

RANGE-KERNEL CHARACTERIZATIONS OF OPERATORS WHICH ARE ADJOINT OF EACH OTHER

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Dedicated to Professor Franciszek Hugon Szafraniec on the occasion of his 80th birthday

ABSTRACT. We provide necessary and sufficient conditions for a pair S, T of Hilbert space operators in order that they satisfy $S^* = T$ and $T^* = S$. As a main result we establish an improvement of von Neumann's classical theorem on the positive self-adjointness of S^*S for two variables. We also give some new characterizations of self-adjointness and skew-adjointness of operators, not requiring their symmetry or skew-symmetry, respectively.

1. INTRODUCTION

The adjoint of an unbounded linear operator was first introduced by John von Neumann in [6] as a profound ingredient for developing a rigorous mathematical framework for quantum mechanics. By definition, the adjoint of a densely defined linear transformation S , acting between two Hilbert spaces, is an operator T with the largest possible domain such that

$$(1.1) \quad (Sx | y) = (x | Ty)$$

holds for every x from the domain of S . The adjoint operator, denoted by S^* , is therefore “maximal” in the sense that it extends every operator T that has property (1.1). On the other hand, every restriction T of S^* fulfills that adjoint relation. Thus, in order to decide whether an operator T is identical with the adjoint of S it seems reasonable to restrict ourselves to investigating those operators T that have property (1.1). This issue was explored in detail in [16] by means of the operator matrix

$$\begin{bmatrix} I & -T \\ S & I \end{bmatrix},$$

cf. also [8, 11, 13, 14].

In the present paper we continue to examine the conditions under which an operator T is equal to the adjoint S^* of S . Nevertheless, as opposed to the situation treated in the cited papers, we do not assume that S and T are adjoint to each

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other in the sense of (1.1). Observe that condition (1.1) is equivalent to identity

$$(1.2) \quad S^* \cap T = T.$$

So, still under condition (1.1), T is equal to the adjoint of S if and only if $S^* \cap T = S^*$. In the present paper we are going to guarantee equality $S^* = T$ by imposing new conditions, weaker than (1.1), by means of the kernel and range spaces. Roughly speaking, we only require that the intersection of the graphs of S^* and T be, in a sense, “large enough”. We also establish a criterion in terms of the norm of the resolvent of the operator matrix

$$M_{S,T} = \begin{bmatrix} 0 & -T \\ S & 0 \end{bmatrix}.$$

As an application we gain some characterizations of self-adjoint, skew-adjoint and unitary operators, thereby generalizing some analogous results by T. Nieminen [5] (cf. also [9]).

2. PRELIMINARIES

Throughout this paper \mathcal{H} and \mathcal{K} will denote real or complex Hilbert spaces. By an operator S between \mathcal{H} and \mathcal{K} we mean a linear map $S : \mathcal{H} \rightarrow \mathcal{K}$ whose domain $\text{dom } S$ is a linear subspace of \mathcal{H} . We stress that, unless otherwise indicated, linear operators are not assumed to be densely defined. However, the adjoint of such an operator can only be interpreted as a “multivalued operator”, that is, a linear relation. Therefore we are going to collect here some basic notions and facts on linear relations.

A linear relation between two Hilbert spaces \mathcal{H} and \mathcal{K} is nothing but a linear subspace S of the Cartesian product $\mathcal{H} \times \mathcal{K}$, respectively, a closed linear relation is just a closed subspace of $\mathcal{H} \times \mathcal{K}$. To a linear relation S we associate the following subspaces

$$\begin{aligned} \text{dom } S &= \{h \in \mathcal{H} : (h, k) \in S\} & \text{ran } S &= \{k \in \mathcal{K} : (h, k) \in S\} \\ \text{ker } S &= \{h \in \mathcal{H} : (h, 0) \in S\} & \text{mul } S &= \{k \in \mathcal{K} : (0, k) \in S\}, \end{aligned}$$

which are referred to as the domain, range, kernel and multivalued part of S , respectively. Every linear operator when identified with its graph is a linear relation with trivial multivalued part. Conversely, a linear relation whose multivalued part consists only of the vector 0 is (the graph of) an operator.

A notable advantage of linear relations, compared to operators, lies in the fact that one might define the adjoint without any further assumption on the domain. Namely, the adjoint of a linear relation S will be again a linear relation S^* between \mathcal{K} and \mathcal{H} , given by

$$S^* := V(S)^\perp.$$

Here, $V : \mathcal{H} \times \mathcal{K} \rightarrow \mathcal{K} \times \mathcal{H}$ stands for the ‘flip’ operator $V(h, k) := (k, -h)$. It is seen immediately that S^* is automatically a closed linear relation and satisfies the useful identity

$$\overline{S} = S^{**} (= (S^*)^*).$$

On the other hand, a closed linear relation S entails the following orthogonal decomposition of the product Hilbert space $\mathcal{K} \times \mathcal{H}$:

$$S^* \oplus V(S) = \mathcal{K} \times \mathcal{H}.$$

Note that another equivalent definition of S^* is obtained in terms of the inner product as follows:

$$S^* = \{(k', h') \in \mathcal{K} \times \mathcal{H} : (k | k') = (h | h') \text{ for all } (h, k) \in S\}.$$

In other words, $(k', h') \in S^*$ holds if and only if

$$(k | k') = (h | h') \quad \forall (h, k) \in S.$$

In particular, if S is a densely defined operator then the relation S^* coincides with the usual adjoint operator of S . Recall also the dual identities

$$\ker S^* = (\text{ran } S)^\perp, \quad \text{mul } S^* = (\text{dom } S)^\perp,$$

where the second equality tells us that the adjoint of a densely defined linear relation is always a (single valued) operator. For further information on linear relation we refer the reader to [1, 2, 4, 10].

3. OPERATORS WHICH ARE ADJOINT OF EACH OTHER

R. Arens [1] characterized the equality $S = T$ of two linear relations in terms of their kernel and range (see Corollary 3.3). Below we provide a similar characterization of $S \subset T$. Observe that the intersection $S \cap T$ of the linear relations S and T is again a linear relation, but this is not true for their union $S \cup T$ as it is not a linear subspace in general. The linear span of $S \cup T$ will be denoted by $S \vee T$, which in turn is a linear relation.

Proposition 3.1. *Let S and T be linear relations between two vector spaces. Then the following three statements are equivalent:*

- (i) $S \subset T$,
- (ii) $\ker S \subset \ker T$ and $\text{ran } S \subset \text{ran}(S \cap T)$,
- (iii) $\text{ran } S \subset \text{ran } T$ and $\ker(S \vee T) \subset \ker T$.

Proof. It is clear that (i) implies both (ii) and (iii). Suppose now (ii) and let $(h, k) \in S$ then there exists u with $(u, k) \in T \cap S$. Consequently, $(h - u, 0) \in S$, i.e., $h - u \in \ker S \subset \ker T$. Hence

$$(h, k) = (h - u, 0) + (u, k) \in T + T \subset T,$$

which yields $S \subset T$, so (ii) implies (i). Finally, assume (iii) and take $(h, k) \in S$. Then $(u, k) \in T$ for some u and hence $(h - u, 0) \in S \vee T$, i.e., $h - u \in \ker T$. Consequently,

$$(h, k) = (h - u, 0) + (u, k) \in T,$$

which yields $S \subset T$. □

Corollary 3.2. *Let S and T be two linear relations between the vector spaces. The following three statements are equivalent:*

- (i) $S = T$,
- (ii) $\ker S = \ker T$ and $\text{ran } S + \text{ran } T \subseteq \text{ran}(S \cap T)$,
- (iii) $\text{ran } S = \text{ran } T$ and $\ker(S \vee T) \subseteq \ker(S \cap T)$.

Corollary 3.3. *Let S and T be linear relations between two vector spaces such that $S \subset T$. Then the following assertions are equivalent:*

- (i) $S = T$,
- (ii) $\ker S = \ker T$ and $\text{ran } S = \text{ran } T$.

In [17, Theorem 2.9] M. H. Stone established a simple yet effective sufficient condition for an operator to be self-adjoint: a densely defined symmetric operator S is necessarily self-adjoint provided it is surjective. In that case, it is invertible with bounded and self-adjoint inverse due to the Hellinger–Toeplitz theorem. Here, density of the domain can be dropped from the hypotheses: a surjective symmetric operator is automatically densely defined (see also [16, Corollary 6.7] and [15, Lemma 2.1]).

Below we establish a generalization of Stone’s result for a pair of operators.

Proposition 3.4. *Let \mathcal{H}, \mathcal{K} be real or complex Hilbert spaces and let $S : \mathcal{H} \rightarrow \mathcal{K}$ and $T : \mathcal{K} \rightarrow \mathcal{H}$ be (not necessarily densely defined or closed) linear operators such that*

$$\text{ran}(S \cap T^*) = \mathcal{K} \quad \text{and} \quad \text{ran}(T \cap S^*) = \mathcal{H}.$$

Then S and T are both densely defined operators such that $S^ = T$ and $T^* = S$.*

Proof. For brevity, introduce the following notations

$$S_0 := S \cap T^*, \quad T_0 := T \cap S^*.$$

Observe that S_0 and T_0 are adjoint to each other in the sense that

$$(S_0 x | y) = (x | T_0 y), \quad x \in \text{dom } S_0, y \in \text{dom } T_0.$$

We claim that S_0 and T_0 are densely defined: let $z \in (\text{dom } S_0)^\perp$, then by surjectivity, $z = T_0 v$ for some $v \in \text{dom } T_0$. Hence

$$0 = (x | z) = (x | T_0 v) = (S_0 x | v), \quad x \in \text{dom } S_0,$$

which implies $v = 0$ and also $z = 0$. The same argument shows that T_0 is densely defined too. We see now that S and T^* are densely defined operators such that

$$\ker S \subseteq (\text{ran } S^*)^\perp = \{0\}, \quad \ker T^* = (\text{ran } T)^\perp = \{0\},$$

and $\text{ran}(S \cap T^*) = \mathcal{K}$. Corollary 3.2 applied to S and T^* implies that $S = T^*$. The same argument yields equality $S^* = T$. \square

Corollary 3.5. *Let $S : \mathcal{H} \rightarrow \mathcal{K}$ and $T : \mathcal{K} \rightarrow \mathcal{H}$ be (not necessarily densely defined) surjective operators such that*

$$(Sx | y) = (x | Ty), \quad x \in \text{dom } S, y \in \text{dom } T.$$

Then S and T are both densely defined operators such that $S^ = T$ and $T^* = S$.*

From Proposition 3.4 we gain a sufficient condition of self-adjointness without the assumptions of being symmetric or densely defined:

Corollary 3.6. *Let \mathcal{H} be a Hilbert space and let $S : \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator such that $\text{ran}(S \cap S^*) = \mathcal{H}$. Then S is densely defined and self-adjoint.*

Proof. Apply Proposition 3.4 with $T := S$. \square

Clearly, if S is a symmetric operator then $S \cap S^* = S$. Hence we retrieve [17, Theorem 2.9] by M. H. Stone as an immediate consequence (cf. also [16, Corollary 6.7]):

Corollary 3.7. *Every surjective symmetric operator is densely defined and self-adjoint.*

In the next result we give a necessary and sufficient condition for an operator S to be identical with the adjoint of a given operator T .

Theorem 3.8. *Let \mathcal{H}, \mathcal{K} be real or complex Hilbert spaces and let $S : \mathcal{H} \rightarrow \mathcal{K}$ and $T : \mathcal{K} \rightarrow \mathcal{H}$ be (not necessarily densely defined or closed) linear operators. The following two statements are equivalent:*

- (i) T is densely defined and $S = T^*$,
- (ii) (a) $(\text{ran } T)^\perp = \ker S$,
(b) $\text{ran } S + \text{ran } T^* \subset \text{ran}(S \cap T^*)$.

Proof. It is obvious that (i) implies (ii). Assume now (ii) and for sake of brevity introduce the operator

$$S_0 := S \cap T^*.$$

We start by establishing that T is densely defined. Let $g \in (\text{dom } T^*)^\perp$, then $(0, g) \in T^*$, i.e., $g \in \text{ran } T^*$. By (ii) (a),

$$T^*g = S_0h = Sh$$

for some $h \in \text{dom } S_0$. Then it follows that $(h, Sh) \in T^*$ and therefore

$$(Tk | h) = (k | Sh) = (k | g) = 0, \quad k \in \text{dom } T,$$

whence we infer that $h \in (\text{ran } T)^\perp$. Again by (ii) (a) we have $h \in \ker S$ and thus $g = Sh = 0$. This proves that T is densely defined and as a consequence, T^* is an operator. Next we prove that

$$(3.1) \quad T^* \subset S.$$

To see this consider $g \in \text{dom } T^*$. By (ii) (b),

$$T^*g = S_0h = Sh = T^*h$$

for some $h \in \text{dom } S_0$. Then it follows that $T^*(g - h) = 0$, i.e.,

$$g - h \in \ker T^* = (\text{ran } T)^\perp = \ker S.$$

Consequently, $g = (g - h) + h \in \text{dom } S$ and $Sg = Sh = T^*g$, which proves (3.1). It only remains to show that the converse inclusion

$$(3.2) \quad S \subset T^*$$

holds also true. For let $g \in \text{dom } S$ and choose $h \in \text{dom } S_0$ such that

$$Sg = S_0h = T^*h = Sh.$$

Then $g - h \in \ker S = (\text{ran } T)^\perp = \ker T^*$ whence we get $g = (g - h) + h \in \text{dom } T^*$ and $T^*g = T^*h = Sg$, which proves (3.2). \square

A celebrated theorem by J. von Neumann [7] states that S^*S and SS^* are positive and selfadjoint operators provided that S is a densely defined and closed operator between \mathcal{H} and \mathcal{K} . In that case, $I + S^*S$ and $I + SS^*$ are both surjective. In [12] it has been proved that the converse is also true: If $I + S^*S$ and $I + SS^*$ are both surjective operators then S is necessarily closed (cf. also [3]). Below, as the main result of the paper, we establish an improvement of Neumann's theorem:

Theorem 3.9. *Let \mathcal{H}, \mathcal{K} be real or complex Hilbert spaces and let $S : \mathcal{H} \rightarrow \mathcal{K}$ and $T : \mathcal{K} \rightarrow \mathcal{H}$ be linear operators and introduce the operators $S_0 := S \cap T^*$ and $T_0 := T \cap S^*$. The following statements are equivalent:*

- (i) S, T are both densely defined and they are adjoint of each other: $S^* = T$ and $T^* = S$,
- (ii) $\text{ran}(I + T_0 S_0) = \mathcal{H}$ and $\text{ran}(I + S_0 T_0) = \mathcal{K}$.

Proof. It is clear that (i) implies (ii). To prove the converse implication observe first that

$$(S_0 u | v) = (u | T_0 v), \quad u \in \text{dom } S_0, v \in \text{dom } T_0.$$

We start by showing that S_0 is densely defined. Take a vector $g \in (\text{dom } S_0)^\perp$, then there is $u \in \text{dom } S_0$ such that $g = u + T_0 S_0 u$. Consequently,

$$0 = (u | g) = (u | u) + (T_0 S_0 u | u) = \|u\|^2 + \|S_0 u\|,$$

whence $u = 0$, and therefore also $g = 0$. It is proved analogously that T_0 is densely defined too, and therefore the adjoint relations S_0^* and T_0^* are operators such that $S_0 \subset T_0^*$ and $T_0 \subset S_0^*$.

We are going to prove now that S_0 and T_0 are adjoint of each other, i.e.,

$$(3.3) \quad S_0^* = T_0, \quad T_0^* = S_0.$$

Consider a vector $g \in \text{dom } T_0^*$ and take $u \in \text{dom } S_0$ and $v \in \text{dom } T_0$ such that

$$g = u + T_0 S_0 u \quad \text{and} \quad T_0^* g = v + S_0 T_0 v.$$

Since u is in $\text{dom } T_0^*$ we infer that $T_0 S_0 u \in \text{dom } T_0^*$ and hence

$$T_0^* g = T_0^* u + T_0^* T_0 S_0 u.$$

It follows then that

$$0 = v - T_0^* u + S_0 T_0 v - T_0^* T_0 S_0 u = (I + T_0^* T_0)(v - S_0 u)$$

which yields $v = S_0 u \in \text{dom } T_0$. As a consequence we obtain that

$$g = u + T_0 S_0 u = v + T_0 v,$$

and therefore that $g \in \text{dom } S_0$. This proves the first equality of (3.3). The second one is proved in a similar way.

Now we can complete the proof easily: since $S_0 \subset T^*$ and $T_0 \subseteq T$ it follows that

$$T_0^* = S_0 \subset T^* \subset T_0^*,$$

whence $T^* = T_0^* = S_0$, and therefore $T^* \subset S$. On the other hand, $T_0 \subset S^*$ implies

$$S \subset S^{**} \subset T_0^* = T^*,$$

whence we conclude that $S = T^*$. It can be proved in a similar way that $T = S^*$. \square

As an immediate consequence we conclude the following result:

Corollary 3.10. *Let \mathcal{H} and \mathcal{K} be real or complex Hilbert spaces and let $S : \mathcal{H} \rightarrow \mathcal{K}$ be a densely defined operator. The following statements are equivalent:*

- (i) S is closed,
- (ii) $S^* S$ and $S S^*$ are self-adjoint operators,
- (iii) $\text{ran}(I + S^* S) = \mathcal{H}$ and $\text{ran}(I + S S^*) = \mathcal{K}$.

Proof. Apply Theorem 3.9 with $T := S^*$. \square

In the ensuing theorem we provide a range-kernel characterization of operators T that are identical with the adjoint S^* of a densely defined symmetric operator S . We stress that no condition on the closedness of the operator or density of the domain is imposed. On the contrary: we get those properties from the other conditions.

Theorem 3.11. *Let \mathcal{H} be a real or complex Hilbert space and let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a (not necessarily densely defined or closed) linear operator and let $T_0 := T \cap T^*$. The following two statements are equivalent:*

- (i) *there exists a densely defined symmetric operator S such that $S^* = T$,*
- (ii) (a) $\ker T = (\text{ran } T^*)^\perp$,
 (b) $\text{ran } T_0 = \text{ran } T^{**} = \text{ran } T^*$.

In particular, if any of the equivalent conditions (i), (ii) is satisfied then T is a densely defined and closed operator such that $T^ \subset T$.*

Proof. It is straightforward that (i) implies (ii) so we only prove the converse. We start by proving that T is densely defined. Take $g \in (\text{dom } T)^\perp$, then $g \in \text{mul } T^* \subseteq \text{ran } T^*$. By (ii) (b), there exists $h \in \text{dom } T_0$ such that $g = T_0 h = Th$. Consequently, $(h, g) \in T^*$ and for every $f \in \text{dom } T$,

$$(Tf | h) = (f | g) = 0,$$

which yields $h \in (\text{ran } T)^\perp$. Observe that (ii) (a) and (b) together imply that

$$(3.4) \quad (\text{ran } T)^\perp = \ker T,$$

whence we infer that $h \in \ker T$ and therefore that $g = Th = 0$. This means that T^* is a (single valued) operator. Or next claim is to show that

$$(3.5) \quad T^* \subset T.$$

To this end, let $g \in \text{dom } T^*$, then $T^* g = T_0 h$ for some $h_0 \in \text{dom } T_0$. From inclusion $T_0 \subset T^*$ we conclude that $g - h \in \ker T^* = (\text{ran } T)^\perp$, thus $g = (g - h) + h \in \text{dom } T$ and

$$Tg = Th = T_0 h = T^* g,$$

which proves (3.5). Next we show that T^* is densely defined too, i.e., T is closable. To this end consider a vector $g \in (\text{dom } T^*)^\perp = \text{mul } T^{**}$. Since $\text{mul } T^{**} \subseteq \text{ran } T^{**}$, we can find a vector $h \in \text{dom } T_0$ such that $g = T_0 h$. For every $k \in \text{dom } T^*$,

$$(h | T^* k) = (Th | k) = (g | k) = 0,$$

thus $h \in (\text{ran } T^*)^\perp$. By (ii) (a) we infer that $h \in \ker T$ and hence $g = Th = 0$, hence $(\text{dom } T^*)^\perp = \{0\}$, as it is claimed. Finally we show that T is closed. Take $g \in \text{dom } T^{**}$, then $T^{**} g = Th$ for some $h \in \text{dom } T$, according to assumption (ii) (b). Hence $g - h \in \ker T^{**} = (\text{ran } T^*)^\perp$, thus $g - h \in \ker T$ because of (ii) (a). Consequently, $g = (g - h) + h \in \text{dom } T$ which proves identity $T = T^{**}$. Summing up, $S := T^*$ is a densely defined operator such that $S \subset T = S^*$. In other words, T is identical with the adjoint S^* of the symmetric operator S . \square

4. CHARACTERIZATIONS INVOLVING RESOLVENT NORM ESTIMATIONS

Let \mathcal{H} and \mathcal{K} be real or complex Hilbert spaces. For given two linear operators $S : \mathcal{H} \rightarrow \mathcal{K}$ and $T : \mathcal{K} \rightarrow \mathcal{H}$, let us consider the operator matrix

$$M_{S,T} := \begin{bmatrix} 0 & -T \\ S & 0 \end{bmatrix},$$

acting on the product Hilbert space $\mathcal{H} \times \mathcal{K}$. More precisely, $M_{S,T}$ is an operator acting on its domain $\text{dom } M_{S,T}(\lambda) := \text{dom } S \times \text{dom } T$ by

$$M_{S,T}(h, k) := (-Tk, Sh) \quad (h \in \text{dom } S, k \in \text{dom } T).$$

Assume that a real or complex number $\lambda \in \mathbb{K}$ belongs to the resolvent set $\rho(M_{S,T})$, which means that

$$M_{S,T} - \lambda = \begin{bmatrix} -\lambda & -T \\ S & -\lambda \end{bmatrix}$$

has an everywhere defined bounded inverse. In that case, for brevity's sake, we introduce the notation

$$R_{S,T}(\lambda) := (M_{S,T} - \lambda)^{-1}$$

for the corresponding resolvent operator.

In the present section we are going to establish some criteria, by means of norms of the resolvent operator $R_{S,T}(\lambda)$, under which the operators S and T are adjoint of each other. Our approach is motivated by the classical paper of T. Nieminen [5] (cf. also [9]). We emphasize that our framework is more general than that of [5] for many ways: we do not assume that the operators under consideration are densely defined or closed, and also the underlying space may be real or complex.

Theorem 4.1. *Let $S : \mathcal{H} \rightarrow \mathcal{K}$ and $T : \mathcal{K} \rightarrow \mathcal{H}$ be linear operators between the real or complex Hilbert spaces \mathcal{H} and \mathcal{K} . The following assertions are equivalent:*

- (i) *S and T are densely defined such that $S^* = T$ and $T^* = S$,*
- (ii) *every non-zero real number t belongs to the resolvent set of $M_{S,T}$ and*

$$(4.1) \quad \|R_{S,T}(t)\| \leq \frac{1}{|t|}, \quad \forall t \in \mathbb{R}, t \neq 0.$$

Proof. Let us start by proving that (i) implies (ii). Assume therefore that S is densely defined and closed and that $T = S^*$. Consider a non-zero real number t and a pair of vectors $h \in \text{dom } S$ and $k \in \text{dom } S^*$, then we have

$$\begin{aligned} \left\| \begin{bmatrix} t & -S^* \\ S & t \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} \right\|^2 &= \|th - S^*k\|^2 + \|Sh + tk\|^2 \\ &= t^2[\|h\|^2 + \|k\|^2] + \|Sh\|^2 + \|S^*k\|^2 \\ &\geq t^2 \left\| \begin{bmatrix} h \\ k \end{bmatrix} \right\|^2 \end{aligned}$$

which implies that $M_{S,T} + t$ is bounded from below and the norm of its inverse $R_{S,T}(-t)$ satisfies (4.1). However it is not yet clear that $R_{S,T}(-t)$ is everywhere defined. But observe that

$$\begin{bmatrix} t & -S^* \\ S & t \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = t \left(\begin{bmatrix} h \\ \frac{1}{t}Sh \end{bmatrix} + \begin{bmatrix} -\frac{1}{t}S^*k \\ k \end{bmatrix} \right)$$

whence we get

$$(4.2) \quad \text{ran}(M_{S,T} + t) = \frac{1}{t}S \oplus W(\frac{1}{t}S^*),$$

where W is the ‘flip’ operator $W(k, h) := (-h, k)$. Since S is densely defined and closed according to our hypotheses, the subspace on the right hand side of (4.2) is equal to $\mathcal{H} \times \mathcal{K}$. This proves statement (ii).

For the converse direction, observe that (4.1) implies

$$\left\| \begin{bmatrix} t & -T \\ S & t \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} \right\|^2 \geq t^2 \left\| \begin{bmatrix} h \\ k \end{bmatrix} \right\|^2, \quad h \in \text{dom } S, k \in \text{dom } T.$$

Hence from (4.1) we conclude that

$$0 \geq \|Sx\|^2 + \|Ty\|^2 + t\{(Sx|y) - (x|Ty) + (y|Sx) - (Ty|x)\}$$

$$= \|Sx\|^2 + \|Ty\|^2 + 2t \operatorname{Re}\{(Sx | y) - (x | Ty)\}$$

for every $t \in \mathbb{R}$. Consequently,

$$\operatorname{Re}(Sx | y) = \operatorname{Re}(x | Ty), \quad x \in \operatorname{dom} S, y \in \operatorname{dom} T.$$

In the real Hilbert space case it is straightforward that S and T are adjoint to each other. In the complex case, replace x by ix to get

$$\operatorname{Im}(Sx | y) = \operatorname{Im}(x | Ty), \quad x \in \operatorname{dom} S, y \in \operatorname{dom} T.$$

So, in both real and complex cases, we obtained that $S \subset T^*$ and $T \subset S^*$. With notation of Theorem 3.9 this means that $S_0 = S$ and $T_0 = T$. Since we have

$$\begin{bmatrix} I & -T \\ S & I \end{bmatrix} = M_{S,T} + 1, \quad \begin{bmatrix} I & T \\ -S & I \end{bmatrix} = -[M_{S,T} - 1],$$

we conclude that

$$\begin{bmatrix} I + TS & 0 \\ 0 & I + ST \end{bmatrix} = -M_{S,T}(1)M_{S,T}(-1)$$

is a surjective operator onto $\mathcal{H} \times \mathcal{K}$, which entails $\operatorname{ran}(I + TS) = \mathcal{H}$ and $\operatorname{ran}(I + ST) = \mathcal{K}$. An immediate application of Theorem 3.9 completes the proof. \square

As an immediate consequence of Theorem 4.1 we can establish the following characterizations of self-adjoint, skew-adjoint and unitary operators.

Corollary 4.2. *Let \mathcal{H} be a real or complex Hilbert space. For a linear operator $S : \mathcal{H} \rightarrow \mathcal{H}$ the following assertions are equivalent:*

- (i) S is densely defined and self-adjoint,
- (ii) Every non-zero real number t is in the resolvent set of $M_{S,-S}$ and

$$(4.3) \quad \|R_{S,-S}(t)\| \leq \frac{1}{|t|}, \quad \forall t \in \mathbb{R}, t \neq 0.$$

Proof. Apply Theorem 4.1 with $T := S$ to conclude the the desired equivalence. \square

Corollary 4.3. *Let \mathcal{H} be a real or complex Hilbert space. For a linear operator $S : \mathcal{H} \rightarrow \mathcal{H}$ the following assertions are equivalent:*

- (i) S is densely defined and skew-adjoint,
- (ii) every non-zero real number t is in the resolvent set of $M_{S,S}$ and

$$(4.4) \quad \|R_{S,S}(t)\| \leq \frac{1}{|t|}, \quad \forall t \in \mathbb{R}, t \neq 0.$$

Proof. Apply Theorem 4.1 with $T := -S$. \square

Corollary 4.4. *Let \mathcal{H} and \mathcal{K} be a real or complex Hilbert spaces. For a linear operator $U : \mathcal{H} \rightarrow \mathcal{K}$ the following assertions are equivalent:*

- (i) U is a unitary operator,
- (ii) $\ker U = \{0\}$, every non-zero real number t is in the resolvent set of $M_{U,U^{-1}}$ and

$$(4.5) \quad \|R_{U,U^{-1}}(t)\| \leq \frac{1}{|t|}, \quad \forall t \in \mathbb{R}, t \neq 0.$$

Proof. An application of Theorem 4.1 with $S := U$ and $T := U^{-1}$ shows that U is densely defined and closed such that $U^* = U^{-1}$. Hence, $\operatorname{ran} U^* \subseteq \operatorname{dom} U$. Since we have $\operatorname{ran} U^* + \operatorname{dom} U = \mathcal{H}$ for every densely defined closed operator U , we infer that $\operatorname{dom} U = \mathcal{H}$ and therefore U is a unitary operator. \square

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