to the κ matrices:

$$\lim_{h\to 0} R_i(\lambda/h|i/(ha)) \sim \kappa_i(\lambda|a).$$

2.2.3. Lagrangian and symmetries

In the previous subsection we found reflection matrices parameterized as (11) which leads to the following boundary condition:

$$J_1^R = \frac{1}{2}\left[M, J_0^R\right]. \tag{17}$$

Using the left currents this condition takes the form:

$$J_1^L = \frac{1}{2}\left[gMg^{-1}, J_0^L\right]. \tag{18}$$

One can obtain the same boundary condition in the Lagrangian description. The Lagrangian density of the bulk theory is

$$\mathcal{L}_{PCM} = -\frac{1}{4}\mathrm{Tr}\left[\mathrm{J}^L\wedge *\mathrm{J}^L\right] = -\frac{1}{4}\mathrm{Tr}\left[\mathrm{J}^R\wedge *\mathrm{J}^R\right]$$

Thus if we add a boundary Lagrangian function as

$$L_b = \frac{1}{4}\mathrm{Tr}\left[MJ_0^R\right]\Big|_{x=0} \tag{19}$$

we get the boundary condition (17). This boundary condition was already investigated in [7] and [9]. It was shown that this is a conformal boundary condition for all $M\in\mathfrak{g}$. Now we have just shown that it has a zero curvature representation too for some special Ms which satisfy the conditions (16).

Now let us continue with the residual symmetries. The bulk Lagrangian has $G_L \times G_R$ symmetries which are the left/right multiplications with a constant group element: $g(x) \to g_L g(x)$ and $g(x) \to g(x) g_R$. The transformations of the currents are the following:

$$\begin{aligned} g_L : &\qquad \mathrm{J}^L \to g_L \mathrm{J}^L g_L^{-1}, &\qquad \mathrm{J}^R \to \mathrm{J}^R, \\ g_R : &\qquad \mathrm{J}^L \to \mathrm{J}^L, &\qquad \mathrm{J}^R \to g_R^{-1} \mathrm{J}^R g_R. \end{aligned}$$

We can see that the boundary Lagrangian breaks the G_R symmetry. The remaining symmetry is $H_R < G_R$ where $H_R = \exp(\mathfrak{h})$. Since the current J^R is invariant under G_L, the G_L symmetry is unbroken therefore the residual symmetry is $G_L \times H_R$.

One can derive the Noether charges by the variation of the action but there is an easier way. We know that the J^L and J^R are the Noether currents of the bulk G_L and G_R symmetries. Let us define the following charges:

$$\begin{aligned} Q_L &= \int_{-\infty}^{0} J_0^L \mathrm{d}x, \\ Q_R &= \int_{-\infty}^{0} J_0^R \mathrm{d}x. \end{aligned}$$

By taking their time derivatives we obtain

$$\begin{aligned} \dot{Q}_L &= \int_{-\infty}^{0} \partial_1 J_1^L \mathrm{d}x = J_1^L \Big|_{x=0} = \frac{1}{2}[gMg^{-1}, J_0^L]\Big|_{x=0} = \frac{1}{2}\partial_0 \left(gMg^{-1}\right)\Big|_{x=0} \\ \dot{Q}_R &= \int_{-\infty}^{0} \partial_1 J_1^R \mathrm{d}x = J_1^R \Big|_{x=0} \end{aligned}$$

We can see that

$$\tilde{Q}_L = Q_L - \frac{1}{2}\left(gMg^{-1}\right)\Big|_{x=0} \quad \text{and} \tag{20}$$

$$\tilde{Q}_R = \Pi_{\mathfrak{h}}\left(Q_R\right) \tag{21}$$

are conserved charges.

Finally we note that we could have used the left current J^L with the κ-matrix

$$\kappa_L(\lambda) \sim 1 + \lambda M + \lambda^2 N$$

This implies that the right reflection matrix, the boundary condition and the boundary Lagrangian are

$$\begin{aligned} \kappa_R(\lambda) &\sim 1 + \frac{1}{\lambda} g^{-1} M g + \frac{1}{\lambda^2} g^{-1} N g \\ J_1^L &= \frac{1}{2}[M, J_0^L] \\ L_b &= \frac{1}{4}\mathrm{Tr}[M J_0^L]\Big|_{x=0} \end{aligned}$$

Therefore, in this case the residual symmetry is $H_L \times G_R$.

3. O(N) sigma model on the half line

The new reflection matrices of the PCM can be used to find new ones for the O(N) sigma model. In particular, using the equivalence between SU(2) PCM and the O(4) sigma model we have immediately new reflection matrices for the O(N) sigma model when $N = 4$. This solution then can be generalized for even N.

3.1. Lax formalism for the O(N) *sigma model*

The field variables are $\mathbf{n} : \Sigma \to \mathbb{R}^N$ with the $\mathbf{n}^T\mathbf{n} = 1$ constrain. The bulk Lagrangian is

$$\mathcal{L}_{NL\sigma} = \frac{1}{2}\mathrm{d}\mathbf{n}^T \wedge *\mathrm{d}\mathbf{n} - \frac{1}{2}\sigma(\mathbf{n}^T\mathbf{n} - 1).$$

from which equation of motion follows:

$$\mathrm{d} * \mathrm{d}\mathbf{n} + (\mathrm{d}\mathbf{n}^T \wedge \mathrm{d}\mathbf{n})\mathbf{n} = 0.$$

We can define an O(N) group element as: $h = 1 - 2\mathbf{n}\mathbf{n}^T$ which satisfies the following identities: $h^T h = 1$ and $h = h^T$. Using this, one can define a current: $\hat{\mathrm{J}} = h\mathrm{d}h = 2\mathbf{n}\mathrm{d}\mathbf{n}^T - 2\mathrm{d}\mathbf{n}\mathbf{n}^T$ which is the Noether current of the bulk global SO(N) symmetry. The e.o.m with this current is $\mathrm{d} * \hat{\mathrm{J}} = 0$ and the Lagrangian is

$$\mathcal{L}_{NL\sigma} = -\frac{1}{16}\mathrm{Tr}\left[\hat{\mathrm{J}} \wedge *\hat{\mathrm{J}}\right].$$

The Lax connection is very similar to the PCM but here the current is constrained.

$$\hat{\mathrm{L}}(\lambda) = \frac{1}{1-\lambda^2}\hat{\mathrm{J}} + \frac{\lambda}{1-\lambda^2}(*\hat{\mathrm{J}}).$$

The double row monodromy matrix can be defined similarly as it was in PCMs. In the following we look for solutions of the boundary flatness equation

$$\kappa(\lambda)\hat{\mathcal{M}}(\lambda) - \hat{\mathcal{M}}(-\lambda)\kappa(\lambda) = \dot{\kappa}(\lambda),$$

Let us start with the constant κ-matrices i.e. $\kappa(\lambda) = U$ where $U \in$ O(N) therefore the boundary flatness equation looks like

$$U\left(\hat{J}_0 - \lambda\hat{J}_1\right) - \left(\hat{J}_0 + \lambda\hat{J}_1\right)U = 0$$

which implies the following:

$$\lambda^0: \qquad \hat{J}_0 = U\hat{J}_0U^{-1}, \tag{22}$$

$$\lambda^1: \qquad -\hat{J}_1 = U\hat{J}_1U^{-1}. \tag{23}$$

In this subsection, we assume that $U^2 = \pm 1$ but we do not derive that. We will return to this at the next section. There are two kinds of Us:

1. $U = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$,
2. $U = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix}$ where $n = N/2$.

Let us start with the first case. Let the number of +1s and −1s be $N-k$ and k respectively. Let us use the notation: $\mathbf{n} = \tilde{\mathbf{n}} + \hat{\mathbf{n}}$, with

$$\tilde{\mathbf{n}} = (n_1, \dots, n_{N-k}, 0, \dots, 0) \quad , \qquad \hat{\mathbf{n}} = (0, \dots, 0, n_{N-k+1}, \dots, n_N).$$

Using this, the equation (22) is equivalent to

$$\tilde{\mathbf{n}}\dot{\hat{\mathbf{n}}}^T = \dot{\tilde{\mathbf{n}}}\hat{\mathbf{n}}^T.$$

Multiplying by $\hat{\mathbf{n}}$ from the right and $\tilde{\mathbf{n}}^T$from the left, we can obtain the following two equations

$$\left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}\right)\dot{\tilde{\mathbf{n}}} = \left(\hat{\mathbf{n}}^T\dot{\hat{\mathbf{n}}}\right)\tilde{\mathbf{n}}, \tag{24}$$

$$\left(\tilde{\mathbf{n}}^T\tilde{\mathbf{n}}\right)\dot{\hat{\mathbf{n}}} = \left(\tilde{\mathbf{n}}^T\dot{\tilde{\mathbf{n}}}\right)\hat{\mathbf{n}}. \tag{25}$$

Similarly, from (23) we can get

$$\left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}\right)\hat{\mathbf{n}}' = \left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}'\right)\hat{\mathbf{n}}, \tag{26}$$

$$\left(\tilde{\mathbf{n}}^T\tilde{\mathbf{n}}\right)\tilde{\mathbf{n}}' = \left(\tilde{\mathbf{n}}^T\tilde{\mathbf{n}}'\right)\tilde{\mathbf{n}}. \tag{27}$$

Let us assume that $\hat{\mathbf{n}}^T\hat{\mathbf{n}} = 0$ which is equivalent to $\tilde{\mathbf{n}}^T\tilde{\mathbf{n}} = 1$ and $\hat{\mathbf{n}} = 0$. From this, the equations (24) and (26) are satisfied trivially and the equations (25) and (27) look like

$$\dot{\tilde{\mathbf{n}}} = 0,$$
$$\tilde{\mathbf{n}}' = 0,$$

where we used that $0 = \mathbf{n}^T\mathbf{n} = \hat{\mathbf{n}}^T\hat{\mathbf{n}}' + \tilde{\mathbf{n}}^T\tilde{\mathbf{n}}' = \tilde{\mathbf{n}}^T\tilde{\mathbf{n}}'$. We can see that this is the restricted boundary condition to a sphere S^k with maximal radius. Analogously, if we assume that $\tilde{\mathbf{n}}^T\tilde{\mathbf{n}} = 0$ then

$$\dot{\hat{\mathbf{n}}} = 0,$$
$$\hat{\mathbf{n}}' = 0,$$

which is the restricted bc to S^{N-k} with maximal radius.

What happens when $\hat{\mathbf{n}}^T\hat{\mathbf{n}} \neq 0$ and $\tilde{\mathbf{n}}^T\tilde{\mathbf{n}} \neq 0$. Let us multiply (24) with $\tilde{\mathbf{n}}^T$ form the left:

$$\left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}\right)\left(\tilde{\mathbf{n}}^T\dot{\tilde{\mathbf{n}}}\right) = \left(\hat{\mathbf{n}}^T\dot{\hat{\mathbf{n}}}\right)\left(\tilde{\mathbf{n}}^T\tilde{\mathbf{n}}\right),$$

Using that $\left(\hat{\mathbf{n}}^T\dot{\hat{\mathbf{n}}}\right) + \left(\tilde{\mathbf{n}}^T\dot{\tilde{\mathbf{n}}}\right) = 0$

$$0 = \left(\hat{\mathbf{n}}^T\hat{\mathbf{n}} + \tilde{\mathbf{n}}^T\tilde{\mathbf{n}}\right)\left(\tilde{\mathbf{n}}^T\dot{\tilde{\mathbf{n}}}\right) = \left(\tilde{\mathbf{n}}^T\dot{\tilde{\mathbf{n}}}\right)$$

therefore $\tilde{\mathbf{n}}^T\dot{\tilde{\mathbf{n}}} = \hat{\mathbf{n}}^T\dot{\hat{\mathbf{n}}} = 0$ which implies

$$\dot{\tilde{\mathbf{n}}} = 0,$$
$$\dot{\hat{\mathbf{n}}} = 0.$$

From this and equations (26), (27), we can see that there are too many boundary conditions therefore the $\hat{J}_0 = U\hat{J}_0U^{-1}$ and $\hat{J}_1 = -U\hat{J}_1U^{-1}$ are consistent boundary conditions if and only if $\hat{\mathbf{n}} = 0$ or $\tilde{\mathbf{n}} = 0$. In Subsection (4.3) we will see that $\kappa = \text{diag}(1, \dots, 1, -1, \dots, -1)$ satisfies the classical boundary Yang-Baxter equation if and only if $\hat{\mathbf{n}} = 0$ or $\tilde{\mathbf{n}} = 0$.

Let us continue with the second case i.e. $U^{\mathrm{T}} = -U$. Let us start with equation (25):

$$\mathbf{n}\mathbf{n}'^T - \mathbf{n}'\mathbf{n}^T = U\mathbf{n}\mathbf{n}'^T U - U\mathbf{n}'\mathbf{n}^T U$$

Let us multiply this with $\mathbf{n}$ from the right:

$$\mathbf{n}' = U\mathbf{n}\left(\mathbf{n}^T U\mathbf{n}'\right)$$

From this we can obtain the following two equations

$$\begin{aligned}\mathbf{n}'\mathbf{n}^T &= +U\mathbf{n}\mathbf{n}^T\left(\mathbf{n}^T U\mathbf{n}'\right)\\ \mathbf{n}\mathbf{n}'^T &= -\mathbf{n}\mathbf{n}^T U\left(\mathbf{n}^T U\mathbf{n}'\right)\end{aligned}$$

therefore

$$J_1 = -2\left(\mathbf{n}\mathbf{n}^T U + U\mathbf{n}\mathbf{n}^T\right)\left(\mathbf{n}^T U\mathbf{n}'\right)$$

Let us multiply this with U from the left and U^T from the right.

$$UJ_1U^T = -2\left(U\mathbf{n}\mathbf{n}^T + \mathbf{n}\mathbf{n}^T U\right)\left(\mathbf{n}^T U\mathbf{n}'\right) = J_1$$

Using this and the original equation (25) we can obtain that $J_1 = 0$ which is equivalent to $\mathbf{n}' = 0$. But we also have equation (24) therefore we have too many boundary condition which means that $\hat{J}_0 = U\hat{J}_0U^{-1}$ and $\hat{J}_1 = -U\hat{J}_1U^{-1}$ are not consistent boundary conditions at the second case. We will also see at Subsection (4.3) that the κ-matrix of the second case do not satisfy the classical boundary Yang-Baxter equation.

3.2. Spectral parameter dependent solution for $N = 4$

In the last section, we found a new spectral parameter dependent reflection matrix for the SU(2) PCM. Since this model is equivalent to the O(4) sigma model we can obtain a new non-constant κ-matrix for the O(4) sigma model by changing the notation to the O(4) sigma model language. We will see that this is a *spectral parameter and field (!) dependent* reflection matrix.

Thus we need to develop a dictionary between the SU(2) PCM and the O(4) sigma model. Let us introduce the following tensor:

$$\sigma^1_{\alpha\dot{\alpha}} = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \qquad \sigma^2_{\alpha\dot{\alpha}} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad \sigma^3_{\alpha\dot{\alpha}} = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \qquad \sigma^4_{\alpha\dot{\alpha}} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

which satisfies the following relations:

$$\begin{aligned}\sigma^i_{\alpha\dot{\alpha}}\bar{\sigma}^{\beta\dot{\beta}}_i &= 2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}},\\ \sigma^i_{\alpha\dot{\alpha}}\bar{\sigma}^{\alpha\dot{\alpha}}_j &= 2\delta^j_i,\end{aligned}$$

where $\bar{\sigma}^{\alpha\dot{\alpha}}_i$ is the complex conjugate of $\sigma^i_{\alpha\dot{\alpha}}$. Using this we can change the basis in which the group element $g_4 = \mathrm{SO}(4)$ is factorized.

$$\frac{1}{2}\sigma^i_{\alpha\dot{\alpha}}(g_4)^j_i\bar{\sigma}^{\beta\dot{\beta}}_j = (g_L)^\beta_\alpha(g_R)^{\dot{\beta}}_{\dot{\alpha}}, \qquad\qquad (g_4)^j_i = \frac{1}{2}\bar{\sigma}^{\alpha\dot{\alpha}}_i(g_L)^\beta_\alpha(g_R)^{\dot{\beta}}_{\dot{\alpha}}\sigma^j_{\beta\dot{\beta}}.$$

In this basis:

$$\mathbf{n} = g_4\mathbf{n}_0 \quad\rightarrow\quad \mathbf{n} = g_L g_R^T = \begin{pmatrix}n_4 + in_3 & in_1 + n_2\\ in_1 - n_2 & n_4 - in_3\end{pmatrix} = g \in \mathrm{SU}(2), \qquad (28)$$

if $\mathbf{n}_0 = (0,0,0,1)$.

We can also find the relation between the variables of the O(4) model $(h, \hat{\mathrm{J}})$ and the SU(2) PCM $(g, \mathrm{J}^{L/R})$. Using $\mathbf{n} = g_4\mathbf{n}_0$ and $h = 1 - 2\mathbf{n}\mathbf{n}^T$ we obtain that $h = g_4 j g_4^T$ where $j = 1 - 2\mathbf{n}_0\mathbf{n}_0^T = \mathrm{diag}(1,1,1,-1) \in \mathrm{O}(4)$. Since $\det(j) = -1$, j is not factorized in the new basis:

$$j \to (\sigma_2 \otimes \sigma_2^\dagger)P,$$

where P is the permutation operator.

The group element h in the new basis takes the form:

$$h = (g_L \otimes g_R)(\sigma_2 \otimes \sigma_2^\dagger)P(g_L^\dagger \otimes g_R^\dagger) = ((g\sigma_2) \otimes (g\sigma_2)^\dagger)P = P((g\sigma_2)^\dagger \otimes (g\sigma_2)).$$

(g was defined in (28)) In the last line we used the following property: $\sigma_2 g \sigma_2^\dagger = \bar{g}$ and $\bar{g}$ denotes the complex conjugate of g. We can see that h is not factorized. This is because h is not an element of SO(4). It is convenient to introduce a new notation:

$$h_2 = g\sigma_2, \quad \to \quad h = h_2 \otimes h_2^\dagger P.$$

Let us calculate $\hat{\mathrm{J}}$ in the new basis.

$$\hat{\mathrm{J}} = h\mathrm{d}h = \mathrm{J}^L \otimes 1 + 1 \otimes \bar{\mathrm{J}}^R, \tag{29}$$

where $\bar{\mathrm{J}}^R$ denotes the complex conjugate of J^R. The Lax connection in the new basis is:

$$\hat{\mathrm{L}}(\lambda) = \left(\frac{1}{1-\lambda^2}\mathrm{J}^L + \frac{\lambda}{1-\lambda^2} * \mathrm{J}^L\right) \otimes 1 + 1 \otimes \left(\frac{1}{1-\lambda^2}\bar{\mathrm{J}}^R + \frac{\lambda}{1-\lambda^2} * \bar{\mathrm{J}}^R\right) = \mathrm{L}^L(\lambda) \otimes 1 + 1 \otimes \bar{\mathrm{L}}^R(\lambda).$$

Therefore the monodromy matrix of the O(4) sigma model factorized in the following way:

$$\hat{T}(\lambda) = T_L(\lambda) \otimes \bar{T}_R(\lambda).$$

The double row monodromy matrix in the new basis reads:

$$\hat{\Omega}(\lambda) = (T_L(-\lambda)^{-1} \otimes \bar{T}_R(-\lambda)^{-1})\kappa_4(\lambda)(T_L(\lambda) \otimes \bar{T}_R(\lambda)).$$

Before we calculate the new κ-matrix let us apply the formula above to the known constant reflection matrices. The simplest known κ_4 is the identity matrix. This is factorized in the spinor basis: $\kappa^L = \kappa^R = 1$. Another known reflection matrix is $\kappa = \mathrm{diag}(-1,-1,1,1)$ in the vector basis. If we change the basis we get:

$$\kappa = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus $\kappa^R = \kappa^L = \mathrm{diag}(1,-1)$. These two reflection factors are consistent if they satisfy the inversion property (6) i.e.

$$\kappa_L(\lambda) = g(0)\kappa_R(1/\lambda)g^\dagger(0)$$

which means that g has to commute with them therefore g is restricted to $H = \mathrm{U}(1)$ at the boundary.

There is another known reflection matrix: $\kappa = \mathrm{diag}(1,1,1,-1)$ in the vector basis. If we change the basis we get:

$$\kappa = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = (\sigma_2 \otimes \sigma_2^\dagger)P.$$

We can see this matrix is not factorized. Using this formula for the monodromy matrix, we obtain that

$$\hat{\Omega}(\lambda) = P(\bar{T}_R^{-1}(-\lambda) \otimes T_L^{-1}(-\lambda))(\sigma_2^\dagger \otimes \sigma_2)(T_L(\lambda) \otimes \bar{T}_R(\lambda)) = \\ = P(\sigma_2^\dagger T_R^{-1}(-\lambda) T_L(\lambda)) \otimes (T_L^{-1}(-\lambda) T_R(\lambda)\sigma_2).$$

This theory is consistent in the principal model language if $g = g^\dagger$ at the boundary which is the boundary conditions (10).

These were the relations of the well known reflection matrices of the SU(2) PCM and the O(4) sigma model. Let us continue with the new one. In the last section we found new reflection matrices for the PCM model which for $\mathfrak{g} = \mathfrak{su}(2)$ simplifies to

$$\kappa^R(\lambda) \sim (1 + \lambda M_R)\,,$$

where M_R is an arbitrary element of $\mathfrak{su}(2)$. Without loss of generality one can choose $M_R = a\sigma_2$. We have seen that $\kappa^L(\lambda) = g\kappa^R(1/\lambda)g^\dagger$ so we have

$$\kappa(\lambda) \sim \left(1 + \frac{1}{\lambda} g M_R g^\dagger\right) \otimes \left(1 + \lambda \bar{M}_R\right) = 1\otimes 1 + \lambda 1 \otimes \bar{M}_R + \frac{1}{\lambda}(gM_R g^\dagger)\otimes 1 + (gM_R g^\dagger)\otimes \bar{M}_R, \quad (30)$$

Let us denote $1 \otimes \bar{M}_R$ in the vector representation by M. In the spinor basis hMh looks like

$$hMh \to ((g\sigma_2) \otimes (g\sigma_2)^\dagger) P (1 \otimes \bar{M}_R) P((g\sigma_2)^\dagger \otimes (g\sigma_2)) = (gM_R g^\dagger) \otimes 1, \quad (31)$$

therefore

$$MhMh = hMhM = \frac{1}{2}\left[M, hMh\right]_+ \to (gM_R g^\dagger) \otimes \bar{M}_R$$

Based on the above formulas, the new κ-matrix for O(4) takes the following form:

$$\kappa(\lambda) \sim 1 + \lambda M + \frac{1}{\lambda} hMh + \frac{1}{2}\left[M, hMh\right]_+\,, \quad (32)$$

where the matrix M looks like

$$M = a \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We can see that this κ is spectral parameter and field dependent too. We can give the boundary condition which correspond to this κ from the boundary conditions of SU(2) PCM (17),(18) and (29).

$$\hat{J}_1 = J_1^L \otimes 1 + 1 \otimes \bar{J}_1^R = \frac{1}{2}[gM_R g^\dagger, J_0^L] \otimes 1 + \frac{1}{2} 1 \otimes [\bar{M}_R, \bar{J}_0^R]$$

Using the definition of M

$$[M, \hat{J}_0] = [1 \otimes \bar{M}_R, J_0^L \otimes 1 + 1 \otimes \bar{J}_0^R] = 1 \otimes [\bar{M}_R, \bar{J}_0^R]$$

and using (31)

$$[hMh, \hat{J}_0] = [(gM_R g^\dagger) \otimes 1, J_0^L \otimes 1 + 1 \otimes \bar{J}_0^R] = [gM_R g^\dagger, J_0^L] \otimes 1$$

Therefore the boundary condition in language of the O(4) model is:

$$\hat{J}_1 = \frac{1}{2}[M + hMh, \hat{J}_0]. \quad (33)$$

This boundary condition was investigated in [9]. Using the definition $\hat{J} = h\mathrm{d}h = 2\mathbf{n}\mathrm{d}\mathbf{n}^T - 2\mathrm{d}\mathbf{n}\mathbf{n}^T$, we can get an equivalent form :

$$\mathbf{n}' = M\dot{\mathbf{n}} - (\mathbf{n}^T M\dot{\mathbf{n}})\mathbf{n}. \tag{34}$$

From the boundary Lagrangian of the SU(2) PCM we get

$$L_b = \frac{1}{4}\mathrm{Tr}[M_R J_0^R] = \frac{1}{8}\mathrm{Tr}[(1\otimes \bar{M}_R)(J_0^L \otimes 1 + 1 \otimes \bar{J}_0^R)],$$

therefore

$$L_b = \frac{1}{8}\mathrm{Tr}[M\hat{J}_0] \tag{35}$$

which agrees with [9]. Using the variables $\mathbf{n}$:

$$L_b = -\frac{1}{2}\mathbf{n}^T M\dot{\mathbf{n}}. \tag{36}$$

Finally, we can see that the residual symmetry is $\mathrm{U}(2) \cong \mathrm{SU}(2)_L \times \mathrm{U}(1)_R$ which is a subgroup of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \cong \mathrm{SO}(4)$. We saw in the PCMs that we have conserved charges $\tilde{Q}_L$ and $\tilde{Q}_R$. The conserved charge in the SO(4) language are:

$$\tilde{Q} = \tilde{Q}_L \otimes 1 + 1 \otimes \bar{\tilde{Q}}_R = Q_L \otimes 1 + 1 \otimes \overline{\Pi_{\mathfrak{h}_R}(Q_R)} - \frac{1}{2}(gM_R g^\dagger)\Big|_{x=0} \otimes 1.$$

which is equivalent to

$$\tilde{Q} = \Pi_{\mathfrak{h}}(Q) - \frac{1}{2}hMh\Big|_{x=0} = \Pi_{\mathfrak{h}}\left(Q - \frac{1}{2}hMh\Big|_{x=0}\right), \tag{37}$$

where $\mathfrak{h} = \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_R$, and Q is the bulk part of the charge:

$$Q = \int_{-\infty}^{0} \hat{J}_0 \mathrm{d}x.$$

3.3. Generalization for $N = 2n$

The result for $N = 4$ can be generalized for any even N. We assume that equation (32) can be used as κ matrix for $N = 2n$ i.e.

$$\kappa(\lambda) \sim 1 + \lambda M + \frac{1}{\lambda}hMh + \frac{1}{2}[M, hMh]_+, \tag{38}$$

where

$$M = a\begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n}\end{pmatrix}.$$

We have to prove that the time derivative of the double row monodromy matrix is zero when the boundary condition is satisfied. The quantity $\partial_0\hat{\Omega}$ is zero when the boundary flatness condition is satisfied

$$\kappa(\lambda)\hat{\mathcal{M}}(\lambda)\Big|_{x=0} - \hat{\mathcal{M}}(-\lambda)\Big|_{x=0}\kappa(\lambda) = \dot{\kappa}(\lambda), \tag{39}$$

Now the RHS is not zero since the κ has field dependence.

$$\dot{\kappa}(\lambda) \sim \partial_0\left(1 + \lambda M + \frac{1}{\lambda}hMh + \frac{1}{2}[M, hMh]_+\right) = \frac{1}{\lambda}\left[hMh, \hat{J}_0\right] + \frac{1}{2}\left[M, \left[hMh, \hat{J}_0\right]\right]_+.$$

Using this, equation (39) leads to the following three equations:

$$\begin{aligned}
\lambda^0: &\quad \frac{1}{2}\left[[M,hMh]_+,\hat{J}_0\right]-\left[hMh,\hat{J}_1\right]_+=\frac{1}{2}\left[M,\left[hMh,\hat{J}_0\right]\right]_+\\
\lambda^1: &\quad \left[M,\hat{J}_0\right]-2\hat{J}_1-\frac{1}{2}\left[[M,hMh]_+,\hat{J}_1\right]_+=-\left[hMh,\hat{J}_0\right]\\
\lambda^2: &\quad -\left[M,\hat{J}_1\right]_+=-\frac{1}{2}\left[M,\left[hMh,\hat{J}_0\right]\right]_+
\end{aligned}$$

If we take the anti-commutator of the boundary condition (33) with M then we will see that the third equation is satisfied. If we use the following identity

$$\left[[M,hMh]_+,\hat{J}_0\right]+\left[\left[\hat{J}_0,M\right],hMh\right]_+-\left[\left[hMh,\hat{J}_0\right],M\right]_+=0$$

then the first equation can be written as

$$\left[hMh,\hat{J}_1\right]_+=\frac{1}{2}\left[hMh,\left[M,\hat{J}_0\right]\right]_+.$$

This is also follows from the boundary condition.

Only the second equation remained. We have to prove that the following term vanish:

$$\frac{1}{2}\left[[M,hMh]_+,\hat{J}_1\right]_+ \tag{40}$$

Using the definition of h, we obtain that

$$MhMh=M(M-2\mathbf{n}\mathbf{n}^TM-2M\mathbf{n}\mathbf{n}^T)=-a^2h-2M\mathbf{n}\mathbf{n}^TM=hMhM.$$

Therefore

$$\frac{1}{2}[M,hMh]_+=-a^2h-2M\mathbf{n}\mathbf{n}^TM.$$

Since $\hat{J}_1$ is anti-commuting with h by definition, we only have to prove only that $M\mathbf{n}\mathbf{n}^TM$ is anti-commuting with $\hat{J}_1$ too. For this, we have to use the boundary condition (33) which can be written as

$$\hat{J}_1=-2M\dot{\mathbf{n}}\mathbf{n}^T-2\mathbf{n}\dot{\mathbf{n}}^TM$$

Using this, we obtain that

$$\left[M\mathbf{n}\mathbf{n}^TM,\hat{J}_1\right]_+=\left[M\mathbf{n}\mathbf{n}^TM,-2M\dot{\mathbf{n}}\mathbf{n}^T-2\mathbf{n}\dot{\mathbf{n}}^TM\right]_+=0.$$

Therefore the expression (40) is vanishing so the second equation is satisfied too which implies that the double row monodromy matrix is conserved if the boundary condition (33) is satisfied.

After this derivation, let us continue with the symmetries. Now the residual symmetry is $\mathrm{U}(n)<\mathrm{SO}(2n)$ where $H=\mathrm{U}(n)$ is the subgroup which commutes with M. Since $\mathrm{SO}(2n)/\mathrm{U}(n)$ is a symmetric space we have a $\mathbb{Z}_2$ graded decomposition $\mathfrak{so}(2n)=\mathfrak{h}\oplus\mathfrak{f}$ where $\mathfrak{h}=$ is the Lie-algebra of $\mathrm{U}(n)$ so $\mathfrak{h}=\mathfrak{su}(n)\oplus\mathfrak{u}(1)$. The $\mathfrak{u}(1)$ is generated by M so $[M,\mathfrak{h}]=0$ and $[M,\mathfrak{f}]\subset\mathfrak{h}$ therefore

$$[M,X]\in\mathfrak{f}, \tag{41}$$

for any $X\in\mathfrak{so}(2n)$.

For conserved charges, we can generalize the formula (37).

$$\tilde{Q}=\Pi_{\mathfrak{h}}\left(Q-\frac{1}{2}hMh\Big|_{x=0}\right)=\Pi_{\mathfrak{h}}\left(\int_{-\infty}^{0}\hat{J}_0\mathrm{d}x-\frac{1}{2}hMh\Big|_{x=0}\right).$$

We can check the conservation of these charges.

$$\dot{\dot{Q}} = \Pi_{\mathfrak{h}}\left(\dot{Q} - \frac{1}{2}(\dot{h}Mh + hM\dot{h})\Big|_{x=0}\right) = \Pi_{\mathfrak{h}}\left(\hat{J}_1 - \frac{1}{2}\left[hMh, \hat{J}_0\right]\right)\Big|_{x=0} = \frac{1}{2}\Pi_{\mathfrak{h}}\left[M, \hat{J}_0\right]\Big|_{x=0} = 0,$$

where we used (41).

The boundary Lagrangian can be written in the same form as we had for the case $N = 4$ (35) or (36):

$$L_b = \frac{1}{8}\mathrm{Tr}[M\hat{J}_0] = -\frac{1}{2}\mathbf{n}^T M\dot{\mathbf{n}}.$$

These have been studied earlier in [9] where it was showed that this is a conform boundary condition for any $M \in \mathfrak{so}(2n)$ but in this paper we showed more, namely that it has a zero curvature representation only when $M^2 \sim 1$.

4. Poisson algebra of double row monodromy matrices

In the previous sections we found new zero curvature representation of PCMs and $\mathrm{O}(N)$ sigma models on a half line. This implies the existence of infinitely many conserved charges. In this section we want to prove that these conserved charges are in involution. For this we determine the Poisson algebra of the double row monodromy matrices (whose trace is the generating function of these charges). In the first subsection we summarize the formulas of general "bulk" non-ultralocal theories based on [15]. After that we derive the Poisson-algebra of the double row monodromy matrices and their consistency condition (which is the classical boundary Yang-Baxter equation) when the Poisson-bracket of the reflection matrix and the Lax-connection is not zero. This is a new result because, so far Poisson-algebras of non-ultralocal theories with boundaries were investigated only when the κ-matrix was field independent [4, 8].

In the second and the third subsection we apply these general formulas for PCMs and non linear sigma models. We will use the following notations:

$$X_1 = X \otimes 1 \qquad\qquad X_2 = 1 \otimes X$$
$$Y_{12} = Y \otimes 1 \qquad\qquad Y_{23} = 1 \otimes Y$$

where $X \in \mathrm{End}(V)$ and $Y \in \mathrm{End}(V) \otimes \mathrm{End}(V)$ for a vector space V.

4.1. The double-row monodromy matrices of non-ultralocal theories

The general Poisson-brackets of the space-like components of the Lax-connection for non-ultralocal theories are the following [15]:

$$\begin{aligned}\{\mathcal{L}_1(x|\lambda_1), \mathcal{L}_2(y|\lambda_2)\} = &-\left[r_{12}(x|\lambda_1,\lambda_2), \mathcal{L}_1(x|\lambda_1) + \mathcal{L}_2(x|\lambda_2)\right]\delta(x-y) + \\ &+\left[s_{12}(x|\lambda_1,\lambda_2), \mathcal{L}_1(x|\lambda_1) - \mathcal{L}_2(x|\lambda_2)\right]\delta(x-y) - \\ &-\left(r_{12}(x|\lambda_1,\lambda_2) + s_{12}(x|\lambda_1,\lambda_2) - r_{12}(y|\lambda_1,\lambda_2) + s_{12}(y|\lambda_1,\lambda_2)\right)\delta'(x-y),\end{aligned} \tag{42}$$

From the anti-symmetry of the Poisson bracket (43) we obtain the following constraints on r- and s-matrices:

$$r_{12}(\lambda_1,\lambda_2) = -r_{21}(\lambda_2,\lambda_1),$$
$$s_{12}(\lambda_1,\lambda_2) = +s_{21}(\lambda_2,\lambda_1).$$

We can generalize the one row monodromy matrix for general paths from y to x:

$$T(x,y|\lambda) = \mathcal{P}\overleftarrow{\exp}\left(-\int_y^x \mathcal{L}(z|\lambda)\mathrm{d}z\right).$$

Let x_1, x_2, y_1, y_2 be different positions and $x_{1,2} > y_{1,2}$ then the general non-ultralocal Poisson-brackets of the monodromy matrices are the following [15]:

$$\{T_1(x_1, y_1|\lambda_1), T_2(x_2, y_2|\lambda_2)\} = t^-_{12}\left(R^-_{12}t_{12} - t_{12}R^+_{12}\right)t^+_{12}. \tag{43}$$

where $x_0 = min(x_1, x_2)$, $y_0 = max(y_1, y_2)$ and

$$\begin{aligned} t^-_{12} &= T_1(x_1, x_0|\lambda_1)T_2(x_2, x_0|\lambda_2) \\ t_{12} &= T_1(x_0, y_0|\lambda_1)T_2(x_0, y_0|\lambda_2) \\ t^+_{12} &= T_1(y_0, y_1|\lambda_1)T_2(y_0, y_2|\lambda_2) \\ R^-_{12} &= r_{12}(x_0|\lambda_1, \lambda_2) + \mathrm{sgn}(x_1 - x_2)s_{12}(x_0|\lambda_1, \lambda_2) \\ R^+_{12} &= r_{12}(y_0|\lambda_1, \lambda_2) + \mathrm{sgn}(y_2 - y_1)s_{12}(y_0|\lambda_1, \lambda_2) \end{aligned}$$

This Poisson-bracket satisfies the Jacobi identity (for not coinciding points) if the generalized classical Yang-Baxter equation is satisfied:

$$\begin{aligned} &[r_{23}(\lambda_2, \lambda_3) + s_{23}(\lambda_2, \lambda_3), r_{13}(\lambda_1, \lambda_3) + s_{13}(\lambda_1, \lambda_3)] + \\ &\quad + [r_{23}(\lambda_2, \lambda_3) + s_{23}(\lambda_2, \lambda_3), r_{12}(\lambda_1, \lambda_2) + s_{12}(\lambda_1, \lambda_2)] + \\ &\quad + [r_{13}(\lambda_1, \lambda_3) + s_{13}(\lambda_1, \lambda_3), r_{12}(\lambda_1, \lambda_2) - s_{12}(\lambda_1, \lambda_2)] + \\ &\qquad + H^{(r+s)}_{123}(\lambda_1, \lambda_2, \lambda_3) - H^{(r+s)}_{213}(\lambda_2, \lambda_1, \lambda_3) = 0 \end{aligned}$$

where

$$\{\mathcal{L}_1(x|\lambda_1), (r_{23}(y|\lambda_2, \lambda_3) + s_{23}(y|\lambda_2, \lambda_3))\} = -H^{(r+s)}_{123}(\lambda_1, \lambda_2, \lambda_3)\delta(x - y).$$

For the calculation of the Poisson bracket of the global monodromy matrices (2) we have to take the limits $x_1 \to x_2$ and $y_1 \to y_2$. However, the Poisson bracket (43) is not continuous due to the non ultra-locality. It is obvious that the equal intervals limit of the canonical brackets does not exist in a strong sense. More precisely, any strong definition implies the breakdown of the Jacobi identity for the canonical brackets of the global monodromy matrices (2).

However, it is possible to define this limit in a weak sense with respect to the canonical brackets based on a split-point procedure and a generalized symmetric limit. We consider canonical brackets of several monodromy matrices defined on intervals having coinciding end points. In order to compute them, let us first split the coinciding points and use (43) which then gives a completely consistent expression. Then if we symmetrize on all the possible splittings and go to the limit of equal points we get the "weak" algebras e.g. the weak algebra of the global monodromy matrices:

$$\{T_1(\lambda_1), T_2(\lambda_2)\} = r_{12}(0|\lambda_1, \lambda_2)T_1(\lambda_1)T_2(\lambda_2) - T_1(\lambda_1)T_2(\lambda_2)r_{12}(-\infty|\lambda_1, \lambda_2).$$

The formulas above can be found in [15] but in this paper we use a different conventions for the Lax-pair i.e. we have to change $\mathcal{L} \to -\mathcal{L}$ to get the formulas in [15]. In the following we derive the Poisson-algebra. For this we need the κ-matrices which were derived in the previous sections. We saw that these matrices can depend on the fields but do not on the derivative of the fields therefore we assume that

$$\{\mathcal{L}_1(x|\lambda_1), \kappa_2(\lambda_2)\} = -G_{12}(\lambda_1, \lambda_2)\delta(x).$$

Let us continue with the generalized double row monodromy matrix:

$$\Omega(x|\lambda) := T^{-1}(0, x| - \lambda)\kappa(\lambda)T(0, x|\lambda) = T(x, 0| - \lambda)\kappa(\lambda)T(0, x|\lambda).$$

The Poisson bracket of $\Omega(x|\lambda)$ and $\Omega(y|\mu)$ are not well defined even when $x \neq y$ therefore we have to use the split-point procedure. For this, we can define a shifted double row monodromy matrix:

$$\Omega^{\Delta}(x|\lambda) = T(x,\Delta|-\lambda)\kappa(\Delta|\lambda)T(\Delta,x|\lambda)$$

where $\Delta < 0$. A general κ-matrix depends on the boundary value of the fields $\phi_a(0)$ (i.e. $\kappa(\lambda) = \kappa(\phi_a(0)|\lambda)$) but we can extend this to arbitrary space coordinate:

$$\kappa(\Delta|\lambda) = \kappa(\phi_a(\Delta)|\lambda).$$

Using these the Poisson bracket of monodromy matrices are

$$\{\Omega_1(x_1|\lambda_1),\Omega_2(x_2|\lambda_2)\} \coloneqq \frac{1}{2}\lim_{\Delta\to 0}\left[\{\Omega_1(x_1|\lambda_1),\Omega_2^{\Delta}(x_2|\lambda_2)\} + \{\Omega_1^{\Delta}(x_1|\lambda_1),\Omega_2(x_2|\lambda_2)\}\right].$$

In the following we assume that

$$\begin{aligned} r(-\lambda_1,-\lambda_2) &= -r(\lambda_1,\lambda_2), \\ s(-\lambda_1,-\lambda_2) &= -s(\lambda_1,\lambda_2). \end{aligned}$$

Now we can calculate the symmetric limit:

$$\begin{aligned} \{\Omega_1(x_1|\lambda_1),\Omega_2(x_2|\lambda_2)\} =& t_{12}^{-}\left([R_{12},\omega_{12}] + \omega_1^{(1)}\tilde{R}_{12}\omega_2^{(2)} - \omega_2^{(2)}\tilde{R}_{12}\omega_1^{(2)}\right)t_{12}^{+} - \\ & -T_{12}^{-}\Bigg(\left[r_{12}(0|\lambda_1,\lambda_2),\kappa_1(\lambda_1)\kappa_2(\lambda_2)\right] + \\ & \quad + \kappa_1(\lambda_1)r_{12}(0|\lambda_1,-\lambda_2)\kappa_2(\lambda_2) - \kappa_2(\lambda_2)r_{12}(0|\lambda_1,-\lambda_2)\kappa_1(\lambda_1) + \\ & \quad + \frac{1}{2}\Big(G_{12}(-\lambda_1,\lambda_2)\kappa_1(\lambda_1) - \kappa_1(\lambda_1)G_{12}(\lambda_1,\lambda_2) - \\ & \quad - G_{21}(-\lambda_2,\lambda_1)\kappa_2(\lambda_2) + \kappa_2(\lambda_2)G_{21}(\lambda_2,\lambda_1)\Big)\Bigg)T_{12}^{+}. \end{aligned} \tag{44}$$

where $x_0 = max(x_1,x_2)$ and

$$\begin{aligned} t_{12}^{-} &= T_1(x_1,x_0|-\lambda_1)T_2(x_2,x_0|-\lambda_2) \\ \omega_{12} &= \Omega_1(x_0|\lambda_1)\Omega_2(x_0|\lambda_2) \\ \omega^{(1)} &= \Omega(x_0|\lambda_1) \\ \omega^{(2)} &= \Omega(x_0|\lambda_2) \\ t_{12}^{+} &= T_1(x_0,x_1|\lambda_1)T_2(x_0,x_2|\lambda_2) \\ R_{12} &= r_{12}(x_0|\lambda_1,\lambda_2) + \mathrm{sgn}(x_2-x_1)s_{12}(x_0|\lambda_1,\lambda_2) \\ \tilde{R}_{12} &= r_{12}(x_0|\lambda_1,-\lambda_2) + \mathrm{sgn}(x_2-x_1)s_{12}(x_0|\lambda_1,-\lambda_2) \\ T_{12}^{-} &= T_1(x_1,0|-\lambda_1)T_2(x_2,0|-\lambda_2) \\ T_{12}^{+} &= T_1(0,x_1|\lambda_1)T_2(0,x_2|\lambda_2) \end{aligned}$$

The existence of infinitely many conserved charges in involution requires that the following expression has to vanish.

$$\begin{aligned} &\left[r_{12}(0|\lambda_1,\lambda_2),\kappa_1(\lambda_1)\kappa_2(\lambda_2)\right] + \kappa_1(\lambda_1)r_{12}(0|\lambda_1,-\lambda_2)\kappa_2(\lambda_2) - \kappa_2(\lambda_2)r_{12}(0|\lambda_1,-\lambda_2)\kappa_1(\lambda_1) + \\ &+ \frac{1}{2}\Big(G_{12}(-\lambda_1,\lambda_2)\kappa_1(\lambda_1) - \kappa_1(\lambda_1)G_{12}(\lambda_1,\lambda_2) - G_{21}(-\lambda_2,\lambda_1)\kappa_2(\lambda_2) + \kappa_2(\lambda_2)G_{21}(\lambda_2,\lambda_1)\Big) = 0 \end{aligned} \tag{45}$$

This is the *classical boundary Yang-Baxter equation* (cbYBE). If the κ-matrix fulfill this equation then the Poisson-bracket of the double row monodromy matrix is

$$\{\Omega_1(x_1|\lambda_1),\Omega_2(x_2|\lambda_2)\} = t_{12}^{-}\left([R_{12},\omega_{12}] + \omega_1^{(1)}\tilde{R}_{12}\omega_2^{(2)} - \omega_2^{(2)}\tilde{R}_{12}\omega_1^{(2)}\right)t_{12}^{+} \tag{46}$$

This Poisson-bracket satisfies the Jacobi identity (this can be derived by a straightforward but very long calculation). Using the split-point procedure and the symmetric limit we can calculate the "weak" Poisson algebra of the global double row monodromy matrix (4).

$$\begin{aligned}\{\Omega_1(\lambda_1),\Omega_2(\lambda_2)\} = {}& [r_{12}(-\infty|\lambda_1,\lambda_2),\Omega_1(\lambda_1)\Omega_2(\lambda_2)] + \\ & + \Omega_1(\lambda_1)r_{12}(-\infty|\lambda_1,-\lambda_2)\Omega_1(\lambda_1) - \Omega_2(\lambda_2)r_{12}(-\infty|\lambda_1,-\lambda_2)\Omega_1(\lambda_1)\end{aligned} \tag{47}$$

Taking trace we get

$$\{\mathrm{Tr}[\Omega(\lambda_1)],\mathrm{Tr}[\Omega(\lambda_2)]\} = 0,$$

which means we have infinite many conserved charges in involution.

4.2. Poisson bracket in PCMs

Let us specify now the previous findings for the PCMs. The Poisson-algebra of the currents is the following [16, 17]:

$$\begin{aligned}\{J_0(x)\overset{\otimes}{,}J_0(y)\} &= \big[C, J_0\otimes 1\big]\delta(x-y),\\ \{J_0(x)\overset{\otimes}{,}J_1(y)\} &= \big[C, J_1\otimes 1\big]\delta(x-y) - C\delta'(x-y),\\ \{J_1(x)\overset{\otimes}{,}J_1(y)\} &= 0\end{aligned} \tag{48}$$

where $\mathrm{J} = \mathrm{J}^A T_A$ if $\{T_A\}$ is a basis in $\mathfrak{g}$ for which we can define an invariant bilinear form $\langle T_A, T_B\rangle = -\frac{1}{2}\mathrm{Tr}[T_A,T_B] = C_{AB}$ and $C = C^{AB}T_A\otimes T_B$ where $C^{AD}C_{DB} = \delta^A_B$. This form can be used to define a totally anti-symmetric tensor from the structure constant $f_{ABC} = C_{AD}f^D_{BC}$ where $[T_A,T_B] = f^C_{AB}T_C$. For semi-simple Lie-algebras there exists a basis for which $C_{AB} = \delta_{AB}$. In this basis the structure constant is totally anti-symmetric $f_{ABC} = f^A_{BC} = f^{ABC}$ and the Poisson bracket looks like

$$\begin{aligned}\{J_0^A(x),J_0^B(y)\} &= f^{ABC}J_0^C\delta(x-y),\\ \{J_0^A(x),J_1^B(y)\} &= f^{ABC}J_1^C\delta(x-y) - \delta^{AB}\delta'(x-y),\\ \{J_1^A(x),J_1^B(y)\} &= 0\end{aligned}$$

In the following we will need the Poisson-bracket of the group element g and the current $J_0^{L/R}$. For this, we can use the following formula

$$g(x) = g(-\infty)\mathcal{P}\overrightarrow{\exp}\int_{-\infty}^{x} J_1^R(y)\mathrm{d}y = g(-\infty)t(-\infty,x),$$

where we used the definition:

$$t(x,y) = \mathcal{P}\overrightarrow{\exp}\int_x^y J_1^R(z)\mathrm{d}z$$

and (48):

$$\begin{aligned}\{J_0^R(x)\overset{\otimes}{,}g(y)\} &= (1\otimes g(-\infty))\int_{-\infty}^{y}(1\otimes t(-\infty,z))\{J_0^R(x)\overset{\otimes}{,}J_1^R(z)\}(1\otimes t(z,y))\,\mathrm{d}z = \\ &= (1\otimes g(-\infty))\int_{-\infty}^{y}(1\otimes t(-\infty,z))\Big(-\left[C,1\otimes J_1^R(z)\right]\delta(x-z)+ \\ &\qquad + C\partial_z\delta(z-x)\Big)(1\otimes t(z,y))\,\mathrm{d}z = \\ &= (1\otimes g(-\infty))\int_{-\infty}^{y}\partial_z\left((1\otimes t(-\infty,z))\left(C\delta(z-x)\right)(1\otimes t(z,y))\right)\mathrm{d}z = \\ &= (1\otimes g)\,C\delta(x-y).\end{aligned}$$

Therefore

$$\begin{aligned}\left\{J_0^R(x)\overset{\otimes}{,}g(y)\right\} &= (1\otimes g)\,C\delta(x-y) & \left\{J_0^R(x)\overset{\otimes}{,}g^{-1}(y)\right\} &= -\,C\left(1\otimes g^{-1}\right)\delta(x-y) \\ \left\{J_0^L(x)\overset{\otimes}{,}g(y)\right\} &= -C\,(1\otimes g)\,\delta(x-y) & \left\{J_0^L(x)\overset{\otimes}{,}g^{-1}(y)\right\} &= \left(1\otimes g^{-1}\right)C\delta(x-y)\end{aligned} \tag{49}$$

The Poisson brackets of the space-like component of the Lax operator is [17]:

$$\begin{aligned}\{\mathcal{L}_1(x|\lambda_1),\mathcal{L}_2(y|\lambda_2)\} = -&\left[r_{12}(\lambda_1,\lambda_2),\mathcal{L}_1(\lambda_1)+\mathcal{L}_2(\lambda_2)\right]\delta(x-y)+ \\ +&\left[s_{12}(\lambda_1,\lambda_2),\mathcal{L}_1(\lambda_1)-\mathcal{L}_2(\lambda_2)\right]\delta(x-y)- \\ -&2s(\lambda_1,\lambda_2)\delta'(x-y),\end{aligned}$$

where

$$\begin{aligned}r(\lambda_1,\lambda_2) &= -\frac{1}{2}\frac{1}{\lambda_1-\lambda_2}\frac{\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2}{(\lambda_1^2-1)(\lambda_2^2-1)}C, \\ s(\lambda_1,\lambda_2) &= -\frac{1}{2}\frac{\lambda_1+\lambda_2}{(\lambda_1^2-1)(\lambda_2^2-1)}C.\end{aligned}$$

In [17] a different convention is used which can be obtained by the following changes: $\mathcal{L}\to-\mathcal{L}$,$\lambda\to-\lambda$, $\gamma\to-1$. This Poisson-bracket is the same as (42) but in this special case the r- and s-matrices are space independent.

Furthermore, we can find a consistency check for the classical boundary Yang-Baxter equation (cbYBE) in Appendix C where we prove that if $\kappa_R(\lambda)$ satisfies the cbYBE then $\kappa_L(\lambda) = g\kappa_R(1/\lambda)g^{-1}$ also does which has to follow from the inversion property of the reflection matrices. In this derivation we have to use a non-trivial identity of the r-matrix

$$r_{12}(\lambda_1,\lambda_2) = r_{12}(1/\lambda_1,1/\lambda_2)-\frac{1}{2}\left(\frac{\lambda_1}{1-\lambda_1^2}-\frac{\lambda_2}{1-\lambda_2^2}\right)C_{12}. \tag{50}$$

In Appendix C we also show that this identity is a consequence of the inversion property and the s-matrix has a similar property:

$$s_{12}(\lambda_1,\lambda_2) = s_{12}(1/\lambda_1,1/\lambda_2)-\frac{1}{2}\left(\frac{\lambda_1}{1-\lambda_1^2}+\frac{\lambda_2}{1-\lambda_2^2}\right)C_{12}.$$

In the following we solve the classical boundary Yang-Baxter equation for constant κ-matrices.

4.2.1. Constant κ-matrices

Let $\kappa(\lambda) = U$ where $U \in G$ is a constant matrix. The cbYBE can be written as

$$\frac{1}{\lambda_1 - \lambda_2}[C_{12}, U_1U_2] + \frac{1}{\lambda_1 + \lambda_2}(U_1C_{12}U_2 - U_2C_{12}U_1) = 0$$

This equation has to be satisfied for every $\lambda_1, \lambda_2 \in \mathbb{C}$ therefore

$$[C_{12}, U_1U_2] = 0 \qquad \text{and} \qquad U_1C_{12}U_2 = U_2C_{12}U_1$$

The first equation is satisfied trivially because C_{12} is invariant i.e. $C_{12} = U_1U_2C_{12}U_1^{-1}U_2^{-2}$. Let us multiply the second by U_1 from the left and by U_2^{-1} from the right

$$U_1^2C_{12} = U_1U_2C_{12}U_1U_2^{-1} = C_{12}U_1^2$$

Using the explicit form of C_{12} we obtain that

$$C^{AB}\left[X_A, U^2\right] \otimes X_B = 0 \qquad \Rightarrow \qquad \left[X, U^2\right] = 0$$

for all $X \in \mathfrak{g}$. Because we work with the defining representation (which is irreducible), U^2 has to be proportional to the identity. This is the same solution which we obtained from the analysis of the boundary flatness equation. Therefore we can conclude that the consistent solution of the flatness condition and the cbYBE are the same for the constant κ-matrix.

In the end of the Subsection 2.1, we saw that there is an other way to define a double row monodromy matrix:

$$\Omega(\lambda) = T_L(-\lambda)^{-1}UT_R(\lambda).$$

For this definition we should modify the formulas (45) and (47). However, this would require a long calculation. Fortunately, we saw that there is another equivalent formalism of this boundary condition:

$$\Omega(\lambda) = T_R(-1/\lambda)\left(g^{-1}(0)U\right)T_R(\lambda) = T_R(-1/\lambda)\kappa(\lambda)T_R(\lambda).$$

Using this, the generalization of (45) and (47) are the following:

$$\begin{aligned} r_{12}(1/\lambda_1, 1/\lambda_2)\kappa_1(\lambda_1)\kappa_2(\lambda_2) - \kappa_1(\lambda_1)\kappa_2(\lambda_2)r_{12}(\lambda_1, \lambda_2) + \\ + \kappa_1(\lambda_1)r_{12}(\lambda_1, -1/\lambda_2)\kappa_2(\lambda_2) + \kappa_2(\lambda_2)r_{12}(-1/\lambda_1, \lambda_2)\kappa_1(\lambda_1) + \\ + \frac{1}{2}\Big(G_{12}(-1/\lambda_1, \lambda_2)\kappa_1(\lambda_1) - \kappa_1(\lambda_1)G_{12}(\lambda_1, \lambda_2) - \\ - G_{21}(-1/\lambda_2, \lambda_1)\kappa_2(\lambda_2) + \kappa_2(\lambda_2)G_{21}(\lambda_2, \lambda_1)\Big) = 0. \end{aligned} \tag{51}$$

$$\begin{aligned} \{\Omega_1(\lambda_1), \Omega_2(\lambda_2)\} = r_{12}(1/\lambda_1, 1/\lambda_2)\Omega_1(\lambda_1)\Omega_2(\lambda_2) - \Omega_1(\lambda_1)\Omega_2(\lambda_2)r_{12}(\lambda_1, \lambda_2) \\ + \Omega_1(\lambda_1)r_{12}(1/\lambda_1, -\lambda_2)\Omega_1(\lambda_1) - \Omega_2(\lambda_2)r_{12}(\lambda_1, -1/\lambda_2)\Omega_1(\lambda_1) \end{aligned} \tag{52}$$

Let us check that the modified cbYBE (51) is satisfied. At first, let us calculate the Gs.

$$\{\mathcal{L}_1(\lambda_1|x), \kappa_2(\lambda_2)\} = -\frac{\lambda_1}{1-\lambda_1^2}\left\{J_0(x) \overset{\otimes}{,} g^{-1}U\right\} = \frac{\lambda_1}{1-\lambda_1^2}C_{12}(g^{-1}U)_2\delta(x)$$

therefore

$$G_{12}(\lambda_1, \lambda_2) = -\frac{\lambda_1}{1-\lambda_1^2}C_{12}(g^{-1}U)_2 = -\frac{\lambda_1}{1-\lambda_1^2}C_{12}\kappa_2.$$

Using this, the modified cbYBE (51) looks like

$$r_{12}(1/\lambda_1,1/\lambda_2)\kappa_1\kappa_2-\kappa_1\kappa_2 r_{12}(\lambda_1,\lambda_2)+\kappa_1 r_{12}(\lambda_1,-1/\lambda_2)\kappa_2+\kappa_2 r_{12}(-1/\lambda_1,\lambda_2)\kappa_1+ \\ -\frac{1}{2}\left(\frac{\lambda_1}{1-\lambda_1^2}-\frac{\lambda_2}{1-\lambda_2^2}\right)C_{12}\kappa_1\kappa_2+\frac{1}{2}\frac{\lambda_1}{1-\lambda_1^2}\kappa_1 C_{12}\kappa_2-\frac{1}{2}\frac{\lambda_2}{1-\lambda_2^2}\kappa_2 C_{12}\kappa_1=0 \quad (53)$$

Using the identities (50) and

$$r_{12}(\lambda_1,-1/\lambda_2)+\frac{1}{2}\frac{\lambda_1}{1-\lambda_1^2}=-\left(r_{12}(-1/\lambda_1,\lambda_2)-\frac{1}{2}\frac{\lambda_2}{1-\lambda_2^2}\right)=\tilde{r}_{12}(\lambda_1,\lambda_2),$$

the equation (53) can be written as

$$[r_{12}(\lambda_1,\lambda_2),\kappa_1\kappa_2]+\kappa_1\tilde{r}_{12}(\lambda_1,\lambda_2)\kappa_2-\kappa_2\tilde{r}_{12}(\lambda_1,\lambda_2)\kappa_1=0$$

where

$$\tilde{r}_{12}(\lambda_1,\lambda_2)=\frac{1}{2}\frac{\lambda_1\lambda_2-1}{\lambda_1\lambda_2+1}\frac{\lambda_1+\lambda_2}{(\lambda_1^2-1)(\lambda_2^2-1)}C_{12}.$$

Therefore the modified cbYBE can be written as

$$(\lambda_1\lambda_2+1)(2\lambda_1^2\lambda_2^2-\lambda_1^2-\lambda_2^2)\,[C_{12}(\lambda_1,\lambda_2),\kappa_1\kappa_2]-(\lambda_1-\lambda_2)(\lambda_1+\lambda_2)(\lambda_1\lambda_2-1)\,(\kappa_1C_{12}\kappa_2-\kappa_2C_{12}\kappa_1)=0$$

Since the coefficients are linearly independent polynomials we have

$$[C_{12},\kappa_1\kappa_2]=0 \qquad \text{and} \qquad \kappa_1C_{12}\kappa_2=\kappa_2C_{12}\kappa_1.$$

We have already solved these equations and the solution is $\kappa^2=e$ i.e. $g^{-1}Ug^{-1}U=e$ which is the same constraint what we get from the boundary flatness equation.

There is another consequence of the fact that we had to modify the equation (47) to (52). Now, the traces of double row monodromy matrices are not in involution i.e.

$$\{\mathrm{Tr}[\Omega_1(\lambda_1)],\mathrm{Tr}[\Omega_1(\lambda_1)]]\}\neq 0.$$

Nevertheless one can show that there exists a conserved quantity $\mathcal{F}(\lambda)$ for which

$$\{\mathcal{F}(\lambda_1),\mathcal{F}(\lambda_2)]\}=0.$$

The explicit form being

$$\mathcal{F}(\lambda)=\mathrm{Tr}\left[\Omega(1/\lambda)\Omega(\lambda)\right]=\mathrm{Tr}\left[T^{-1}(-\lambda)\kappa T(1/\lambda)T^{-1}(-1/\lambda)\kappa T(\lambda)\right]$$

4.2.2. Spectral parameter dependent κ-matrix

The κ-matrices described in Section 2 fulfill the classical boundary Yang-Baxter equation (45). The derivation can be found in Appendix B.

In [12] the following theorem was proven.

Theorem. *Let $U\in G$ for which Ad_U defines a Lie-algebra involution and $\mathfrak{h}:=\left\{X\in\mathfrak{g}|UXU^{-1}=X\right\}$. If $\kappa(\lambda)$ is a solutions of the following cbYBE*

$$\frac{1}{\lambda_1-\lambda_2}\left[C_{12},\kappa_1(\lambda)\kappa_2(\lambda)\right]+\frac{1}{\lambda_1+\lambda_2}\left(\kappa_1(\lambda)C_{12}\kappa_2(\lambda)-\kappa_2(\lambda)C_{12}\kappa_1(\lambda)\right)=0$$

then $\kappa(\lambda)=U$ for semi-simple $\mathfrak{h}$ or $\kappa(\lambda)=U+\frac{1}{\lambda}X_0U+\mathcal{O}(\lambda^{-2})$ for reductive $\mathfrak{h}$ where X_0 is a central element of $\mathfrak{h}$. The κ-matrix $\kappa(\lambda)$ is unique for a given U (up to normalization) if we fix the norm of X_0.

Previously we showed that these solutions exist therefore we classified the field independent solutions of the cbYBE.

We close this subsection with the Poisson-algebra of the Noether charges of the global symmetries. Let us start with the right charges

$$\tilde{Q}_R^{(0)} = \Pi_{\mathfrak{h}}\left(Q_R^{(0)}\right) = \int_{-\infty}^{0} \Pi_{\mathfrak{h}}\left(J_0^R(x)\right)\mathrm{d}x$$

Using the Poisson-algebra of the current we can obtain that

$$\left\{\tilde{Q}_R^{(0)} \overset{\otimes}{,} \tilde{Q}_R^{(0)}\right\} = \left(\Pi_{\mathfrak{h}} \otimes \Pi_{\mathfrak{h}}\right) \circ \left[C, \tilde{Q}_R^{(0)} \otimes 1\right],$$

We can decompose the basis $\{T_A\}$ into $\{T_a \in \mathfrak{h}\}$ and $\{T_\alpha \in \mathfrak{f}\}$. Using these, the equation above can be written as

$$\left\{\tilde{Q}_R^{(0)a} \overset{\otimes}{,} \tilde{Q}_R^{(0)b}\right\} = f^{abc}\tilde{Q}_R^{(0)c}$$

therefore they form the Lie-algebra $\mathfrak{h}$ as expected. Let us continue with the Noether charges of the left multiplication

$$\tilde{Q}_L = Q_L - \frac{1}{2}(gMg^{-1})\Big|_{x=0} = \int_{-\infty}^{0} J_0^L(x)\mathrm{d}x - \frac{1}{2}(gMg^{-1})\Big|_{x=0}$$

The Poisson-bracket $\left\{\tilde{Q}_L^{(0)} \overset{\otimes}{,} \tilde{Q}_L^{(0)}\right\}$ is not well defined because it contains the following expression

$$\left\{\int_{-\infty}^{0} J_0^L(x)\mathrm{d}x \overset{\otimes}{,} \left(gMg^{-1}\right)\Big|_{x=0}\right\}$$

therefore we have to use the symmetric limit ($\Delta < 0$):

$$\left\{\int_{-\infty}^{0} J_0^L(x)\mathrm{d}x \overset{\otimes}{,} \left(gMg^{-1}\right)\Big|_{x=0}\right\} :=$$
$$\frac{1}{2}\lim_{\Delta\to 0}\left(\left\{\int_{-\infty}^{0} J_0^L(x)\mathrm{d}x \overset{\otimes}{,} \left(gMg^{-1}\right)\Big|_{x=\Delta}\right\} + \left\{\int_{-\infty}^{\Delta} J_0^L(x)\mathrm{d}x \overset{\otimes}{,} \left(gMg^{-1}\right)\Big|_{x=0}\right\}\right) =$$
$$= \frac{1}{2}\lim_{\Delta\to 0}\left\{\int_{-\infty}^{0} J_0^L(x)\mathrm{d}x \overset{\otimes}{,} \left(gMg^{-1}\right)\Big|_{x=\Delta}\right\} = \frac{1}{2}\left[C, \left(gMg^{-1}\right)\Big|_{x=0} \otimes 1\right]$$

Using this, we can obtain the following equation

$$\left\{\tilde{Q}_L^{(0)} \overset{\otimes}{,} \tilde{Q}_L^{(0)}\right\} = \left[C, \tilde{Q}_L^{(0)} \otimes 1\right],$$

which can be written as

$$\left\{\tilde{Q}_L^{(0)A} \overset{\otimes}{,} \tilde{Q}_L^{(0)B}\right\} = f^{ABC}\tilde{Q}_L^{(0)C}.$$

Clearly these charges form the Lie-algebra $\mathfrak{g}$ as expected. This calculation shows the importance of the symmetric limit because if we do not use it properly then we cannot get the proper Poisson-algebra of the Noether charges of the symmetry G_L.

4.3. Poisson bracket in O(N) *sigma models*

The Poisson-algebra of the fields n_i is the following

$$\begin{aligned} \{n_i(x), n_j(y)\} &= 0 \\ \{\dot{n}_i(x), n_j(y)\} &= (\delta_{ij} - n_i n_j)\, \delta(x-y) \\ \{\dot{n}_i(x), \dot{n}_j(y)\} &= (n_i \dot{n}_j - \dot{n}_i n_j)\, \delta(x-y) \end{aligned}$$

From this one can calculate the Poisson-algebra of the currents [18]:

$$\begin{aligned} \left\{\hat{J}_0(x) \overset{\otimes}{,} \hat{J}_0(y)\right\} &= \left[C, \hat{J}_0(x) \otimes 1\right] \delta(x-y) \\ \left\{\hat{J}_0(x) \overset{\otimes}{,} \hat{J}_1(y)\right\} &= \left[C, \hat{J}_1(x) \otimes 1\right] \delta(x-y) - 2\Gamma(y)\delta'(x-y) \\ \left\{\hat{J}_1(x) \overset{\otimes}{,} \hat{J}_1(y)\right\} &= 0 \end{aligned}$$

where

$$\begin{aligned} C &= 2(K-P) \\ \Gamma(x) &= C\,(Z(x)\otimes 1) + (Z(x)\otimes 1)\,C = C\,(1\otimes Z(x)) + (1\otimes Z(x))\,C \end{aligned}$$

and $(P)_{ij,kl} = \delta_{il}\delta_{jk}$, $(K)_{ij,kl} = \delta_{ik}\delta_{jl}$ are the permutation and the trace operators and $(Z)_{ij} = n_i n_j$.

Using this, one can obtain the non-ultralocal Poisson-algebra of the space-like component of the Lax-connection (42) where the r- and s-matrices are

$$\begin{aligned} r(x|\lambda_1,\lambda_2) &= \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1\lambda_2-1)}C + \frac{(\lambda_1-\lambda_2)}{(\lambda_1^2-1)(\lambda_2^2-1)}\frac{(\lambda_1\lambda_2+1)}{(\lambda_1\lambda_2-1)}\Gamma(x) \\ s(x|\lambda_1,\lambda_2) &= \frac{(\lambda_1+\lambda_2)}{(\lambda_1^2-1)(\lambda_2^2-1)}\Gamma(x) \end{aligned}$$

At first, we solve the cbYBE for constant κ-matrices and after that we check the spectral parameter and field dependent κ-matrix.

4.3.1. Constant κ-matrix

For $\kappa(\lambda) = U \in$ O(N), the cbYBE looks like

$$[r_{12}(\lambda_1,\lambda_2), U_1U_2] + U_1 r_{12}(\lambda_1,-\lambda_2)U_2 - U_2 r_{12}(\lambda_1,-\lambda_2)U_1 = 0.$$

After substitution, we obtain the following four equations:

$$\begin{aligned} [C_{12}, U_1U_2] &= 0 \\ U_1C_{12}U_2 &= U_2C_{12}U_1 \\ [\Gamma_{12}, U_1U_2] &= 0 \\ U_1\Gamma_{12}U_2 &= U_2\Gamma_{12}U_1 \end{aligned}$$

The first equation follows from the fact that $U \in$ O(N). From the second equation if follows that $U^2 = \pm 1$ i.e. $U = \pm U^T$. Multiplying the fourth one by U_1 from the left and right, we can see that the third one comes from the fourth. Let us write the third one explicitly.

$$C_{12}Z_2U_1U_2 + Z_2C_{12}U_1U_2 = U_1U_2C_{12}Z_2 + U_1U_2Z_2C_{12}$$

Multiplying by $U_1^T U_2^T$from the left, we obtain the following

$$C_{12}U_2^T Z_2 U_1 + U_2^T Z_2 U_2 C_{12} = C_{12}Z_2 + Z_2 C_{12}.$$

Using the explicit form of C_{12}, we can obtain that

$$(P_{12} - K_{12})(Z_2 - U_2^T Z_2 U_2) = (Z_2 - U_2^T Z_2 U_2)(K_{12} - P_{12}).$$

Let us multiply by P_{12} from the left.

$$\tilde{Z}_2 - K_{12}\tilde{Z}_2 = K_{12}\tilde{Z}_2 - \tilde{Z}_1$$

where $\tilde{Z} = Z - U^T Z U$. Taking the trace on the first site:

$$N\tilde{Z} - \tilde{Z} = \tilde{Z}^T - \mathrm{Tr}\left(\tilde{Z}\right)1$$

Using that $\tilde{Z}_2^T = \tilde{Z}_2$, $\mathrm{Tr}\left(\tilde{Z}\right) = 0$ and $N > 2$, we obtain that

$$\tilde{Z} = 0.$$

Since U can be $U = \pm U^T$, there are two cases.

1. $U = U^T$. Using a global symmetry transformation U can be diagonalized as

   $$U = \begin{pmatrix} 1_{N-k} & 0_{N-k\times k} \\ 0_{k\times k} & -1_k \end{pmatrix}$$

   and Z in the same block diagonal form looks like

   $$Z = \begin{pmatrix} \tilde{\mathbf{n}}\tilde{\mathbf{n}}^T & \tilde{\mathbf{n}}\hat{\mathbf{n}}^T \\ \hat{\mathbf{n}}\tilde{\mathbf{n}}^T & \hat{\mathbf{n}}\hat{\mathbf{n}}^T \end{pmatrix}$$

   therefore $\tilde{Z}$ looks like

   $$\tilde{Z} = \begin{pmatrix} 0 & 2\tilde{\mathbf{n}}\hat{\mathbf{n}}^T \\ 2\hat{\mathbf{n}}\tilde{\mathbf{n}}^T & 0 \end{pmatrix}.$$

   From this explicit form we can see that $\tilde{Z} = 0$ if and only if $\tilde{\mathbf{n}} = 0$ or $\hat{\mathbf{n}} = 0$.

2. $U = -U^T$. Using a global symmetry transformation U can be diagonalized as

   $$U = \begin{pmatrix} 0_{n\times n} & -1_n \\ 1_n & 0_{n\times n} \end{pmatrix}$$

   where $n = N/2$ and $\tilde{Z}$ looks like

   $$\tilde{Z} = \begin{pmatrix} \tilde{\mathbf{n}}\tilde{\mathbf{n}}^T - \hat{\mathbf{n}}\hat{\mathbf{n}}^T & \tilde{\mathbf{n}}\hat{\mathbf{n}}^T + \hat{\mathbf{n}}\tilde{\mathbf{n}}^T \\ \tilde{\mathbf{n}}\hat{\mathbf{n}}^T + \hat{\mathbf{n}}\tilde{\mathbf{n}}^T & \hat{\mathbf{n}}\hat{\mathbf{n}}^T - \tilde{\mathbf{n}}\tilde{\mathbf{n}}^T \end{pmatrix}.$$

   Multiplying the off-diagonal terms by $\hat{\mathbf{n}}$ form the right, we obtain

   $$\tilde{\mathbf{n}}\left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}\right) = -\hat{\mathbf{n}}\left(\tilde{\mathbf{n}}^T\hat{\mathbf{n}}\right)$$

   and multiplying this by $\hat{\mathbf{n}}^T$ form the left, we obtain

   $$\left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}\right)\left(\tilde{\mathbf{n}}^T\hat{\mathbf{n}}\right) = 0.$$

   At first, let us assume that $\hat{\mathbf{n}} \neq 0$ therefore $\tilde{\mathbf{n}}^T\hat{\mathbf{n}} = 0$. Substituting this to the previous equation, we obtain that $\tilde{\mathbf{n}} = 0$. Using this in the diagonal term, we obtain that $\hat{\mathbf{n}}\hat{\mathbf{n}}^T = 0$ which contradicts to $\hat{\mathbf{n}} \neq 0$. Therefore $\hat{\mathbf{n}} = 0$. From $\mathbf{n}^T\mathbf{n} = 1$ and from the diagonal, we obtain that $\tilde{\mathbf{n}}^T\tilde{\mathbf{n}} = 1$ and $\tilde{\mathbf{n}}\tilde{\mathbf{n}}^T$ which is a contradiction. Therefore anti-symmetric U cannot be a solution of the cbYBE.

We can conclude that we have obtained the same constant κ-matrices from the cbYBE as we got from the boundary flatness condition.

4.3.2. Spectral parameter and field dependent κ-matrix

If we want to check that the new κ-matrix (32) satisfy the classical boundary Yang-Baxter equation (45) then we have to compute $G_{12}(\lambda_1,\lambda_2)$. For this, we will need the following Poisson brackets:

$$\begin{aligned}\left\{(J_0(x))_{ij}\,, n_k(y)\right\} &= 2\left(\delta_{jk}n_i-\delta_{ik}n_j\right)\delta(x-y)\\ \left\{J_0(x)\overset{\otimes}{,}h(y)\right\} &= [1\otimes h, C]\,\delta(x-y)\\ \left\{J_0(x)\overset{\otimes}{,}(hMh)(y)\right\} &= \left((1\otimes h)\,[C,1\otimes M]\,(1\otimes h)+[1\otimes hMh,C]\right)\delta(x-y)\end{aligned}$$

From this

$$G(\lambda,\mu)\delta(x)=-\left\{\mathcal{L}(x|\lambda)\overset{\otimes}{,}\kappa(\mu)\right\}=\frac{\lambda}{1-\lambda^2}\left(1\otimes\left(\frac{1}{\mu}+M\right)\right)\left\{J_0(x)\overset{\otimes}{,}(hMh)(0)\right\}$$

therefore

$$G_{12}(\lambda,\mu)=\frac{1}{\mu}\frac{\lambda}{1-\lambda^2}\left(1+\mu M_2\right)\left(h_2\left[C_{12},M_2\right]h_2+\left[(hMh)_2\,,C_{12}\right]\right)$$

We checked the cbYBE for $O(4)$ and $O(6)$ sigma models with explicit calculations using Wolfram Mathematica. For this, we parameterized the sphere with stereo-graphic coordinates:

$$\begin{aligned}n_a &= \frac{2\xi_a}{1+\xi^2} \qquad \text{for } a=1,\ldots,N-1\\ n_N &= \frac{1-\xi^2}{1+\xi^2}\end{aligned}$$

where

$$\xi^2=\sum_{a=1}^{N-1}\xi_a\xi_a.$$

Using this parameterization we can calculate explicitly the matrices $r(\lambda,\mu), G(\lambda,\mu), \kappa(\lambda)$ and we can substitute these into the cbYBE. Using Mathematica we have checked that the cbYBE is satisfied for $O(4)$ and $O(6)$ sigma models.

5. Conclusion

In this paper new double row monodromy matrices have been determined for the principal chiral models. The corresponding integrable boundary conditions break one chiral half of the symmetry to $G_L\times H_R$ where H_R was not arbitrary but G/H_R had to be a symmetric space and the Lie algebra of H_R was not semi-simple. We determined the boundary conditions which correspond to these monodromy matrices. Both the monodromy matrices and boundary conditions contain free parameters.

We used these results for finding new monodromy matrices for the $O(N)$ sigma models. At first, the $SO(4)\cong SU(2)_L\times SU(2)_R$ isometry was used to determine the $SU(2)_L\times U(1)_R$ symmetric κ matrices for $SO(4)$ sigma models. These new spectral parameter dependent κ matrices were then generalized for $O(2n)$ sigma models. They corresponds to $U(n)$ symmetric boundary conditions.

We also showed that these κ-matrices satisfy the classical boundary Yang-Baxter equation therefore there exist infinitely many conserved charges in involution i.e. the boundary conditions proportional to these κs are classically integrable.

There exist quantum $O(4)$ sigma models which have reflection matrix with two free parameters and the residual symmetry is $O(2)\times O(2)$ [10]. Therefore one interesting direction to pursue would be to find the classical field theoretical description of these quantum theories i.e. κ matrices and boundary conditions which have two independent parameters and residual symmetry $O(2)\times O(2)$. In the language of the SU(2) PCM, this means boundary conditions which independently break left and right symmetries. These results could be then generalized to general PCMs.

As a last remark, it would be interesting to check that the quantum version of the κ matrices determined in the paper are really the known reflection matrices. This could be done in the large-N limit. Recently, the large-N limit was studied for the $\mathbb{C}P^N$ sigma models on finite intervals e.g. [19][20]. These methods may also be applicable to the models studied in this paper.

Acknowledgment

I thank Zoltán Bajnok and László Palla for the useful discussions and for reading the manuscript. The work was supported by the NKFIH 116505 Grant.

Appendix A. Non-local conserved charges

If we expand the monodromy matrix around $\lambda=\lambda_0$ we get infinitely many conserved charges which are generally non-local. In this section we will deal with the expansions around $\lambda=\infty$ and $\lambda=0$ and we will give the first two terms of these series.

Appendix A.1. Expansion around $\lambda=\infty$

We will start with the expansion of the one row monodromy matrix

$$T_R(\lambda)=\mathcal{P}\overleftarrow{\exp}\left(\int_{-\infty}^{0}-\mathcal{L}^R(\lambda)\mathrm{d}x\right)=\exp\left(\sum_{r=0}^{\infty}\left(-\frac{1}{\lambda}\right)^{r+1}Q_R^{(r)}\right)= \\ 1-\frac{1}{\lambda}Q_R^{(0)}+\frac{1}{\lambda^2}\left(Q_R^{(1)}+\frac{1}{2}Q_R^{(0)2}\right)+\dots \quad \text{(A.1)}$$

Since

$$\mathcal{L}^R(\lambda)=\frac{1}{1-\lambda^2}J_1^R-\frac{\lambda}{1-\lambda^2}J_0^R=\frac{1}{\lambda}J_0^R-\frac{1}{\lambda^2}J_1^R+\dots$$

the expansion leads to

$$T_R(\lambda)=1-\frac{1}{\lambda}\int_{-\infty}^{0}J_0^R(x)\mathrm{d}x+\frac{1}{\lambda^2}\left(\int_{-\infty}^{0}J_1^R(x)\mathrm{d}x+\int_{-\infty}^{0}\int_{-\infty}^{x_1}J_0^R(x_1)J_0^R(x_2)\mathrm{d}x_1\mathrm{d}x_2\right)+\dots$$

which gives the first two charges

$$Q_R^{(0)}=\int_{-\infty}^{0}J_0^R(x)\mathrm{d}x,$$
$$Q_R^{(1)}=\int_{-\infty}^{0}J_1^R(x)\mathrm{d}x+\frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{x_1}\left[J_0^R(x_1),J_0^R(x_2)\right]\mathrm{d}x_1\mathrm{d}x_2.$$

In order to calculate the expansion of the monodromy matrix we will also need the following series:

$$T_R^{-1}(-\lambda)=\exp\left(-\sum_{r_0}^{\infty}\left(\frac{1}{\lambda}\right)^{r+1}Q_R^{(r)}\right)=1-\frac{1}{\lambda}Q_R^{(0)}+\frac{1}{\lambda^2}\left(-Q_R^{(1)}+\frac{1}{2}Q_R^{(0)2}\right)+\dots, \quad \text{(A.2)}$$

In Subsection 2.2, the classification of the new κ-matrices are showed. For $\mathfrak{g} = \mathfrak{so}(2n), \mathfrak{sp}(n)$, $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{su}(n)$ the form of these are the same:

$$\kappa(\lambda) = \frac{1}{\sqrt{1 + a^2\lambda^2}} \left(1 + \lambda M\right),$$

where M generates the $\mathfrak{u}(1)$ and $M^2 = -a^2 1$ and $M = -M^T$. The generalization for other κ matrices follows straightforwardly. The expansion of the κ is the following:

$$\kappa(\lambda) = U + \frac{1}{a\lambda} - \frac{1}{2}\frac{1}{(a\lambda)^2}U + \dots \tag{A.3}$$

where $M = aU$. The conserved charges come from the expansion of the double row monodromy matrix.

$$\Omega(\lambda) = T_R^{-1}(-\lambda)\kappa(\lambda)T_R(\lambda) = U \cdot \exp\left(2\sum_{r_0}^{\infty}\left(-\frac{1}{\lambda}\right)^{r+1}\tilde{Q}_R^{(r)}\right) = \\ U - \frac{2}{\lambda}U\tilde{Q}_R^{(0)} + \frac{2}{\lambda^2}\left(U\tilde{Q}_R^{(1)} + U\tilde{Q}_R^{(0)2}\right) + \dots,$$

where $\{\tilde{Q}_R^{(r)}\}$ is the infinite set of conserved charges. In the above equation multiplication with U is necessary for the proper normalization because

$$\lim_{\lambda\to\infty} \Omega(\lambda) = U.$$

Using (A.1), (A.2) and (A.3):

$$\Omega(\lambda) = \left[1 - \frac{1}{\lambda}Q_R^{(0)} + \frac{1}{\lambda^2}\left(-Q_R^{(1)} + \frac{1}{2}Q_R^{(0)2}\right) + \dots\right] \cdot \\ \left[U + \frac{1}{a\lambda} - \frac{1}{2}\frac{1}{(a\lambda)^2}U + \dots\right] \cdot \left[1 - \frac{1}{\lambda}Q_R^{(0)} + \frac{1}{\lambda^2}\left(Q_R^{(1)} + \frac{1}{2}Q_R^{(0)2}\right) + \dots,\right] = \\ = U - \frac{1}{\lambda}U\left(Q_R^{(0)} + U^TQ_R^{(0)}U - \frac{1}{a}U^T\right) + \\ + \frac{1}{\lambda^2}U\left(Q_R^{(1)} - U^TQ_R^{(1)}U + \frac{1}{2}(Q_R^{(0)2} + U^TQ_R^{(0)2}U) + U^TQ_R^{(0)}UQ_R^{(0)} - \frac{2}{a}U^TQ_R^{(0)} - \frac{1}{2a^2}\right)$$

From this the first two conserved charges are the following:

$$\tilde{Q}_R^{(0)} = \Pi_{\mathfrak{h}}\left(Q_R^{(0)}\right) + \frac{1}{2a}U, \\ \tilde{Q}_R^{(1)} = \Pi_{\mathfrak{f}}\left(Q_R^{(1)}\right) + \frac{1}{2}\left[\Pi_{\mathfrak{h}}\left(Q_R^{(0)}\right), \Pi_{\mathfrak{f}}\left(Q_R^{(0)}\right)\right] + \frac{1}{2a}\left[U, Q_R^{(0)}\right] = \\ = \Pi_{\mathfrak{f}}\left(Q_R^{(1)}\right) + \frac{1}{2}\left[\Pi_{\mathfrak{h}}\left(Q_R^{(0)}\right) + \frac{1}{a}U, \Pi_{\mathfrak{f}}\left(Q_R^{(0)}\right)\right].$$

The first charge is equivalent to the charge (21) (up to a constant). $\tilde{Q}_R^{(1)}$ is very similar to the charge for the $g \in H$ restricted boundary condition but there is an extra term: $\left[U, Q_R^{(0)}\right]$ [21]. These charges also satisfy the relations: $\tilde{Q}_R^{(0)} \in \mathfrak{h}$ and $\tilde{Q}_R^{(1)} \in \mathfrak{f}$.

For a crosscheck we can take the time derivative of these charges and we will see that they all vanish.

Appendix A.2. Expansion around $\lambda = 0$

For the expansion around $\lambda = 0$, we can use the inversion property of the double row monodromy matrix (7):

$$\Omega_R(\lambda) = g^{-1}(-\infty)\left(T_L^{-1}(-1/\lambda)\left(g\kappa(\lambda)g^{-1}\right)\Big|_{x=0} T_L(1/\lambda)\right)g(-\infty) = \\ g^{-1}(-\infty)\exp\left(2\sum_{r_0}^{\infty}(-\lambda)^{r+1}\tilde{Q}_L^{(r)}\right)g(-\infty) \quad \text{(A.4)}$$

We can do the same calculation as before:

$$T_L^{-1}(-1/\lambda)\left(g\kappa(\lambda)g^{-1}\right)\Big|_{x=0} T_L(1/\lambda) = \\ = 1 - 2\lambda\left(Q_L^{(0)} - \frac{1}{2}gMg^{-1}\Big|_{x=0}\right) + 2\lambda^2\left(Q_L^{(0)2} - \frac{1}{2}\left[Q_L^{(0)}, gMg^{-1}\Big|_{x=0}\right]_+ - \frac{1}{4}a^2\right) + \dots$$

therefore the conserved charges are the following:

$$\tilde{Q}_L^{(0)} = Q_L^{(0)} - \frac{1}{2}gMg^{-1}\Big|_{x=0}, \\ \tilde{Q}_L^{(1)} = 0.$$

We can see that the first conserved charge is equal to the Noether charge of the left multiplication symmetry (20): $\tilde{Q}_L^{(0)} = \tilde{Q}_L$. The second set of charges vanish. This is similar to the case of the free boundary condition ($\mathfrak{g} = \mathfrak{h}$) in [21].

Appendix B. Classical boundary Yang-Baxter equation for the new κs

In this section, we prove that matrices described in Subsection 2.2 fulfill the cbYBE (45).
We start with the $N = 0$ case. For this, the cbYBE (45) looks like:

$$\frac{1}{\lambda_1 - \lambda_2}\left[C_{12}, (1 + \lambda_1 M_1)(1 + \lambda_2 M_2)\right] + \\ + \frac{1}{\lambda_1 + \lambda_2}\Big((1 + \lambda_1 M_1)C_{12}(1 + \lambda_2 M_2) - (1 + \lambda_2 M_2)C_{12}(1 + \lambda_1 M_1)\Big) \stackrel{?}{=} 0. \quad \text{(B.1)}$$

This equation is satisfied thanks to the following identities:

$$[C_{12}, M_1] = -[C_{12}, M_2], \quad \text{(B.2)}$$
$$[C_{12}, M_1 M_2] = 0 \quad \text{(B.3)}$$
$$M_1 C_{12} M_2 = M_2 C_{12} M_1. \quad \text{(B.4)}$$

Equation (B.2) follows from $M \in \mathfrak{g}$.

$$[C_{12}, M_1] = [T_A, M^B T_B] \otimes T^A = f_{AB}^C M^B T_C \otimes T^A = -M^B T_C \otimes [T^C, T_B] = -[C_{12}, M_2].$$

Equation (B.3) and (B.4) follows from $M^2 \sim 1$ which means $MT_aM^{-1} = T_a$ and $MT_\alpha M^{-1} = -T_\alpha$ where $T_a \in \mathfrak{h}$ and $T_\alpha \in \mathfrak{f}$.

$$[C_{12}, M_1 M_2] = T_A M \otimes T^A M - MT_A \otimes MT^A = \\ = T_A M \otimes T^A M - (T_a M \otimes T^a M + (-T_\alpha M) \otimes (-T^\alpha M)) = 0.$$

The derivation of (B.4) is similar.

In the following we will continue with the $N \neq 0$ case. The cbYBE looks like:

$$\frac{1}{\lambda_1-\lambda_2}[C_{12},(1+\lambda_1 M_1+\lambda_1^2 N_1)(1+\lambda_2 M_2+\lambda_2^2 N_2)]+ \\ +\frac{1}{\lambda_1+\lambda_2}\Big((1+\lambda_1 M_1+\lambda_1^2 N_1)C_{12}(1+\lambda_2 M_2+\lambda_2^2 N_2)- \\ -(1+\lambda_2 M_2+\lambda_2^2 N_2)C_{12}(1+\lambda_1 M_1+\lambda_1^2 N_1)\Big) \stackrel{?}{=} 0. \quad \text{(B.5)}$$

The matrices M and N satisfy the following identities:

$$[C_{12},M_1] = -[C_{12},M_2], \quad \text{(B.6)}$$
$$[C_{12},M_1M_2] = -[C_{12},N_1]-[C_{12},N_2], \quad \text{(B.7)}$$
$$M_1C_{12}M_2 - M_2C_{12}M_1 = -[C_{12},N_1]+[C_{12},N_2], \quad \text{(B.8)}$$
$$[C_{12},M_1N_2] = -[C_{12},N_1M_2], \quad \text{(B.9)}$$
$$[C_{12},M_1N_2] = M_1C_{12}N_2 - N_2C_{12}M_1. \quad \text{(B.10)}$$
$$[C_{12},N_1N_2] = 0, \quad \text{(B.11)}$$
$$N_1C_{12}N_2 = N_2C_{12}N_1. \quad \text{(B.12)}$$

Using these, the equation (B.5) is satisfied. The identity (B.6) is satisfied because $M \in \mathfrak{g}$. Let us see (B.7) and (B.8).

$$[C_{12},M_1M_2] - M_1C_{12}M_2 + M_2C_{12}M_1 = [[C_{12},M_1],M_2]_+ = \\ = -[[C_{12},M_2],M_2]_+ = -[C_{12},M_2^2] = -2[C_{12},N_2]$$

$$[C_{12},M_1M_2] + M_1C_{12}M_2 - M_2C_{12}M_1 = [[C_{12},M_2],M_1]_+ = \\ = -[[C_{12},M_1],M_1]_+ = -[C_{12},M_1^2] = -2[C_{12},N_1]$$

where we used (16). By adding and subtracting the equations above we can get (B.7) and (B.8). Equations (B.11) and (B.12) follows from $N^2 \sim 1$ similarly to (B.3) and (B.4).

Now we only have to prove the equation (B.9) and (B.10). This can be done by using the explicit forms of M and N which were shown in Subsection 2.2. When $N \neq 0$, $(MN - c1) \in \mathfrak{g}$ where c is a number. For ($\mathfrak{g} = \mathfrak{su}(n)$, $\mathfrak{h} = \mathfrak{u}(1)\oplus\mathfrak{su}(m)\oplus\mathfrak{su}(n-m)$) $c = i\frac{4km}{\lambda n}$ and for ($\mathfrak{g} = \mathfrak{so}(n)$, $\mathfrak{h} = \mathfrak{so}(2)\oplus\mathfrak{so}(n-2)$) $c = 0$.

$$[C_{12},M_1N_2] = (C_{12}M_1N_1^{-1} - M_1N_2C_{12}N_1^{-1}N_2^{-1})N_1N_2 = \\ = (C_{12}M_1N_1^{-1} - M_1N_1^{-1}C_{12})N_1N_2 = [C_{12},M_1N_1^{-1}]N_1N_2 = -[C_{12},M_2N_2^{-1}]N_1N_2 = \\ = -(C_{12}N_1M_2 - M_2N_2^{-1}C_{12}N_1N_2) = -(C_{12}N_1M_2 - M_2N_1C_{12}) = -[C_{12},N_1M_2], \quad \text{(B.13)}$$

where we used $MN - c1 \in \mathfrak{g}$ and (B.12). Finally let us see the derivation of (B.10):

$$[C_{12},M_1N_2] = [X_a,M]\otimes X^aN + [X_\alpha,M]_+\otimes X^\alpha N = \\ = [M,X_a]\otimes X^aN + [X_\alpha,M]_+\otimes X^\alpha N = M_1C_{12}N_2 - N_2C_{12}M_1$$

where we used that $[M,X_a] = 0$ for all $X_a \in \mathfrak{h}$.

Appendix C. Consistency check of the cbYBE

In the PCM we can work with right or left currents. For a general boundary condition the κ-matrices can be different using right or left currents. Let κ^L and κ^R be these two κ-matrices. We saw that the double row monodromy matrices and the κ-matrices have the inversion property:

$$\Omega_L(\lambda) = g(-\infty)\Omega_R(1/\lambda)g^{-1}(-\infty),$$
$$\kappa^L(\lambda) = g(0)\kappa^R(1/\lambda)g^{-1}(0).$$

The classical boundary Yang-Baxter equation (cbYBE) for $\kappa^R(\lambda)$ and $\kappa^L(\lambda)$ are the following:

$$\begin{aligned}&\left[r_{12}(\lambda_1,\lambda_2),\kappa_1^{L/R}(\lambda_1)\kappa_2^{L/R}(\lambda_2)\right]+\\ &\quad+\kappa_1^{L/R}(\lambda_1)r_{12}(\lambda_1,-\lambda_2)\kappa_2^{L/R}(\lambda_2)-\kappa_2^{L/R}(\lambda_2)r_{12}(\lambda_1,-\lambda_2)\kappa_1^{L/R}(\lambda_1)+\\ &\quad+\frac{1}{2}\Big(G_{12}^{L/R}(-\lambda_1,\lambda_2)\kappa_1^{L/R}(\lambda_1)-\kappa_1^{L/R}(\lambda_1)G_{12}^{L/R}(\lambda_1,\lambda_2)-\\ &\quad-G_{21}^{L/R}(-\lambda_2,\lambda_1)\kappa_2^{L/R}(\lambda_2)+\kappa_2^{L/R}(\lambda_2)G_{21}^{L/R}(\lambda_2,\lambda_1)\Big)=0\end{aligned}\tag{C.1}$$

where we assumed that

$$\left\{\mathcal{L}_1^{L/R}(x|\lambda_1),\kappa_2^{L/R}(\lambda_2)\right\}=-G_{12}^{L/R}(\lambda_1,\lambda_2)\delta(x).$$

This assumption implicitly contains that $\kappa^{L/R}$ does not depend on the time derivative of the fields.

In the following we prove that if $\kappa^L(\lambda)$ satisfies the cbYBE then $\kappa^R(\lambda)=g^{-1}\kappa^L(1/\lambda)g$ also does. At first let us assume that $\kappa^L(\lambda)$ satisfies the cbYBE (C.1). Let us see G_{12}^L:

$$\begin{aligned}G_{12}^L(\lambda_1,\lambda_2)\delta(x)&=-\left\{\mathcal{L}^L(x|\lambda_1)\overset{\otimes}{,}\kappa^L(\lambda_2)\right\}=-\left\{g(x)\mathcal{L}^R(x|1/\lambda_1)g^{-1}(x)+J_1^L(x)\overset{\otimes}{,}g\kappa^R(1/\lambda_2)g^{-1}\right\}=\\ &g\otimes g\left(G^R(1/\lambda_1,1/\lambda_2)-\frac{\lambda_1}{1-\lambda_1^2}\left[C,1\otimes\kappa^R(1/\lambda_2)\right]\right)g^{-1}\otimes g^{-1}\delta(x)\end{aligned}\tag{C.2}$$

where we used that κ^R does not depend on the time derivative of the fields and equation (49):

$$\left\{J_0^R(x)\overset{\otimes}{,}g(y)\right\}=(1\otimes g)\,C\delta(x-y)\qquad\left\{J_0^R(x)\overset{\otimes}{,}g^{-1}(y)\right\}=-C\left(1\otimes g^{-1}\right)\delta(x-y)$$

Since the r-matrices are proportional to C then $g_1^{-1}g_2^{-1}r_{12}g_1g_2=r_{12}$. Using this and (C.2) in (C.1) we can obtain the following:

$$\begin{aligned}&\left[r_{12}(\lambda_1,\lambda_2),\kappa_1^R(1/\lambda_1)\kappa_2^R(1/\lambda_2)\right]+\\ &\quad+\kappa_1^R(1/\lambda_1)r_{12}(\lambda_1,-\lambda_2)\kappa_2^R(1/\lambda_2)-\kappa_2^R(1/\lambda_2)r_{12}(\lambda_1,-\lambda_2)\kappa_1^R(1/\lambda_1)+\\ &\quad+\frac{1}{2}\Big(G_{12}^R(-1/\lambda_1,1/\lambda_2)\kappa_1^R(1/\lambda_1)-\kappa_1^R(1/\lambda_1)G_{12}^R(1/\lambda_1,1/\lambda_2)-\\ &\quad-G_{21}^R(-1/\lambda_2,1/\lambda_1)\kappa_2^R(1/\lambda_2)+\kappa_2^R(1/\lambda_2)G_{21}^R(1/\lambda_2,1/\lambda_1)\Big)+\\ &+\frac{1}{2}\frac{\lambda_1}{1-\lambda_1^2}\left[\left[C_{12},\kappa_2^R(1/\lambda_2)\right],\kappa_1^R(1/\lambda_1)\right]_+-\frac{1}{2}\frac{\lambda_2}{1-\lambda_2^2}\left[\left[C_{12},\kappa_1^R(1/\lambda_1)\right],\kappa_2^R(1/\lambda_2)\right]_+=0\end{aligned}\tag{C.3}$$

Let us see the last two terms

$$\begin{aligned}&\frac{1}{2}\frac{\lambda_1}{1-\lambda_1^2}\left[\left[C_{12},\kappa_2^R(1/\lambda_2)\right],\kappa_1^R(1/\lambda_1)\right]_+-\frac{1}{2}\frac{\lambda_2}{1-\lambda_2^2}\left[\left[C_{12},\kappa_1^R(1/\lambda_1)\right],\kappa_2^R(1/\lambda_2)\right]_+=\\ &\quad\frac{1}{2}\left(\frac{\lambda_1}{1-\lambda_1^2}-\frac{\lambda_2}{1-\lambda_2^2}\right)\left[C_{12},\kappa_1^R(1/\lambda_1)\kappa_2^R(1/\lambda_2)\right]+\\ &\quad+\frac{1}{2}\left(\frac{\lambda_1}{1-\lambda_1^2}+\frac{\lambda_2}{1-\lambda_2^2}\right)\left(\kappa_1^R(1/\lambda_1)C_{12}\kappa_2^R(1/\lambda_2)-\kappa_2^R(1/\lambda_2)C_{12}\kappa_1^R(1/\lambda_1)\right)\end{aligned}\tag{C.4}$$

The second line of (C.4) can be merged with the first line of (C.3) and the third line of (C.4) with the second line of (C.3):

$$r_{12}(\lambda_1,\lambda_2)+\frac{1}{2}\left(\frac{\lambda_1}{1-\lambda_1^2}-\frac{\lambda_2}{1-\lambda_2^2}\right)C_{12}=r_{12}(1/\lambda_1,1/\lambda_2) \tag{C.5}$$

$$r_{12}(\lambda_1,-\lambda_2)+\frac{1}{2}\left(\frac{\lambda_1}{1-\lambda_1^2}+\frac{\lambda_2}{1-\lambda_2^2}\right)C_{12}=r_{12}(1/\lambda_1,-1/\lambda_2) \tag{C.6}$$

Using this in (C.3) we get

$$\begin{aligned}&\big[r_{12}(1/\lambda_1,1/\lambda_2),\kappa_1^R(1/\lambda_1)\kappa_2^R(1/\lambda_2)\big]+\\ &\quad+\kappa_1^R(1/\lambda_1)r_{12}(1/\lambda_1,-1/\lambda_2)\kappa_2^R(1/\lambda_2)-\kappa_2^R(1/\lambda_2)r_{12}(1/\lambda_1,-1/\lambda_2)\kappa_1^R(1/\lambda_1)+\\ &\qquad+\frac{1}{2}\Big(G_{12}^R(-1/\lambda_1,1/\lambda_2)\kappa_1^R(1/\lambda_1)-\kappa_1^R(1/\lambda_1)G_{12}^R(1/\lambda_1,1/\lambda_2)-\\ &\qquad\qquad-G_{21}^R(-1/\lambda_2,1/\lambda_1)\kappa_2^R(1/\lambda_2)+\kappa_2^R(1/\lambda_2)G_{21}^R(1/\lambda_1,1/\lambda_2)\Big)=0\end{aligned}$$

After changing $1/\lambda_1$ and $1/\lambda_2$ to λ_1 and λ_2, the last equation is the cbYBE for κ^R. Therefore we proved that if $\kappa^L(\lambda)$ satisfies the cbYBE then $\kappa^R(\lambda)=g^{-1}\kappa^L(1/\lambda)g$ also does.

Finally, we prove that equation (C.5) follows from the Poisson algebras of $\mathcal{L}^L$ and $\mathcal{L}^R$ (42):

$$\begin{aligned}\{\mathcal{L}_1^{L/R}(x|\lambda_1),\mathcal{L}_2^{L/R}(y|\lambda_2)\}=&-\big[r_{12}(\lambda_1,\lambda_2),\mathcal{L}_1^{L/R}(\lambda_1)+\mathcal{L}_2^{L/R}(\lambda_2)\big]\delta(x-y)+\\ &+\big[s_{12}(\lambda_1,\lambda_2),\mathcal{L}_1^{L/R}(\lambda_1)-\mathcal{L}_2^{L/R}(\lambda_2)\big]\delta(x-y)-\\ &-2s_{12}(\lambda_1,\lambda_2)\delta'(x-y)\end{aligned} \tag{C.7}$$

and the inversion property

$$\mathcal{L}^L(\lambda)=g\mathcal{L}^R(1/\lambda)g^{-1}+U^L$$

where we used the notation: $U^{L/R}=J_1^{L/R}$. Let us start with the left connections.

$$\{\mathcal{L}_1^L(x|\lambda_1),\mathcal{L}_2^L(y|\lambda_2)\}=\left\{\left(g\mathcal{L}^R(1/\lambda)g^{-1}+U^L\right)_1(x),\left(g\mathcal{L}^R(1/\lambda)g^{-1}+U^L\right)_2(y)\right\}.$$

The r.h.s. is equal to the sum of the following three terms

$$\begin{aligned}&\Big\{\left(g\mathcal{L}^R(1/\lambda_1)g^{-1}\right)_1(x),\left(g\mathcal{L}^R(1/\lambda_2)g^{-1}\right)_2(y)\Big\}=\\ &\qquad=g_1(x)g_2(y)\left\{\mathcal{L}_1^R(x|1/\lambda_1),\mathcal{L}_2^R(y|1/\lambda_2)\right\}g_1^{-1}(x)g_2^{-1}(y)+\\ &\qquad+\frac{\lambda_1}{1-\lambda_1^2}g_1g_2\left[C_{12},\mathcal{L}_2^R(1/\lambda_2)\right]g_1^{-1}g_2^{-1}\delta(x-y)-\\ &\qquad-\frac{\lambda_2}{1-\lambda_2^2}g_1g_2\left[C_{12},\mathcal{L}_1^R(1/\lambda_2)\right]g_1^{-1}g_2^{-1}\delta(x-y),\end{aligned} \tag{C.8}$$

$$\left\{\left(g\mathcal{L}^R(1/\lambda_1)g^{-1}\right)_1(x),U_2^L(y)\right\}=-\frac{\lambda_1}{1-\lambda_1^2}\left(\left[C_{12},U_1^L\right]\delta(x-y)-C\delta'(x-y)\right), \tag{C.9}$$

$$\left\{U_1^L(x),\left(g\mathcal{L}^R(1/\lambda_2)g^{-1}\right)_2(y)\right\}=-\frac{\lambda_2}{1-\lambda_2^2}\left(\left[C_{12},U_1^L\right]\delta(x-y)-C\delta'(x-y)\right). \tag{C.10}$$

Let us calculate the first term in the r.h.s of (C.8).

$$\begin{aligned}&g_1(x)g_2(y)\left\{\mathcal{L}_1^R(x|1/\lambda_1),\mathcal{L}_2^R(y|1/\lambda_2)\right\}g_1^{-1}(x)g_2^{-1}(y)=\\ &\qquad-\big[r_{12}(1/\lambda_1,1/\lambda_2),\mathcal{L}_1^L(\lambda_1)-U_1^L+\mathcal{L}_2^L(\lambda_2)-U_2^L\big]\delta(x-y)\\ &\qquad+\big[s_{12}(1/\lambda_1,1/\lambda_2),\mathcal{L}_1^L(\lambda_1)-U_1^L-\mathcal{L}_2^L(\lambda_2)+U_2^L\big]\delta(x-y)\\ &\qquad-2g_1(x)g_2(y)s_{12}(1/\lambda_1,1/\lambda_2)g_1^{-1}(x)g_2^{-1}(y)\delta'(x-y)\end{aligned}$$

where we used that $g_1 g_2 C_{12} g_1^{-1} g_2^{-1} = C_{12}$. The third term can be written as

$$
\begin{aligned}
&-2g_1(x)g_2(y)s_{12}(1/\lambda_1, 1/\lambda_2)g_1^{-1}(x)g_2^{-1}(y)\delta'(x-y) = \\
&\qquad = -2s_{12}(1/\lambda_1, 1/\lambda_2)\delta'(x-y) + 2\left[s_{12}(1/\lambda_1, 1/\lambda_2), U_1^L\right]\delta(x-y)
\end{aligned}
$$

where we used that $(f(x) - f(y))\,\delta'(x-y) = -f'(x)\delta(x-y)$. Using these the formula above can be written as

$$
\begin{aligned}
g_1(x)g_2(y)\left\{\mathcal{L}_1^R(x|1/\lambda_1), \mathcal{L}_2^R(y|1/\lambda_2)\right\}g_1^{-1}(x)g_2^{-1}(y) = \\
&-\left[r_{12}(1/\lambda_1, 1/\lambda_2), \mathcal{L}_1^L(\lambda_1) + \mathcal{L}_2^L(\lambda_2)\right]\delta(x-y) + \\
&+\left[s_{12}(1/\lambda_1, 1/\lambda_2), \mathcal{L}_1^L(\lambda_1) - \mathcal{L}_2^L(\lambda_2)\right]\delta(x-y) - \\
&-2s_{12}(1/\lambda_1, 1/\lambda_2)\delta'(x-y).
\end{aligned}
$$

Summing the equations (C.8), (C.9) and (C.10), we can obtain

$$
\begin{aligned}
\{\mathcal{L}_1^L(x|\lambda_1), \mathcal{L}_2^L(y|\lambda_2)\} = &-\left[\left(r_{12}(1/\lambda_1, 1/\lambda_2) - \frac{a_1 - a_2}{2}C_{12}\right), \mathcal{L}_1^L(\lambda_1) + \mathcal{L}_2^L(\lambda_2)\right]\delta(x-y) + \\
&+\left[\left(s_{12}(1/\lambda_1, 1/\lambda_2) - \frac{a_1 + a_2}{2}C_{12}\right), \mathcal{L}_1^L(\lambda_1) - \mathcal{L}_2^L(\lambda_2)\right]\delta(x-y) \\
&-2\left(s_{12}(1/\lambda_1, 1/\lambda_2) - \frac{a_1 + a_2}{2}C_{12}\right)\delta'(x-y)
\end{aligned}
$$

where we used the following notations

$$
a_1 = \frac{\lambda_1}{1 - \lambda_1^2}, \qquad a_2 = \frac{\lambda_2}{1 - \lambda_2^2}.
$$

From the original Poisson bracket (C.7), we can see that the r- and s-matrices satisfies the following identities

$$
\begin{aligned}
r_{12}(\lambda_1, \lambda_2) &= r_{12}(1/\lambda_1, 1/\lambda_2) - \frac{1}{2}\left(\frac{\lambda_1}{1 - \lambda_1^2} - \frac{\lambda_2}{1 - \lambda_2^2}\right)C_{12}, \\
s_{12}(\lambda_1, \lambda_2) &= s_{12}(1/\lambda_1, 1/\lambda_2) - \frac{1}{2}\left(\frac{\lambda_1}{1 - \lambda_1^2} + \frac{\lambda_2}{1 - \lambda_2^2}\right)C_{12}.
\end{aligned}
$$

References

[1] A. M. Polyakov, Hidden Symmetry of the Two-Dimensional Chiral Fields, Phys. Lett. 72B (1977) 224–226. doi:10.1016/0370-2693(77)90707-9.

[2] Y. Y. Goldschmidt, E. Witten, Conservation Laws in Some Two-dimensional Models, Phys. Lett. 91B (1980) 392–396.

[3] A. B. Zamolodchikov, A. B. Zamolodchikov, Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models, Annals Phys. 120 (1979) 253–291. doi:10.1016/0003-4916(79)90391-9.

[4] E. Corrigan, Z.-M. Sheng, Classical integrability of the O(N) nonlinear sigma model on a half line, Int. J. Mod. Phys. A12 (1997) 2825–2834. arXiv:hep-th/9612150, doi:10.1142/S0217751X97001560.

[5] N. J. MacKay, The SO(N) principal chiral field on a half line, J. Phys. A32 (1999) L189–L193. arXiv:hep-th/9903137, doi:10.1088/0305-4470/32/17/101.

[6] N. J. MacKay, B. J. Short, Boundary scattering, symmetric spaces and the principal chiral model on the half line, Commun. Math. Phys. 233 (2003) 313–354. arXiv:hep-th/0104212, doi:10.1007/s00220-002-0735-y.

[7] M. Moriconi, Integrable boundary conditions and reflection matrices for the O(N) nonlinear sigma model, Nucl. Phys. B619 (2001) 396–414. arXiv:hep-th/0108039, doi:10.1016/S0550-3213(01)00527-2.

[8] N. Mann, S. E. Vazquez, Classical Open String Integrability, JHEP 04 (2007) 065. arXiv:hep-th/0612038, doi:10.1088/1126-6708/2007/04/065.

[9] I. Aniceto, Z. Bajnok, T. Gombor, M. Kim, L. Palla, On integrable boundaries in the 2 dimensional $O(N)$ σ-models, J. Phys. A50 (36) (2017) 364002. arXiv:1706.05221, doi:10.1088/1751-8121/aa8205.

[10] D. Arnaudon, J. Avan, N. Crampe, A. Doikou, L. Frappat, E. Ragoucy, Classification of reflection matrices related to (super) Yangians and application to open spin chain models, Nucl. Phys. B668 (2003) 469–505. arXiv:math/0304150, doi:10.1016/S0550-3213(03)00503-0.

[11] T. Gombor, Nonstandard Bethe Ansatz equations for open O(N) spin chains, Nucl. Phys. B935 (2018) 310–343. arXiv:1712.03753, doi:10.1016/j.nuclphysb.2018.08.014.

[12] T. Gombor, On the classification of rational K-matrices arXiv:1904.03044.

[13] J. Avan, V. Caudrelier, N. Crampé, From Hamiltonian to zero curvature formulation for classical integrable boundary conditions, J. Phys. A51 (30) (2018) 30LT01. arXiv:1802.07593, doi:10.1088/1751-8121/aac976.

[14] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Pure and Applied Mathematics, Elsevier Science, 1979.

[15] J. M. Maillet, New Integrable Canonical Structures in Two-dimensional Models, Nucl. Phys. B269 (1986) 54–76. doi:10.1016/0550-3213(86)90365-2.

[16] L. D. Faddeev, L. A. Takhtajan, Hamiltonian methods in the theory of solitons, 1987.

[17] J. M. Maillet, Hamiltonian Structures for Integrable Classical Theories From Graded Kac-moody Algebras, Phys. Lett. 167B (1986) 401–405. doi:10.1016/0370-2693(86)91289-X.

[18] J. M. Maillet, Kac-moody Algebra and Extended Yang-Baxter Relations in the $O(N)$ Non-linear σ Model, Phys. Lett. 162B (1985) 137–142. doi:10.1016/0370-2693(85)91075-5.

[19] S. Bolognesi, K. Konishi, K. Ohashi, Large-N $\mathbb{CP}^{N-1}$ sigma model on a finite interval, JHEP 10 (2016) 073. arXiv:1604.05630, doi:10.1007/JHEP10(2016)073.

[20] S. Bolognesi, S. B. Gudnason, K. Konishi, K. Ohashi, Large-N $\mathbb{CP}^{N-1}$ sigma model on a finite interval: general Dirichlet boundary conditions, JHEP 06 (2018) 064. arXiv:1802.08543, doi:10.1007/JHEP06(2018)064.

[21] N. J. MacKay, Introduction to Yangian symmetry in integrable field theory, Int. J. Mod. Phys. A20 (2005) 7189–7218. arXiv:hep-th/0409183, doi:10.1142/S0217751X05022317.