

Lichtenstein’s integral equation for the Stokes problem via conformal mapping¹

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ÖSSZEFOGLALÓ. Síkbeli egyszerűen összefüggő tartományon kitűzött Stokes feladattal kapcsolatos Lichtenstein peremintegrál egyenletet vizsgáljuk a tartomány egységkörre való konform leképezésének segítségével. Az elméleti eredmények mellett néhány numerikus kísérlet tapasztalatait is ismertetjük.

ABSTRACT. We investigate Lichtenstein’s boundary integral equation method for the Stokes problem on a planar simply connected domain by transforming it onto the unit disc via conformal mapping. We also present some simple numerical experiments utilizing the conformal mapping as a prerequisite.

1 Introduction

Lichtenstein’s integral equation [2] transforms the equations of the stationary Stokes problem and also the equation of linear elasticity into a boundary integral equation for the divergence of the displacement field or for the pressure in case of Stokes flows. The kernel in this boundary integral equation is connected to Green’s function for Dirichlet boundary value problems for the Laplacian on the domain. In this article we use conformal mapping in order to transform the problem onto the unit disc and we obtain an equivalent of Lichtenstein’s integral equation with a kernel connected to the conformal mapping instead of Green’s function. Following the theoretical derivation of the boundary integral equation on the unit circle we also examine a simple numerical method for solving it, which we illustrate with some concrete examples.

2 Lichtenstein’s integral equation on the unit disc

Let Ω be a simply connected planar domain with boundary denoted by $\partial\Omega$. Consider the following stationary Stokes problem

$$\Delta_w U(w) = \nabla_w P(w), \text{ for } w \in \Omega, \quad (1)$$

$$\operatorname{div}_w U(w) = \sigma P(w), \text{ for } w \in \Omega, \quad (2)$$

$$U(w) = U_0(w), \text{ for } w \in \partial\Omega, \quad (3)$$

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for the unknown functions U and P , where $\sigma \in \mathbb{R}$ is a real parameter. This problem is connected to linear elasticity (parameter σ connected to the Lamé constants) or to the Cosserat eigenvalue problem (if setting $U_0 = 0$ and σ denoting an eigenvalue) or also to Stokes flows if $\sigma = 0$ and U, P describe flow velocity and pressure, respectively. Therefore we refer to these variables in this note from here on as velocity and pressure. By the divergence theorem the boundary values U_0 of the velocity have to fulfil the compatibility condition

$$\int_{\partial\Omega} U_0 n = \sigma \int_{\Omega} P, \quad (4)$$

where n denotes the unit outward pointing normal vector to the boundary of Ω . This means especially in case of Stokes flows ($\sigma = 0$) that the pressure function P is determined by (1)-(3) only up to an additive constant. Comparing the divergence of (1) with the Laplacian of (2) yields that in case $\sigma \neq 1$ the pressure function P is harmonic in Ω , which observation is important for the considerations of the present note.

If we think Ω as a subset of the complex plane, then the velocity function $U = (U_1; U_2)$ can be identified by the complex valued function $U = U_1 + iU_2$. The divergence of U and the gradient of P become $\operatorname{div}_w U(w) = 2 \operatorname{Re} \{ \partial_w U(w) \}$ and $\nabla_w P(w) = 2 \partial_{\bar{w}} P(w)$, respectively, where ∂_w and $\partial_{\bar{w}}$ denote the Wirtinger derivatives.

According to the Riemann mapping theorem there is a unique conformal map $g : \mathbb{D} \rightarrow \Omega$ with $g(0) = w_0$ and $g'(0) > 0$ for each interior point $w_0 \in \Omega$, where \mathbb{D} denotes the unit disc, see e.g. [3]. If $\partial\Omega$ is smooth enough then g extends continuously to $\partial\mathbb{D}$. Moreover, the smoothness properties of the boundary $\partial\Omega$ are related to the boundary behaviour of the conformal mapping on $\partial\mathbb{D}$, c.f. [4].

Using this mapping we can transform the problem (1)-(3) to the unit disc.

$$\Delta_z u(z) = g'(z) \nabla_z p(z), \text{ for } z \in \mathbb{D} \quad (5)$$

$$2 \operatorname{Re} \left\{ \frac{1}{g'(z)} \partial_z u(z) \right\} = \sigma p(z), \text{ for } z \in \mathbb{D} \quad (6)$$

$$u(z) = u_0(z), \text{ for } z \in \partial\mathbb{D}, \quad (7)$$

where the new unknown functions are $u(z) = U(g(z))$ and $p(z) = P(g(z))$ with $w = g(z)$. The transformed pressure function p is also harmonic on the unit disc for $\sigma \neq 1$ because transformations by conformal mappings preserve harmonicity. Using that the unit outer normal to $\partial\Omega$ at $w = g(z)$ is $n(g(z)) = \frac{zg'(z)}{|g'(z)|}$ and that $|dw| = |g'(z)|dz$ there follows $\int_{\partial\Omega} U_0(w)n(w)|dw| = \operatorname{Re} \int_{\partial\mathbb{D}} u_0(z) \overline{zg'(z)} |dz|$, which means that the compatibility condition (4) transforms as

$$\operatorname{Re} \int_{\partial\mathbb{D}} u_0(z) \overline{zg'(z)} |dz| = \sigma \int_{\mathbb{D}} p |g'|^2. \quad (8)$$

In [2] a method is proposed for transforming the problem (1)-(3) into a boundary integral equation on $\partial\Omega$. In the present paper we adapt this method for the transformed problem (5)-(7). Because for $\sigma \neq 1$ the transformed pressure function p is harmonic on the unit disc, we can use in this case the Poisson integral formula

$$p(z) = \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|z - \zeta|^2} \phi(\zeta) \frac{|d\zeta|}{2\pi} \text{ for } z \in \mathbb{D}, \quad (9)$$

where ϕ denotes the unknown boundary value of the pressure function on the unit circle. By the harmonicity of p from (5) there follows

$$\Delta_z \left(u(z) - \frac{1}{2} g(z) p(z) \right) = 0$$

using $\partial_{\bar{z}}g(z) = 0$, i.e. the function $u - \frac{1}{2}gp$ is also harmonic on the unit disc. Again by the Poisson integral formula there follows

$$u(z) - \frac{1}{2}g(z)p(z) = \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|z - \zeta|^2} \left(u_0(\zeta) - \frac{1}{2}g(\zeta)\phi(\zeta) \right) \frac{|d\zeta|}{2\pi} \text{ for } z \in \mathbb{D}.$$

Multiplying (9) with $g(z)$ and combining it with the previous equation we obtain for $z \in \mathbb{D}$

$$u(z) = \frac{1}{2} \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|z - \zeta|^2} (g(z) - g(\zeta))\phi(\zeta) \frac{|d\zeta|}{2\pi} + \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|z - \zeta|^2} u_0(\zeta) \frac{|d\zeta|}{2\pi}. \quad (10)$$

This equation expresses the values of the function u in the interior of the unit disc using its known boundary values and also the unknown boundary values ϕ of the pressure function. Substituting (10) into (6) yields for $z \in \mathbb{D}$:

$$\operatorname{Re} \left\{ \int_{\partial\mathbb{D}} \frac{g(\zeta) - g(z)}{g'(z)(\zeta - z)} \cdot \frac{\zeta}{\zeta - z} \phi(\zeta) - \frac{2\zeta}{g'(z)(\zeta - z)^2} u_0(\zeta) \frac{|d\zeta|}{2\pi} \right\} = (1 - \sigma)p(z). \quad (11)$$

Here we used the derivatives of the involved kernels:

$$\begin{aligned} \partial_z \left(\frac{1 - |z|^2}{|z - \zeta|^2} (g(z) - g(\zeta)) \right) &= \frac{1 - |z|^2}{|z - \zeta|^2} g'(z) + \frac{g(z) - g(\zeta)}{z - \zeta} \cdot \frac{\zeta}{z - \zeta} \text{ and} \\ \partial_z \left(\frac{1 - |z|^2}{|z - \zeta|^2} \right) &= \frac{\zeta}{(z - \zeta)^2} \text{ for } z \in \mathbb{D}. \end{aligned}$$

In order to obtain a boundary integral equation for the unknown boundary values ϕ we calculate the limit $z \rightarrow \partial\mathbb{D}$ in (11). For the first term in the integral in (11) we have the decomposition

$$\operatorname{Re} \int_{\partial\mathbb{D}} \frac{g(\zeta) - g(z)}{g'(z)(\zeta - z)} \cdot \frac{\zeta}{\zeta - z} \phi(\zeta) \frac{|d\zeta|}{2\pi} = \int_{\partial\mathbb{D}} L(z, \zeta) \phi(\zeta) \frac{|d\zeta|}{2\pi} + \operatorname{Re} \int_{\partial\mathbb{D}} \frac{\zeta}{\zeta - z} \phi(\zeta) \frac{|d\zeta|}{2\pi}, \quad (12)$$

where $L(z, \zeta)$ denotes the kernel

$$L(z, \zeta) = \operatorname{Re} \left(\left(\frac{g(\zeta) - g(z)}{g'(z)(\zeta - z)} - 1 \right) \cdot \frac{\zeta}{\zeta - z} \right). \quad (13)$$

If the conformal mapping g is smooth enough in the sense that its second derivative is continuous in the closed unit disc then the kernel (13) is also continuous in $\bar{\mathbb{D}}$ and we obtain

$$L(z, \zeta) = \begin{cases} \operatorname{Re} \left(\left(\frac{g(\zeta) - g(z)}{g'(z)(\zeta - z)} - 1 \right) \cdot \frac{\zeta}{\zeta - z} \right) & \text{for } z \neq \zeta, \\ \operatorname{Re} \left(\frac{zg''(z)}{2g'(z)} \right) & \text{for } z = \zeta, \end{cases} \quad (14)$$

for $z, \zeta \in \partial\mathbb{D}$.

Remark 1. The kernel (14) can be also expressed for $z, \zeta \in \partial\mathbb{D}$ as

$$L(z, \zeta) = \operatorname{Re} \left(\frac{1}{g'(z)} \sum_{m=2}^{\infty} a_m \zeta^{m-1} \sum_{k=0}^{m-2} (k+1)(z\bar{\zeta})^k \right)$$

using the series expansion $g(z) = \sum_{m=0}^{\infty} a_m z^m$ of the conformal mapping.

On the other hand by the properties of the kernel $\frac{\zeta}{\zeta-z} = \frac{1}{1-\bar{z}\zeta}$ for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ and by (9) we have

$$\operatorname{Re} \left\{ \int_{\partial\mathbb{D}} \frac{\zeta}{\zeta-z} \phi(\zeta) \frac{|d\zeta|}{2\pi} \right\} = \frac{1}{2}p(0) + \frac{1}{2}p(z)$$

for the second term in (12). Utilizing these equations we obtain from (11) the boundary integral equation

$$\left(\frac{1}{2} - \sigma\right) \phi(z) = p(0) + \int_{\partial\mathbb{D}} L(z, \zeta) \phi(\zeta) \frac{|d\zeta|}{2\pi} - \lim_{x \rightarrow z} \operatorname{Re} \int_{\partial\mathbb{D}} \frac{2\zeta}{g'(x)(\zeta-x)^2} u_0(\zeta) \frac{|d\zeta|}{2\pi} \quad (15)$$

for $z \in \partial\mathbb{D}$. By the mean value property $\int_{\partial\mathbb{D}} \phi(\zeta) \frac{|d\zeta|}{2\pi} = p(0)$ for the real valued harmonic function p there follows

$$\left(\frac{1}{2} - \sigma\right) \phi(z) = \int_{\partial\mathbb{D}} (1 + L(z, \zeta)) \phi(\zeta) \frac{|d\zeta|}{2\pi} - \lim_{x \rightarrow z} \operatorname{Re} \int_{\partial\mathbb{D}} \frac{2\zeta}{g'(x)(\zeta-x)^2} u_0(\zeta) \frac{|d\zeta|}{2\pi}. \quad (16)$$

This boundary integral equation can be used to obtain the unknown boundary values ϕ for the pressure function p using the prescribed boundary data u_0 for the velocity u . Then substituting the ϕ data into (11) and (9) we can calculate the velocity and pressure values at any interior point of the unit disc. These are also the velocity and pressure values at the corresponding conformal image of this interior point of the unit disc.

Remark 2. If we set the parameter $\sigma = 0$ then the pressure p is determined by the system (5)-(7) only up to an additive constant. In order to assure uniqueness we have to prescribe the value of the pressure in a point or we have to prescribe its integral over the domain to be some constant (e.g. zero). That is, in case $\sigma = 0$ we can set for example $p(0) = 0$ in (15). In case $\sigma \neq 0$ then we have to use (16). If we set $u_0 = 0$ and we consider (5)-(7) as an eigenvalue problem then we also have the same uniqueness issue regarding the pressure p . In this case $\sigma = 0$ is an eigenvalue with the constant pressure as eigenfunction (and u being any divergence free velocity function.)

Remark 3. If the conformal mapping g is not smooth enough, i.e. it does not have continuous second derivative in the closed unit disc, but if $\partial\Omega$ has a Dini-smooth corner of opening $\pi\alpha$ with $0 < \alpha \leq 2$ at the point $g(\zeta)$ for some $\zeta \in \partial\mathbb{D}$ then according to Theorem 3.9. in [4] the functions

$$\frac{g(z) - g(\zeta)}{(z - \zeta)^\alpha} \text{ and } \frac{g'(z)}{(z - \zeta)^{\alpha-1}}$$

are $\neq 0, \infty$ in a neighbourhood of ζ within the closed unit disc. Hence their quotient has a finite angular limit for $z \rightarrow \zeta$. Considering the fact that this quotient is the reciprocal of the Visser-Ostrowski quotient for the conformal mapping g (see equation (3) in Chapter 11 of [4]), we obtain the limit

$$\frac{g(z) - g(\zeta)}{g'(z)(z - \zeta)} \rightarrow \frac{1}{\alpha} \text{ for } z \rightarrow \zeta.$$

We can use this for calculating the limit in (11) when $z \rightarrow \mathbb{D}$, i.e. for obtaining the boundary integral equation from (11). It turns out that instead of (13) we have to use the kernel

$$L(z, \zeta) = \operatorname{Re} \left(\frac{g(\zeta) - g(z)}{g'(z)(\zeta - z)} - \frac{1}{\alpha} \right) \cdot \frac{\zeta}{\zeta - z}. \quad (17)$$

In the smooth case we have $\alpha = 1$ and therefore (17) reduce to (13).

Remark 4. In [1] another integral equation for the transformed pressure on \mathbb{D} was derived also using the conformal mapping of the unit disc onto the problem domain, see Theorem 3 in [1].

Remark 5. In case of the polynomial mapping $g(z) = a_1z + \dots + a_nz^n$ setting $u_0 = 0$ we have the Cosserat eigenvalue problem for the image domain $g(\mathbb{D})$ as discussed in the introduction. Equation (16) simplifies to

$$\left(\frac{1}{2} - \sigma\right) \phi(z) = \operatorname{Re} \left\{ \frac{1}{g'(z)} \sum_{k=1}^{n-1} \left(\sum_{\ell=1}^{n-k} a_{k+\ell} \bar{\phi}_\ell \right) kz^{k-1} \right\},$$

where $\phi(z) = 2 \operatorname{Re} \left\{ \sum_{k=0}^{\infty} \phi_k z^k \right\}$. This can be written in a block matrix form if we have the Taylor series expansion $\frac{1}{g'(z)} = \sum_{k=0}^{\infty} b_k z^k$ for the reciprocal of the derivative of the conformal map.

$$\begin{pmatrix} B & 0 \\ B' & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \overline{\begin{pmatrix} \phi \\ \phi' \end{pmatrix}} = (1 - 2\sigma) \begin{pmatrix} \phi \\ \phi' \end{pmatrix}, \quad (18)$$

where (ϕ, ϕ') is the block vector composed from $\phi = (\phi_1, \dots, \phi_{n-1})$ and $\phi' = (\phi_n, \phi_{n+1}, \dots)$, the two finite matrix blocks $A, B \in \mathbb{C}^{(n-1) \times (n-1)}$ are defined by

$$A_{k,\ell} = \begin{cases} ka_{k+\ell} & \text{if } k + \ell \leq n, \\ 0 & \text{otherwise,} \end{cases} \quad B_{k,\ell} = \begin{cases} b_{k-\ell}, & \text{if } k \geq \ell, \\ 0, & \text{otherwise,} \end{cases}$$

and the infinite matrix block B' has the k -th row $(b_{n+k-2}, \dots, b_k, 0, 0, \dots)$ for $k = 1, 2, \dots$. The block A is a Hankel matrix multiplied by a diagonal matrix and the block B is a Toeplitz matrix, they are very similar to those investigated in [6, 7]. Because of this block structure (18) can be decomposed into two parts:

- if $\phi = 0$ and ϕ' is a vector having only one non zero entry, then $\sigma = \frac{1}{2}$ is the eigenvalue,
- if $\phi' = 0$, then the finite matrix eigenvalue problem $BA\bar{\phi} = (1 - 2\sigma)\phi$ gives other eigenvalues.

Another possibility for reducing the infinite matrix eigenvalue problem to a finite one is that the reciprocal of the derivative of the conformal map is itself a polynomial, see [7]. In these cases the domain $\Omega = g(\mathbb{D})$ is a quadrature domain meaning that integrals of holomorphic functions over the domain reduce to a finite quadrature rule similar to the mean value property of holomorphic functions on the unit disc.

Example 6. Exact solution of (16) is not possible for each conformal mapping only for the simplest ones. For the univalent quadratic polynomial $g(z) = z + az^2$, where $|a| \leq \frac{1}{2}$, which maps the unit disc onto a cardioid, one easily computes the kernel (13) as

$$L(z, \zeta) = \operatorname{Re} \left(\frac{a\zeta}{1 + 2az} \right),$$

which yields

$$\int_{\partial\mathbb{D}} L(z, \zeta) \phi(\zeta) \frac{|d\zeta|}{2\pi} = \operatorname{Re} \left(\frac{a\bar{\phi}_1}{1 + 2az} \right),$$

where we have set the series expansion $\phi(z) = 2 \operatorname{Re} \sum_{j=0}^{\infty} \phi_j z^j$. Prescribing the boundary values $u_0(\zeta) = \sum_{j=0}^{\infty} u_{0,j} \zeta^j + \sum_{j=0}^{\infty} u_{0,-j} \bar{\zeta}^j$ for the velocity and substituting these along with $\sigma = 0$ into (15) yields

$$\frac{1}{2} \phi(z) - \operatorname{Re} \frac{a\bar{\phi}_1}{1 + 2az} = - \operatorname{Re} \frac{2}{1 + 2az} \sum_{j=0}^{\infty} u_{0,j+1} (j+1) z^j.$$

Using the fact, that the functions z^j for $j \in \mathbb{Z}$ constitute a complete orthonormal system on $\partial\mathbb{D}$ w.r.t. $\frac{1}{2\pi}dt$, there follows

$$\phi_1 + 2a^2\overline{\phi_1} - 4(au_{0,1} - u_{0,2}) = 0,$$

which has the solution

$$\phi_1 = \frac{(au_{0,1} - u_{0,2}) - 2a^2\overline{(au_{0,1} - u_{0,2})}}{\frac{1}{4} - |a|^4}.$$

Now, that we have computed ϕ_1 from the coefficients of the velocity boundary data and of the conformal map, we also obtain the whole boundary function $\phi(z)$ of the pressure p as

$$\phi(z) = \operatorname{Re} \frac{2a\overline{\phi_1} - 4 \sum_{j=0}^{\infty} u_{0,j+1}(j+1)z^j}{1+2az} \text{ for } z \in \partial\mathbb{D}.$$

For example the velocity boundary values corresponding to $u_0(\zeta) = -izg'(z) = -i\zeta - 2ia\zeta^2$ are tangential to the boundary of the cardioid in each point and therefore obviously fulfil the compatibility condition (8). In this case we obtain the exact pressure boundary values as

$$\phi(z) = -4 \frac{1 - 4|a|^2}{1 - 2|a|^2} \operatorname{Im} \frac{1}{1 + 2az}.$$

This can be utilized as a benchmark for numerical solutions of (16) for example.

Example 7. For the case of a cardioid from the previous example the eigenvalue problem (18) reduces to

$$\operatorname{Re} \left\{ \frac{a}{1 + 2az} \overline{\phi_1} \right\} = \left(\frac{1}{2} - \sigma \right) \phi(z),$$

from which one has the eigenvalue $\frac{1}{2}$ with infinite multiplicity and the two additional eigenvalues $\frac{1}{2} \pm |a|^2$. Hence the Cosserat constant, that is the least positive Cosserat eigenvalue of the cardioid is $\sigma(g(\mathbb{D})) = \frac{1}{2} - |a|^2$, see also the corresponding example in Remark 12 of [7].

3 Numerical experiments

In this section we examine a simple numerical treatment of the boundary integral equation (16). We set $N \geq 2$ points $z_k = e^{it_k}$ ($k = 0, 1, \dots, N-1$) on the unit circle so that their angles satisfy $0 \leq t_1 < \dots < t_{N-1} < 2\pi$. These points partition the unit circle in N disjoint arcs. We denote the arc between the points z_k and z_{k+1} by A_k , i.e. $A_k = \{z = e^{it} \mid t_k \leq t \leq t_{k+1}\}$, where we set $t_N = t_0 + 2\pi$ meaning $z_N = z_0$. We approximate the unknown function $\phi(z)$ on $\partial\mathbb{D}$ by a piecewise constant function on the arcs of the given partition of the unit circle, that is, $\phi(z) = \phi_\ell \in \mathbb{R}$ for $z \in A_\ell$. We intend to calculate these approximate values by discretizing (16), especially the integral with the kernel (13). It means that for each point $z_k \in \mathbb{D}$ we approximate by

$$\int_{\partial\mathbb{D}} L(z_k, \zeta) \phi(\zeta) \frac{|d\zeta|}{2\pi} \approx \sum_{\ell=0}^{N-1} \phi_\ell \int_{A_\ell} L(z_k, \zeta) \frac{|d\zeta|}{2\pi} = [\mathcal{L}\phi]_k,$$

where by a slight abuse of notation $\phi \in \mathbb{R}^N$ denotes the vector composed of the approximate values of $\phi(z)$ on the arcs A_ℓ ($\ell = 0, 1, \dots, N-1$) and $\mathcal{L} \in \mathbb{R}^{N \times N}$ is the matrix composed of the entries

$$\mathcal{L}_{k,\ell} = \int_{A_\ell} L(z_k, \zeta) \frac{|d\zeta|}{2\pi}. \quad (19)$$

These entries are calculated from the auxiliary information about the conformal mapping g by numerically evaluating its defining integral (19) over the arc A_ℓ . This auxiliary information about the mapping g can be its Taylor expansion coefficients or its values at some points of the arcs A_ℓ . If the domain which g maps the unit disc on is a simple domain, then we know the exact mapping function and its Taylor series coefficients, as for example in the cases of a cardioid or a square. For more complicated domains we have to calculate the mapping numerically, see e.g. [5] for the theoretically and also numerically important case of the Schwarz-Christoffel mapping of the unit disc onto polygonal domains. Given the images $g(z_k)$ of each point z_k in the partition then we can use the trapezoid rule for example to approximate the integral over the arc A_ℓ in (19) as

$$\mathcal{L}_{k,\ell} \approx \frac{L(z_k, z_{\ell+1}) + L(z_k, z_\ell)}{2} \cdot \frac{\Delta t_\ell}{2\pi}.$$

Here the values $L(z_k, z_\ell)$ are calculated utilizing (14), wherein for the approximate calculation of the derivatives $g'(z_k)$ and $g''(z_k)$ we can use some Taylor coefficients of g or if we are not given any of them then we first compute them using for example discrete Fourier transform.

For the other integral in (16) involving the boundary data u_0 there follows for $z \in \mathbb{D}$ that

$$\int_{\partial\mathbb{D}} \frac{\zeta}{(\zeta - z)^2} u_0(\zeta) \frac{|d\zeta|}{2\pi} = \int_{\partial\mathbb{D}} \frac{\bar{\zeta}}{(1 - z\bar{\zeta})^2} u_0(\zeta) \frac{|d\zeta|}{2\pi} = \sum_{m=1}^{\infty} u_{0,m} m z^{m-1},$$

where $u_0(\zeta) = \sum_{m=-\infty}^{\infty} u_{0,m} \zeta^m$ again as in Example 6. Here the coefficients $u_{0,m}$ can be again calculated from the boundary data $u_0(\zeta)$ given in some discrete points on the unit circle approximately by discrete Fourier transform for example. Having these coefficients for $m = 1, 2, \dots, n$ we conclude

$$\mathcal{M}u_0(z_k) = \operatorname{Re} \lim_{x \rightarrow z_k} \int_{\partial\mathbb{D}} \frac{2\zeta}{g'(x)(\zeta - x)^2} u_0(\zeta) \frac{|d\zeta|}{2\pi} \approx \operatorname{Re} \frac{2}{g'(z_k)} \sum_{m=1}^n u_{0,m} m z_k^{m-1}.$$

Combining the previous approximating terms we obtain the matrix equation

$$\frac{1}{2}\phi - \mathcal{L}\phi = -\mathcal{M}u_0 \quad (20)$$

with the matrices $\mathcal{L}, \mathcal{M} \in \mathbb{R}^{N \times N}$ defined by the equations above for $\phi \in \mathbb{R}^N$ instead of the integral equation (16) for the function $\phi(z)$. The solution of (20) yields approximate boundary values of the pressure, which can be substituted into (9) and (10) in order to compute the pressure and velocity at any interior point of interest.

In the rest of this section we present some numerical experiments which are calculated by the NumPy package for Python, see <https://numpy.org/>. For the calculations we used the quadratic conformal mapping of the unit disc onto a cardioid as in the previous examples.

Example 8. The matrices \mathcal{L} and $\frac{1}{2}\mathcal{I} - \mathcal{L}$ are near singular practically for any number of partition points on the unit circle. The condition number of the matrix $\frac{1}{2}\mathcal{I} - \mathcal{L}$ in (20) is well behaved only if the magnitude of the derivative of the conformal mapping is bounded from below on the boundary, i.e. $|g'(z)| > \epsilon > 0$. If $|g'(z_0)|$ is near zero on the boundary and this point is near to a point in the partition then this condition number becomes very large and the solution of the system (20) will be very sensitive for small changes in the input values. This is illustrated on the Figure 1. Without suitable preconditioning in this ill-conditioned case the numerical solution deviates from the exact solution in Example 6 as shown in Figure 2.

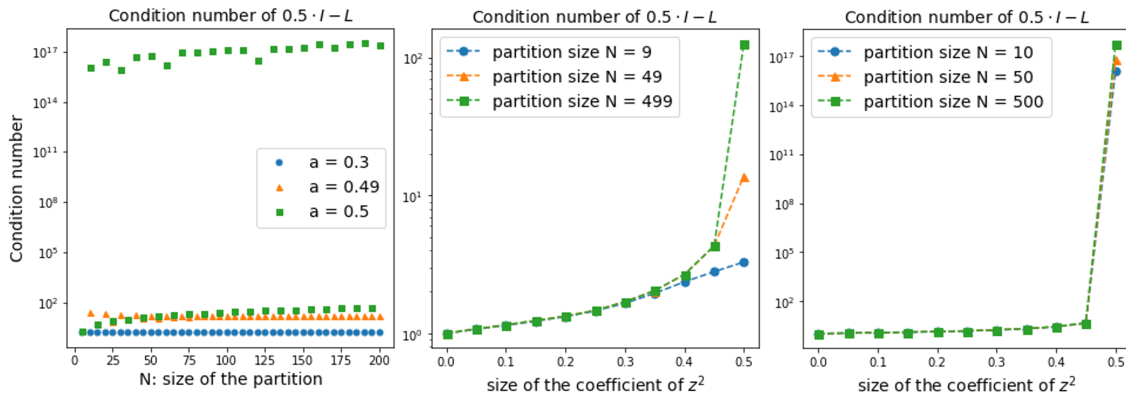


Figure 1: Condition number of $\frac{1}{2}\mathcal{I} - \mathcal{L}$

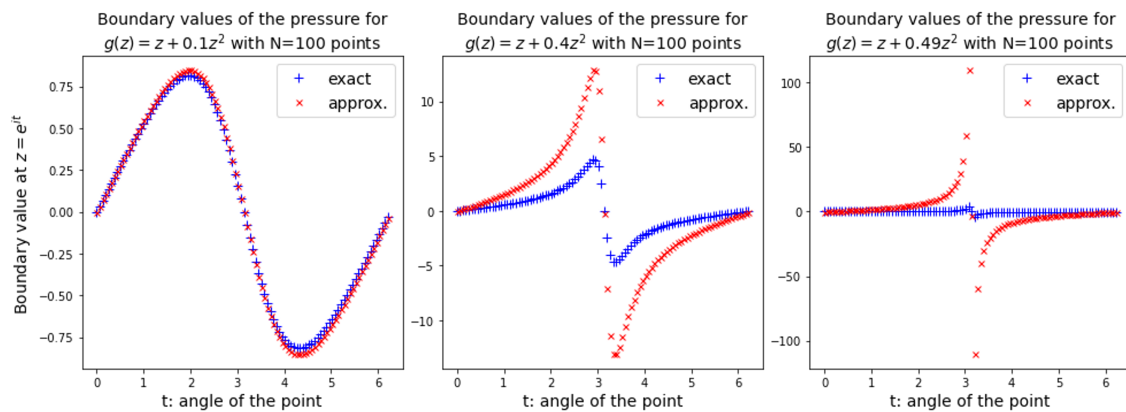


Figure 2: Comparison of the exact and approximate pressure boundary values

Example 9. The Cosserat constant, i.e. the least positive Cosserat eigenvalue, of the image domain $g(\mathbb{D})$ can be deduced from the maximum eigenvalue of the problem (15) (i.e. setting $u_0 = 0$), see also Remark 5 and Example 7. The eigenvalue structure of the matrix \mathcal{L} is practically independent of the partition size as the comparison of the middle and right diagrams on Figure 3 shows. The vast majority of the eigenvalues is practically zero except for few which are also several magnitude smaller than the maximal eigenvalue. If any of the partition points

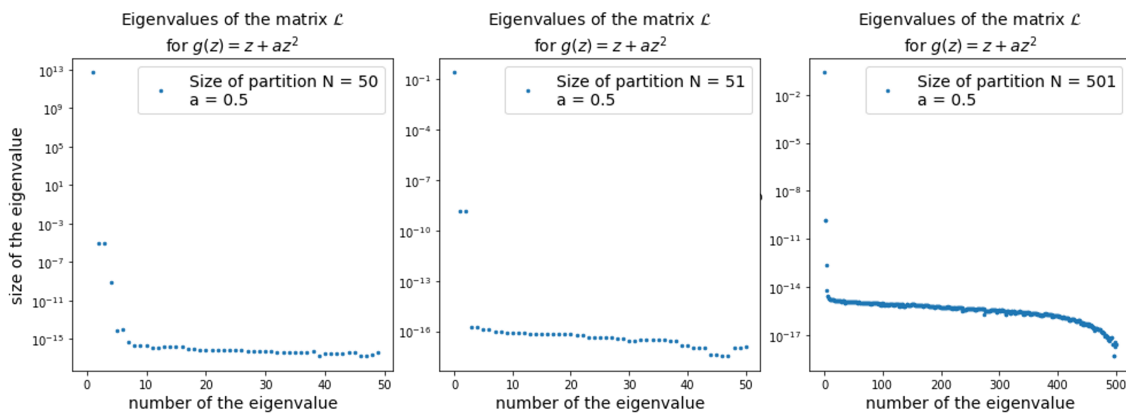


Figure 3: Eigenvalues of \mathcal{L}

is nearly a zero of the derivative of the conformal map then the numerical determination of the

spectrum of \mathcal{L} is also inaccurate: compare the left and middle diagrams on Figure 3. Therefore this affects the numerical approximation of the Cosserat constant of the domain via the maximal eigenvalue of the matrix \mathcal{L} as shown on Figure 4.

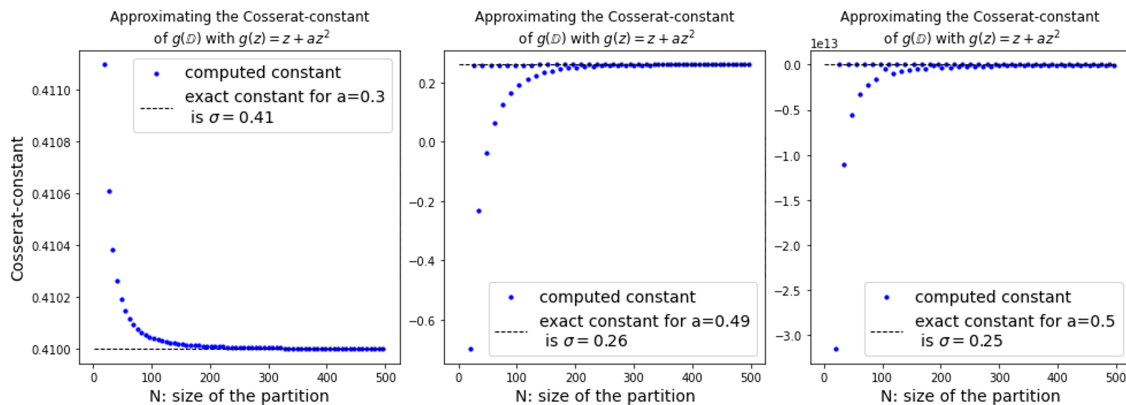


Figure 4: Approximating the Cosserat constant of the cardioid

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