

ON HARDY TYPE INEQUALITIES FOR WEIGHTED QUASIDEVIATION MEANS

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ABSTRACT. Using recent results concerning the homogenization and the Hardy property of weighted means, we establish sharp Hardy constants for concave and monotone weighted quasideviation means and for a few particular subclasses of this broad family. More precisely, for a mean \mathcal{D} like above and a sequence (λ_n) of positive weights such that $\lambda_n/(\lambda_1 + \dots + \lambda_n)$ is nondecreasing, we determine the smallest number $H \in (1, +\infty]$ such that

$$\sum_{n=1}^{\infty} \lambda_n \mathcal{D}((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n)) \leq H \cdot \sum_{n=1}^{\infty} \lambda_n x_n \text{ for all } x \in \ell_1(\lambda).$$

It turns out that H depends only on the limit of the sequence $(\lambda_n/(\lambda_1 + \dots + \lambda_n))$ and the behaviour of the mean \mathcal{D} near zero.

1. INTRODUCTION

In 1920's several authors, motivated by a conjecture of Hilbert, proved that

$$(1.1) \quad \sum_{n=1}^{\infty} \mathcal{P}_p(x_1, \dots, x_n) \leq C(p) \sum_{n=1}^{\infty} x_n$$

for every sequences $(x_n)_{n=1}^{\infty}$ with positive terms, where \mathcal{P}_p denotes the p -th *power mean* (extended to the limiting cases $p = \pm\infty$),

$$C(p) := \begin{cases} 1 & p = -\infty, \\ (1-p)^{-1/p} & p \in (-\infty, 0) \cup (0, 1), \\ e & p = 0, \\ \infty & p \in [1, \infty], \end{cases}$$

and this constant is sharp, i.e., it cannot be diminished.

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The first result of this type with a nonoptimal constant was established by Hardy in [13]. Later this result was improved and extended by Landau [17], Knopp [15], and Carleman [3] whose results are summarized in the inequality (1.1). Meanwhile, Copson [4] adopted Elliott's [11] proof of the Hardy inequality and showed (in an equivalent form) that if $\mathcal{P}_p(x, \lambda)$ denotes the p -th λ -weighted power mean of the vector x , then

$$(1.2) \quad \sum_{n=1}^{\infty} \lambda_n \mathcal{P}_p((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n)) \leq C(p) \sum_{n=1}^{\infty} \lambda_n x_n$$

for all $p \in (0, 1)$, and sequences $(x_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$ with positive terms. For more details about the history of the developments related to Hardy type inequalities, see papers Pečarić–Stolarsky [26], Duncan–McGregor [10], and the book of Kufner–Maligranda–Persson [16].

Obviously, the constant $C(p)$ is sharp if we require the inequality to be valid for all positive sequences λ and x . One of the main goal of this presentation is to determine the best possible constant $C_{\lambda}(p)$ such that the inequality (1.2) be valid with $C(p)$ replaced by $C_{\lambda}(p)$ for all positive sequences x . Moreover, we will extend this result also for the case $p \leq 0$. In fact, under some additional assumptions, we will show that $C_{\lambda}(p)$ is function of p and the limit of the sequence $(\frac{\lambda_n}{\lambda_1 + \dots + \lambda_n})$. On the other hand, our results will be developed not only for power means, but in a much larger class of weighted means, in the class of weighted quasideviation means which includes quasiarithmetic and also Gini means. The motivation for this paper originates from the paper [38] related to the nonweighted and homogeneous case.

2. WEIGHTED MEANS

For $n \in \mathbb{N}$, define the set of n -dimensional real weight vectors W_n by

$$W_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1, \dots, \lambda_n \geq 0, \lambda_1 + \dots + \lambda_n > 0\}$$

and let

$$W_0 := \{(\lambda_n)_{n \in \mathbb{N}} \mid \lambda_1 > 0 \text{ and } \lambda_2, \dots, \lambda_n, \dots \geq 0\}.$$

Now we recall the concept of a weighted mean as it was introduced in the paper [37].

For a given subinterval $I \subset \mathbb{R}$, a *weighted mean on I* is a function

$$\mathcal{M}: \bigcup_{n=1}^{\infty} I^n \times W_n \rightarrow I$$

which is nullhomogeneous in the weights, admits the reduction principle, the mean value property, and the elimination principle (see [37] for the details). For $n \in \mathbb{N}$

and $(x, \lambda) \in I^n \times W_n$, we will frequently use the sum type abbreviation:

$$\mathcal{M}_{i=1}^n(x_i, \lambda_i) := \mathcal{M}((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n)).$$

Let us now introduce some important properties of weighted means. A weighted mean \mathcal{M} is said to be *symmetric*, if for all $n \in \mathbb{N}$, $(x, \lambda) \in I^n \times W_n$, and $\sigma \in S_n$,

$$\mathcal{M}(x, \lambda) = \mathcal{M}(x \circ \sigma, \lambda \circ \sigma).$$

We will call a weighted mean \mathcal{M} *Jensen concave* if, for all $n \in \mathbb{N}$, $x, y \in I^n$ and $\lambda \in W_n$,

$$(2.1) \quad \mathcal{M}\left(\frac{x+y}{2}, \lambda\right) \geq \frac{1}{2}(\mathcal{M}(x, \lambda) + \mathcal{M}(y, \lambda)).$$

If the above inequality holds with reversed inequality sign, then we speak about the *Jensen convexity* of \mathcal{M} . Using that the mapping $x \mapsto \mathcal{M}(x, \lambda)$ is locally bounded, the Bernstein–Doetsch Theorem [2] implies that \mathcal{M} is in fact concave or convex, respectively.

A weighted mean \mathcal{M} is said to be *monotone* (or *nondecreasing*) if, for all $n \in \mathbb{N}$ and $\lambda \in W_n$, the mapping $x_i \mapsto \mathcal{M}(x, \lambda)$ is nondecreasing for all $i \in \{1, \dots, n\}$.

Assuming that I is a subinterval of \mathbb{R}_+ , we call a weighted mean \mathcal{M} *homogeneous*, if for all $t > 0$, $n \in \mathbb{N}$ and $(x, \lambda) \in (I \cap \frac{1}{t}I)^n \times W_n$,

$$\mathcal{M}(tx, \lambda) = t\mathcal{M}(x, \lambda).$$

For a given subinterval I of \mathbb{R}_+ with $\inf I = 0$ and a weighted mean \mathcal{M} on I , we define two functions $\mathcal{M}_\#, \mathcal{M}^\# : \bigcup_{n=1}^\infty \mathbb{R}_+^n \times W_n \rightarrow \mathbb{R}_+$ by

$$\mathcal{M}_\#(x, \lambda) := \liminf_{t \rightarrow 0^+} \frac{1}{t} \mathcal{M}(tx, \lambda) \quad \text{and} \quad \mathcal{M}^\#(x, \lambda) := \limsup_{t \rightarrow 0^+} \frac{1}{t} \mathcal{M}(tx, \lambda).$$

We call $\mathcal{M}_\#$ and $\mathcal{M}^\#$ the *lower and upper homogenization* of the weighted mean \mathcal{M} , respectively. It is obvious that $\mathcal{M}_\#$ and $\mathcal{M}^\#$ are homogeneous weighted means on \mathbb{R}_+ , furthermore, we have the inequality $\mathcal{M}_\# \leq \mathcal{M}^\#$ on $\bigcup_{n=1}^\infty \mathbb{R}_+^n \times W_n$. It is also easy to see that if \mathcal{M} is symmetric (monotone), then also $\mathcal{M}_\#$ and $\mathcal{M}^\#$ are symmetric (monotone). Moreover, in the case when \mathcal{M} is concave, we have a few additional properties.

Lemma 2.1 ([35], Theorem 2.1). *Let I be a subinterval of \mathbb{R}^+ with $\inf I = 0$ and \mathcal{M} be a Jensen concave weighted mean on I . Then $\mathcal{M}_\# = \mathcal{M}^\#$ and these means are also Jensen concave. In addition, $\mathcal{M} \leq \mathcal{M}_\# = \mathcal{M}^\#$ on the domain of \mathcal{M} .*

In what follows, we recall several particular classes of weighted means. For a parameter $p \in \mathbb{R}$, define the *weighted power mean* $\mathcal{P}_p: \bigcup_{n=1}^{\infty} \mathbb{R}_+^n \times W_n \rightarrow \mathbb{R}_+$ by

$$\mathcal{P}_p(x, \lambda) := \begin{cases} \left(\frac{\lambda_1 x_1^p + \cdots + \lambda_n x_n^p}{\lambda_1 + \cdots + \lambda_n} \right)^{1/p} & \text{if } p \neq 0, \\ (x_1^{\lambda_1} \cdots x_n^{\lambda_n})^{1/(\lambda_1 + \cdots + \lambda_n)} & \text{if } p = 0. \end{cases}$$

In a more general setting, we can define weighted quasiarithmetic means in the spirit of [14]. Given an interval I and a continuous strictly monotone function $f: I \rightarrow \mathbb{R}$, the *weighted quasiarithmetic mean* $\mathcal{A}_f: \bigcup_{n=1}^{\infty} I^n \times W_n \rightarrow I$ is defined by

$$(2.2) \quad \mathcal{A}_f(x, \lambda) := f^{-1} \left(\frac{\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)}{\lambda_1 + \cdots + \lambda_n} \right).$$

Another important generalization of power means was introduced in the paper [12]. For two real parameters p, q , the Gini mean $\mathcal{G}_{p,q}: \bigcup_{n=1}^{\infty} \mathbb{R}_+^n \times W_n \rightarrow \mathbb{R}_+$ is defined by

$$\mathcal{G}_{p,q}(x, \lambda) := \begin{cases} \left(\frac{\lambda_1 x_1^p + \cdots + \lambda_n x_n^p}{\lambda_1 x_1^q + \cdots + \lambda_n x_n^q} \right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \exp \left(\frac{\lambda_1 x_1^p \log x_1 + \cdots + \lambda_n x_n^p \log x_n}{\lambda_1 x_1^p + \cdots + \lambda_n x_n^p} \right) & \text{if } p = q. \end{cases}$$

In a sequence papers, further generalizations were obtained: *Bajraktarević means* [1], *deviation (or Daróczy) means* [5] and *quasideviation means* [27]. For more details, we just refer the reader to a series of papers by Losonczi [18–23] (for Bajraktarević means), Daróczy [5, 6], Daróczy–Losonczi [7], Daróczy–Páles [8, 9] (for deviation means), Páles [27–33] (for deviation and quasideviation means) and Páles–Pasteczka [35] (for semideviation means).

In what follows, we recall the notions of a quasideviation and the related weighted quasideviation mean (cf. [27], [34] and [35]).

Definition 2.2. A function $E: I \times I \rightarrow \mathbb{R}$ is said to be a *quasideviation* if

- (a) for all elements $x, y \in I$, the sign of $E(x, y)$ coincides with that of $x - y$,
- (b) for all $x \in I$, the map $y \mapsto E(x, y)$ is continuous and,
- (c) for all $x < y$ in I , the mapping $(x, y) \ni t \mapsto \frac{E(y, t)}{E(x, t)}$ is strictly increasing.

By the results of the paper [27], for all $n \in \mathbb{N}$ and $(x, \lambda) \in I^n \times W_n$, the equation

$$(2.3) \quad \lambda_1 E(x_1, y) + \cdots + \lambda_n E(x_n, y) = 0$$

has a unique solution y , which will be called the *E -quasideviation mean* of (x, λ) and denoted by $\mathcal{D}_E(x, \lambda)$.

One can easily notice that power means, quasiarithmetic means, Gini means are quasideviation means.

We say that a quasideviation $E: I \times I \rightarrow \mathbb{R}$ is *normalizable* if, for all $x \in I$, the function $y \mapsto E(x, y)$ is differentiable at x and the mapping $x \mapsto \partial_2 E(x, x)$ is strictly negative and continuous on I . The normalization $E^*: I \times I \rightarrow \mathbb{R}$ of E is defined by

$$E^*(x, y) := \frac{E(x, y)}{-\partial_2 E(y, y)} \quad (x, y \in I).$$

The quasideviation means generated by E and E^* are identical. In [35, Lemma 5.1] we proved that, for a normalized quasideviation E , the partial derivative $\partial_2 E$ is identically equal to -1 on the diagonal of $I \times I$, hence E^* is also a normalizable quasideviation and $(E^*)^* = E^*$ holds.

The following two results of the papers [38] and [35] are instrumental for us.

Lemma 2.3 ([38], Theorem 2.3). *Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be concave such that $\text{sign}(f(x)) = \text{sign}(x - 1)$ for all $x \in \mathbb{R}_+$. Then the function $E: \mathbb{R}_+^2 \rightarrow \mathbb{R}$ defined by $E(x, y) := f(\frac{x}{y})$ is a quasideviation and the weighted quasideviation mean $\mathcal{E}_f := \mathcal{D}_E$ is homogeneous, continuous, nondecreasing and concave.*

Lemma 2.4 ([35], Theorem 6.3). *Let $E: I \times I \rightarrow \mathbb{R}$ be a normalizable quasideviation such that E^* is concave. Assume that $\lim_{t \rightarrow 0^+} E^*(xt, t) = 0$ for all $x \in \mathbb{R}_+$. Then, for all $x \in \mathbb{R}_+$, the limit*

$$(2.4) \quad h_E(x) := \lim_{t \rightarrow 0^+} t^{-1} E^*(xt, t)$$

exists, $\text{sign}(h_E(x)) = \text{sign}(x - 1)$, and the function $h_E: \mathbb{R}_+ \rightarrow \mathbb{R}$ so defined is concave and nondecreasing on \mathbb{R}_+ , and is strictly increasing on $(0, 1)$. Furthermore, the weighted quasideviation mean \mathcal{D}_E is Jensen concave, monotone, and

$$\mathcal{E}_{h_E} = (\mathcal{D}_E)_\# = (\mathcal{D}_E)^\#.$$

3. HARDY TYPE INEQUALITIES FOR GENERAL WEIGHTED MEANS

We recall several definitions and results of the papers [39] and [35]. Throughout the rest of the paper, let I be an interval with $\inf I = 0$.

Definition 3.1 (Weighted Hardy property). For a weighted mean \mathcal{M} on I and a weight sequence $\lambda \in W_0$, let C be the smallest extended real number such that

$$\sum_{n=1}^{\infty} \lambda_n \cdot \mathcal{M}_{i=1}^n(x_i, \lambda_i) \leq C \cdot \sum_{n=1}^{\infty} \lambda_n x_n \quad \text{for all sequences } (x_n) \text{ in } I.$$

We call C to be the λ -weighted Hardy constant of \mathcal{M} or the λ -Hardy constant of \mathcal{M} and denote it by $\mathcal{H}_\lambda(\mathcal{M})$. Whenever this constant is finite, then \mathcal{M} is called a λ -weighted Hardy mean or simply a λ -Hardy mean.

Extending some previous results by Elliott [11] and Copson [4], we have obtained in [39] that, in a large class of weighted means, the Hardy constant corresponding to the weight sequence $\mathbf{1} := (1, 1, \dots)$ is the maximal one.

Theorem 3.2. *For every symmetric and monotone weighted mean \mathcal{M} on I , we have*

$$\mathcal{H}_{\mathbf{1}}(\mathcal{M}) = \sup_{\lambda \in W_0} \mathcal{H}_{\lambda}(\mathcal{M}).$$

The following lemma from [39] will be used.

Lemma 3.3. *Let \mathcal{M} be a weighted mean on I and $\lambda \in W_0$. Then, for all $n \in \mathbb{N}$ and $x \in I^n$,*

$$(3.1) \quad \sum_{i=1}^n \lambda_i \cdot \mathcal{M}_{\#}^i(x_j, \lambda_j) \leq \mathcal{H}_{\lambda}(\mathcal{M}) \sum_{i=1}^n \lambda_i x_i.$$

Based on this lemma and Lemma 2.1, we can compare the λ -Hardy constant of the weighted mean \mathcal{M} and its lower homogenization $\mathcal{M}_{\#}$.

Theorem 3.4. *Let \mathcal{M} be a weighted mean on I . Then, for all $\lambda \in W_0$,*

$$(3.2) \quad \mathcal{H}_{\lambda}(\mathcal{M}_{\#}) \leq \mathcal{H}_{\lambda}(\mathcal{M}).$$

If, in addition, \mathcal{M} is Jensen concave, then (3.2) holds with equality.

Proof. Let (x_m) be a sequence in \mathbb{R}_+ . For any fixed $n \in \mathbb{N}$, there exists a positive number τ_n such that $t(x_1, \dots, x_n) \in I^n$ for $t \in (0, \tau_n]$. Using Lemma 3.3, it follows that

$$\sum_{i=1}^n \lambda_i \cdot \mathcal{M}_{\#}^i(tx_j, \lambda_j) \leq \mathcal{H}_{\lambda}(\mathcal{M}) \sum_{i=1}^n \lambda_i tx_i.$$

Dividing by $t \in (0, \tau_n]$, and then taking the liminf of the left hand side of the inequality so obtained as $t \rightarrow 0^+$, (by the superadditivity of the liminf operation), we arrive at

$$\sum_{i=1}^n \lambda_i \cdot \liminf_{t \rightarrow 0^+} \frac{1}{t} \mathcal{M}_{\#}^i(tx_j, \lambda_j) \leq \mathcal{H}_{\lambda}(\mathcal{M}) \sum_{i=1}^n \lambda_i x_i.$$

This inequality is equivalent to

$$\sum_{i=1}^n \lambda_i \cdot \mathcal{M}_{\#}^i(x_j, \lambda_j) \leq \mathcal{H}_{\lambda}(\mathcal{M}) \sum_{i=1}^n \lambda_i x_i.$$

Finally, passing the limit $n \rightarrow \infty$ in the above inequality, we get

$$\sum_{n=1}^{\infty} \lambda_n \cdot \mathcal{M}_{\#}^n(x_i, \lambda_i) \leq \mathcal{H}_{\lambda}(\mathcal{M}) \cdot \sum_{n=1}^{\infty} \lambda_n x_n,$$

which proves that $\mathcal{H}_\lambda(\mathcal{M}_\#) \leq \mathcal{H}_\lambda(\mathcal{M})$.

If, additionally, \mathcal{M} is Jensen concave, then, by Lemma 2.1, the comparison inequality $\mathcal{M} \leq \mathcal{M}_\#$ is valid, and hence, (3.2) must hold with equality, indeed. \square

The following result of the paper [39], which is a weighted analogue of [36, Thm 3.3], provides a lower bound for the Hardy constant $\mathcal{H}_\lambda(\mathcal{M})$.

Lemma 3.5. *Let \mathcal{M} be a weighted mean on I , $\lambda \in W_0$, and $(x_n)_{n=1}^\infty$ be a sequence of elements in I . If $\sum_{n=1}^\infty \lambda_n x_n = \infty$, then*

$$\mathcal{H}_\lambda(\mathcal{M}) \geq \liminf_{n \rightarrow \infty} \frac{1}{x_n} \mathcal{M}_{i=1}^n(x_i, \lambda_i).$$

By taking $x_n := \frac{y}{\Lambda_n}$ for a fixed $y \in \lambda_1 I$ in the above theorem, the first inequality of the following consequence was deduced in [39]. The second inequality is an application of the first one to the mean $\mathcal{M}_\#$ and Theorem 3.4.

Corollary 3.6. *Let \mathcal{M} be a weighted mean on I and $\lambda \in W_0$ be a weight sequence with $\sum_{n=1}^\infty \lambda_n = \infty$. Then we have the following two lower estimates for the λ -Hardy constant $\mathcal{H}_\lambda(\mathcal{M})$:*

$$\mathcal{H}_\lambda(\mathcal{M}) \geq \sup_{y \in \lambda_1 I} \liminf_{n \rightarrow \infty} \frac{\Lambda_n}{y} \cdot \mathcal{M}_{k=1}^n\left(\frac{y}{\Lambda_k}, \lambda_k\right) =: \mathcal{C}_\lambda(\mathcal{M})$$

and

$$\mathcal{H}_\lambda(\mathcal{M}) \geq \liminf_{n \rightarrow \infty} \mathcal{M}_\#^n\left(\frac{\Lambda_n}{\Lambda_k}, \lambda_k\right) = \mathcal{C}_\lambda(\mathcal{M}_\#).$$

Finally let us recall one of key results from [39].

Proposition 3.7 ([39], Corollary 4.3). *Let \mathcal{M} be a symmetric, monotone and Jensen-concave weighted mean which is continuous in the weights and $\lambda \in W_0$ such that $\left(\frac{\lambda_n}{\lambda_1 + \dots + \lambda_n}\right)_{n=1}^\infty$ is nonincreasing. Then $\mathcal{H}_\lambda(\mathcal{M}) \leq \mathcal{C}_\lambda(\mathcal{M})$. Furthermore, if $\sum_{n=1}^\infty \lambda_n = \infty$, then $\mathcal{H}_\lambda(\mathcal{M}) = \mathcal{C}_\lambda(\mathcal{M})$.*

4. AUXILIARY RESULTS

In this section we prove a number of results which will be instrumental in the forthcoming sections. Throughout this section, let $\lambda \in W_0$ be a fixed weight sequence and $\Lambda_n := \lambda_1 + \dots + \lambda_n$ for $n \in \mathbb{N}$.

Lemma 4.1. *The sequence (Λ_n) and the series $\sum \lambda_n / \Lambda_n$ are equi-convergent (either both of them are convergent or both of them are divergent).*

Proof. If $\Lambda_\infty := \sum_{n=1}^{\infty} \lambda_n < \infty$, then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n} \leq \sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_1} = \frac{\Lambda_\infty}{\Lambda_1} < \infty.$$

Conversely, if $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n} < \infty$, then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $\lambda_n/\Lambda_n < \frac{1}{2}$. Equivalently, $\Lambda_{n-1}/\Lambda_n = 1 - \lambda_n/\Lambda_n > \frac{1}{2}$ for $n \geq n_0$. Thus,

$$\infty > \sum_{n=n_0}^{\infty} \frac{\lambda_n}{\Lambda_n} \geq \frac{1}{2} \sum_{n=n_0}^{\infty} \frac{\lambda_n}{\Lambda_{n-1}} \geq \frac{1}{2} \sum_{n=n_0}^{\infty} \int_{\Lambda_{n-1}}^{\Lambda_n} \frac{1}{x} dx = \frac{1}{2} \int_{\Lambda_{n_0-1}}^{\Lambda_\infty} \frac{1}{x} dx.$$

As this integral is finite, we obtain $\Lambda_\infty < \infty$. □

Lemma 4.2. *If $\lambda_n/\Lambda_n \rightarrow 0$ and $\Lambda_n \rightarrow \infty$, then*

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{\max(\lambda_1, \dots, \lambda_n)}{\Lambda_n} = 0.$$

Proof. Fix $\varepsilon > 0$. There exists $k_0 \in \mathbb{N}$ such that $\lambda_k/\Lambda_k \leq \varepsilon$ for all $k > k_0$. Take $n_0 \geq k_0$ such that $\Lambda_{n_0} \geq \frac{1}{\varepsilon} \cdot \max(\lambda_1, \dots, \lambda_{k_0})$. Fix $n > n_0$ arbitrarily. Then

$$\begin{aligned} \frac{\lambda_k}{\Lambda_n} &\leq \frac{\max(\lambda_1, \dots, \lambda_{k_0})}{\Lambda_{n_0}} \leq \varepsilon, & k \in \{1, \dots, k_0\}, \\ \frac{\lambda_k}{\Lambda_n} &= \frac{\lambda_k}{\Lambda_k} \cdot \frac{\Lambda_k}{\Lambda_n} \leq \varepsilon \cdot 1 = \varepsilon, & k \in \{k_0 + 1, \dots, n\}. \end{aligned}$$

Therefore,

$$\frac{\max(\lambda_1, \dots, \lambda_n)}{\Lambda_n} \leq \varepsilon \quad \text{for every } n \geq n_0,$$

which completes the proof of the statement. □

Lemma 4.3. *Let $\varphi: (0, 1] \rightarrow \mathbb{R}$ be a continuous and nonincreasing function and $q \in (0, 1)$. Then the integral $\int_0^1 \varphi$ and the series $\sum_{k=1}^{\infty} q^k \varphi(q^k)$ are equiconvergent. Furthermore,*

$$(4.2) \quad \frac{q}{1-q} \int_0^1 \varphi \leq \sum_{k=1}^{\infty} q^k \varphi(q^k) \leq \frac{1}{1-q} \int_0^q \varphi.$$

Proof. If φ is constant then both the integral and the series are convergent. Therefore, replacing φ by $\varphi - \varphi(1)$ if necessary, we may assume that $\varphi(1) = 0$. Using the nonincreasingness of φ , for all $k \in \mathbb{N}$, we obtain

$$q \int_{q^k}^{q^{k-1}} \varphi \leq q \int_{q^k}^{q^{k-1}} \varphi(q^k) = (1-q)q^k \varphi(q^k) = \int_{q^{k+1}}^{q^k} \varphi(q^k) \leq \int_{q^{k+1}}^{q^k} \varphi.$$

Summing up these inequalities side by side, the inequality (4.2) follows. which proves the integrability of φ over $(0, 1]$. This inequality also shows the equiconvergence of the integral and the series. \square

Proposition 4.4. *Let $\varphi: (0, 1] \rightarrow \mathbb{R}$ be a continuous and monotone function. If $\Lambda_n \rightarrow \infty$ and the sequence $(\frac{\lambda_n}{\Lambda_n})$ is convergent with a limit η belonging to $[0, 1)$, then*

$$(4.3) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\lambda_k}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) = \begin{cases} \int_0^1 \varphi(x) dx & \text{if } \eta = 0, \\ \sum_{k=0}^{\infty} \eta(1-\eta)^k \varphi((1-\eta)^k) & \text{if } \eta \in (0, 1). \end{cases}$$

Proof. We may suppose without loss of generality that φ is nonincreasing. The equality (4.3) is obvious if φ is a constant function. Therefore, replacing φ by $\varphi - \varphi(1)$, we also can assume that $\varphi(1) = 0$ and then φ is nonnegative. Thus the integral and the sum of the series on the right hand side of formula (4.3) are well-defined, however, their value could be equal to $+\infty$.

Assume first that $\eta = 0$. For $n \in \mathbb{N}$, consider the partition $0 < \frac{\Lambda_1}{\Lambda_n} < \dots < \frac{\Lambda_n}{\Lambda_n} = 1$ of the interval $[0, 1]$. By Lemma 4.2, the mesh size of this partition tends to zero as $n \rightarrow \infty$. The sum on the left hand side of (4.3) is the Lebesgue integral of the step function φ_n defined as $\varphi_n(t) = \varphi(\Lambda_k/\Lambda_n)$ for $t \in (\Lambda_{k-1}/\Lambda_n, \Lambda_k/\Lambda_n]$. Due to the inequality $\varphi_n \leq \varphi$, we have that the left hand side of (4.3) is smaller than or equal to the right side. To prove the reversed inequality, let $c < \int_0^1 \varphi(x) dx$. Then, there exists $0 < \alpha < 1$ such that $c < \int_\alpha^1 \varphi(x) dx$. By the continuity of φ and (4.1), the sequence of functions φ_n pointwise converges to φ and the convergence is uniform on the interval $[\alpha, 1]$. Therefore the sequence of integrals $\int_\alpha^1 \varphi_n(x) dx$ converges to $\int_\alpha^1 \varphi(x) dx$. Thus, for large n , we have that

$$c < \int_\alpha^1 \varphi_n(x) dx \leq \int_0^1 \varphi_n(x) dx = \sum_{k=1}^n \frac{\lambda_k}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right).$$

This proves the reversed inequality in (4.3) in the case $\eta = 0$.

From now on let us assume that $\eta > 0$. We know that

$$\lim_{n \rightarrow \infty} \frac{\Lambda_{n-1}}{\Lambda_n} = 1 - \eta$$

and therefore, by simple induction,

$$\lim_{n \rightarrow \infty} \frac{\Lambda_{n-k}}{\Lambda_n} = (1 - \eta)^k, \quad k \in \mathbb{N}.$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n-k}}{\Lambda_n} = \lim_{n \rightarrow \infty} \frac{\Lambda_{n-k}}{\Lambda_n} - \frac{\Lambda_{n-k-1}}{\Lambda_n} = (1-\eta)^k - (1-\eta)^{k+1} = \eta \cdot (1-\eta)^k.$$

Therefore, for all $k \in \mathbb{N}$,

$$(4.4) \quad \lim_{n \rightarrow \infty} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) = \eta \cdot (1-\eta)^k \cdot \varphi((1-\eta)^k).$$

For all $n > m \geq 1$, we have

$$\sum_{k=1}^n \frac{\lambda_k}{\Lambda_n} \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) = \sum_{k=0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) \geq \sum_{k=0}^m \frac{\lambda_{n-k}}{\Lambda_n} \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right).$$

Thus, using (4.4), the above inequality implies

$$\liminf_{n \rightarrow \infty} \sum_{k=0}^n \frac{\lambda_k}{\Lambda_n} \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) \geq \lim_{n \rightarrow \infty} \sum_{k=0}^m \frac{\lambda_{n-k}}{\Lambda_n} \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) = \sum_{k=0}^m \eta(1-\eta)^k \cdot \varphi((1-\eta)^k).$$

Upon taking the limit $m \rightarrow \infty$, it follows that

$$\liminf_{n \rightarrow \infty} \sum_{k=0}^n \frac{\lambda_k}{\Lambda_n} \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) \geq \sum_{k=0}^{\infty} \eta(1-\eta)^k \cdot \varphi((1-\eta)^k).$$

This implies also the equality in (4.3) if the right-hand-side series is divergent. Therefore, in the rest of the proof, we can assume that this series is convergent.

Fix $\varepsilon > 0$ and choose k_0 such that

$$(4.5) \quad \sum_{k=k_0}^{\infty} \eta \cdot (1-\eta)^k \cdot \varphi((1-\eta)^k) \leq \frac{\varepsilon}{4} \quad \text{and} \quad \int_0^{(1-\eta)^{k_0}} \varphi(x) dx \leq \frac{\varepsilon}{4}.$$

Moreover, by (4.4), there exists n_0 such that for all $k \in \{0, 1, \dots, k_0 - 1\}$ and $n \geq n_0$

$$(4.6) \quad \left| \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) - \eta \cdot (1-\eta)^k \cdot \varphi((1-\eta)^k) \right| \leq \frac{\varepsilon}{4k_0}.$$

Now, applying the nonincreasingness of φ again, for all $n \geq k_0$,

$$(4.7) \quad \sum_{k=k_0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) = \sum_{k=1}^{n-k_0} \frac{\lambda_k}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) \leq \int_0^{\frac{\Lambda_{n-k_0}}{\Lambda_n}} \varphi(x) dx$$

But

$$\lim_{n \rightarrow \infty} \int_0^{\frac{\Lambda_{n-k_0}}{\Lambda_n}} \varphi(x) dx = \int_0^{(1-\eta)^{k_0}} \varphi(x) dx \leq \frac{\varepsilon}{4},$$

so there exists $n_1 \geq \max(n_0, k_0)$ such that

$$\int_0^{\frac{\Lambda_{n-k_0}}{\Lambda_n}} \varphi(x) dx \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq n_1.$$

Thus, by (4.7),

$$(4.8) \quad \sum_{k=k_0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) \leq \frac{\varepsilon}{2} \quad \text{for all } n \geq n_1.$$

Finally, applying (4.6), (4.8), and (4.5), for all $n \geq n_1$,

$$\begin{aligned} & \left| \sum_{k=1}^n \frac{\lambda_k}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) - \sum_{k=0}^{\infty} \eta(1-\eta)^k \varphi((1-\eta)^k) \right| \\ &= \left| \sum_{k=0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) - \sum_{k=0}^{\infty} \eta(1-\eta)^k \varphi((1-\eta)^k) \right| \\ &\leq \sum_{k=0}^{k_0-1} \left| \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) - \eta(1-\eta)^k \varphi((1-\eta)^k) \right| \\ &\quad + \sum_{k=k_0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) + \sum_{k=k_0}^{\infty} \eta(1-\eta)^k \varphi((1-\eta)^k) \\ &\leq k_0 \cdot \frac{\varepsilon}{4k_0} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

This completes the proof in the case $\eta \in (0, 1)$. □

It is worth mentioning that (4.2) with $q = 1 - \eta$ follows that

$$\lim_{\eta \rightarrow 0^+} \sum_{k=0}^{\infty} \eta(1-\eta)^k \varphi((1-\eta)^k) = \int_0^1 \varphi(x) dx,$$

which means, that the right hand side of (4.3) is a continuous function of η . Applying the above result to the power function $\varphi(x) = x^{-p}$ (where $p < 1$), we immediately get

Corollary 4.5. *If $p < 1$, furthermore, $\Lambda_n \rightarrow \infty$ and $\frac{\lambda_n}{\Lambda_n} \rightarrow \eta \in [0, 1)$, then*

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\lambda_k}{\Lambda_n} \cdot \left(\frac{\Lambda_k}{\Lambda_n}\right)^{-p} = \begin{cases} \frac{1}{1-p} & \eta = 0, \\ \frac{\eta}{1 - (1-\eta)^{1-p}} & \eta \in (0, 1). \end{cases}$$

Lemma 4.6. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a concave function such that $\text{sign}(f(x)) = \text{sign}(x - 1)$ holds for all $x \in \mathbb{R}_+$ and the function $x \mapsto f(1/x)$ is integrable over $(0, 1]$. Then the function F given by*

$$F(x, q) := \sum_{k=0}^{\infty} q^k f(q^{-k}x) \quad ((x, q) \in \mathbb{R}_+ \times (0, 1))$$

is well-defined, continuous and nondecreasing in its first variable. Furthermore, for all fixed $q \in (0, 1)$, the equation $F(x, q) = 0$ has a unique solution $x(q) \in (0, 1)$. The mapping $x(\cdot)$ so defined is continuous, and we have the following estimates:

$$(4.9) \quad \frac{q}{1-q} \int_0^{1/q} f\left(\frac{x}{t}\right) dt \leq F(x, q) \leq \frac{1}{1-q} \int_0^1 f\left(\frac{x}{t}\right) dt \quad (x, q) \in \mathbb{R}_+ \times (0, 1).$$

Proof. By elementary considerations, it follows from the concavity and the sign properties that f is nondecreasing on \mathbb{R}_+ and is strictly increasing on $(0, 1)$, furthermore, it is also continuous. It also follows from the concavity that the map

$$u \mapsto \frac{f(u) - f(1)}{u - 1} = \frac{f(u)}{u - 1}$$

is nonincreasing on $\mathbb{R}_+ \setminus \{1\}$. Therefore, if $1 < u_0 \leq u$, then

$$(4.10) \quad \frac{f(u)}{u} \leq \frac{f(u)}{u - 1} \leq \frac{f(u_0)}{u_0 - 1}.$$

To show that F is continuous, let $(x_0, q_0) \in \mathbb{R}_+ \times (0, 1)$ be fixed. The product $q_0^{-k}x$ is bigger than 1 for $k \geq k_0 := 1 + \lceil \log(x_0)/\log(q_0) \rceil$. Therefore, there exist $0 < x_* < x_0 < x^*$ and $0 < q_* < q_0 < q^* < 1$ such that $q^{-k}x > 1$ for all $(x, q) \in V := [x_*, x^*] \times [q_*, q^*]$ and $k \geq k_0$. The expression $\sum_{k=0}^{k_0-1} q^k f(q^{-k}x)$ being a finite sum of continuous functions is obviously continuous at (x_0, q_0) . Therefore, it suffices to show that tail sum

$$F_{k_0}(x, q) := \sum_{k=k_0}^{\infty} q^k f(q^{-k}x)$$

is also continuous at (x_0, q_0) . By the choice of k_0 , each term is positive for $(x, q) \in V$. By the nondecreasingness of f , for all $k \geq 0$ and $(x, q) \in \mathbb{R}_+ \times (0, 1)$, we clearly have

$$(4.11) \quad \begin{aligned} q^k f(q^{-k}x) &= \frac{1}{1-q} \int_{q^{k+1}}^{q^k} f\left(\frac{x}{t}\right) dt \leq \frac{1}{1-q} \int_{q^{k+1}}^{q^k} f\left(\frac{x}{t}\right) dt, \\ q^k f(q^{-k}x) &= \frac{q}{1-q} \int_{q^k}^{q^{k-1}} f\left(\frac{x}{t}\right) dt \geq \frac{q}{1-q} \int_{q^k}^{q^{k-1}} f\left(\frac{x}{t}\right) dt. \end{aligned}$$

Summarizing the first inequality for $k \geq k_0$, it follows that

$$\sum_{k=k_0}^{\infty} q^k f(q^{-k}x) \leq \frac{1}{1-q} \int_0^{q^{k_0}} f\left(\frac{x}{t}\right) dt < +\infty \quad (x, q) \in V,$$

which implies that the series on the left hand side is convergent. To prove that the sum of this series (i.e., $F_{k_0}(x, q)$) is a continuous function of (x, q) at (x_0, q_0) , it suffices to show that the convergence is uniform over V .

Observe that, for $k \geq k_0$ and $(x, q) \in V$, we have

$$u_0 := \frac{x_*}{(q^*)^k} \leq \frac{x}{q^k} := u.$$

Now using the inequality (4.10) for the above u_0 and u , it follows that

$$\begin{aligned} q^k f(q^{-k}x) &= x \frac{f(u)}{u} \leq x^* \frac{f(u_0)}{u_0 - 1} \\ &= \frac{x^*}{x_* - (q^*)^k} \cdot (q^*)^k f((q^*)^{-k}x_*) \leq \frac{x^*}{x_* - (q^*)^{k_0}} \cdot (q^*)^k f((q^*)^{-k}x_*). \end{aligned}$$

This inequality shows that, for $(x, q) \in V$ and $k \geq k_0$, the k th term of the series corresponding to $F_{k_0}(x, q)$ is majorized by a constant multiple of the corresponding term of the series for $F_{k_0}(x_*, q^*)$. Thus, in view of the Weierstrass M -test, the convergence of the series corresponding to $F_{k_0}(x, q)$ is uniform. By the continuity of each term of this series, it follows that the sum function is also continuous at (x_0, q_0) .

Thus, we have proved that F is a well-defined continuous function on $\mathbb{R}_+ \times (0, 1)$. Moreover, as f is nondecreasing on $(1, \infty)$ and strictly increasing on $(0, 1)$, we obtain that $F(\cdot, q)$ is nondecreasing on $(1, \infty)$ and strictly increasing on $(0, 1)$ (as all terms of the sum are nondecreasing and the very first of them is strictly increasing on $(0, 1)$).

Finally, in order to prove that $F(\cdot, q)$ has a unique zero note that $f(q^n) < f(q)$, and $f(q^k) < 0$ for all $k \in \{1, \dots, n-1\}$. Thus we get

$$F(q^n, q) = f(q^n) + \dots + q^{n-1}f(q) + q^n F(1, q) < f(q) + q^n F(1, q).$$

Therefore, for large n , we have that $F(q^n, q) < 0$. This, with the easy-to-see inequality $F(1, q) > 0$, implies that for all $q \in (0, 1)$, the equality $F(x, q) = 0$ has a solution $x = x(q) \in (0, 1)$.

To show that $x(q)$ depends continuously on q , let $q_0 \in (0, 1)$ be fixed and $0 < \varepsilon < \min(x(q_0), 1 - x(q_0))$. Since $F(x(q_0), q_0) = 0$, we have that

$$F(x(q_0) - \varepsilon, q_0) < 0 < F(x(q_0) + \varepsilon, q_0).$$

By the continuity of F , there exists $0 < \delta < \min(q_0, 1 - q_0)$ such that, for all $q \in (q_0 - \delta, q_0 + \delta)$,

$$F(x(q_0) - \varepsilon, q) < 0 < F(x(q_0) + \varepsilon, q).$$

Therefore, the uniquely defined value $x(q)$ must be between $x(q_0) - \varepsilon$ and $x(q_0) + \varepsilon$, that is, $|x(q) - x(q_0)| < \varepsilon$ for all $q \in (q_0 - \delta, q_0 + \delta)$.

Finally, if we sum up (both) inequalities side by side in (4.11) for all $k \in \{0, 1, \dots\}$, we easily obtain (4.9). \square

Proposition 4.7. *Let $\lambda \in W_0$ with $\Lambda_n \rightarrow \infty$, $\lambda_n/\Lambda_n \rightarrow \eta \in [0, 1)$ and let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a concave function such that $\text{sign}(f(x)) = \text{sign}(x - 1)$ holds for all $x \in \mathbb{R}_+$ and the function $x \mapsto f(1/x)$ is integrable over $(0, 1]$. Then $c := \mathcal{C}_\lambda(\mathcal{E}_f)$ is the unique solution of the equation*

$$(4.12) \quad \begin{aligned} \int_0^1 f\left(\frac{1}{cx}\right) dx &= 0 & \text{for } \eta = 0, \\ \sum_{k=0}^{\infty} (1 - \eta)^k f\left(\frac{1}{c(1 - \eta)^k}\right) &= 0 & \text{for } \eta > 0. \end{aligned}$$

Proof. The first equation in (4.12) is equivalent to

$$\int_0^c f\left(\frac{1}{x}\right) dx = 0,$$

which, by [38, Theorem 3.4] has a unique solution c in the interval $(1, \infty)$. On the other hand, putting $q := 1 - \eta$, the second equation in (4.12) is equivalent to the $F(1/c, 1 - \eta) = 0$, which, according to Lemma 4.6, also has a unique solution in the interval $(1, \infty)$.

Fix any $K \in (0, c)$. Then there exists $n_K \in \mathbb{N}$ such that

$$K < \mathcal{E}_f^n\left(\frac{\Lambda_n}{\Lambda_k}, \lambda_k\right) \quad \text{for all } n > n_K.$$

Equivalently,

$$0 < \sum_{k=1}^n \frac{\lambda_k}{\Lambda_n} f\left(\frac{\Lambda_n}{K\Lambda_k}\right) \quad \text{for all } n > n_K.$$

Then, with $\varphi_K(x) := f(\frac{1}{Kx})$, we have

$$0 < \sum_{k=1}^n \frac{\lambda_k}{\Lambda_n} \varphi_K\left(\frac{\Lambda_k}{\Lambda_n}\right) \quad \text{for all } K \in (0, c) \text{ and } n \geq n_K.$$

Observe that φ_K is a nonincreasing, continuous and integrable function on $(0, 1]$, therefore upon taking the limit $n \rightarrow \infty$ and using Proposition 4.4, it follows that

$$(4.13) \quad 0 \leq \begin{cases} \int_0^1 \varphi_K(x) dx & \text{if } \eta = 0, \\ \sum_{k=0}^{\infty} \eta(1-\eta)^k \varphi_K((1-\eta)^k) & \text{if } \eta \in (0, 1). \end{cases}$$

Similarly, for all $L \in (c, +\infty)$, there exist a sequence of integers $n_i \rightarrow \infty$, such that, for all $i \in \mathbb{N}$,

$$0 > \sum_{k=1}^{n_i} \frac{\lambda_k}{\Lambda_{n_i}} \varphi_L\left(\frac{\Lambda_k}{\Lambda_{n_i}}\right).$$

Upon taking the limit $i \rightarrow \infty$ and again using Proposition 4.4, we obtain that

$$(4.14) \quad 0 \geq \begin{cases} \int_0^1 \varphi_L(x) dx & \text{if } \eta = 0, \\ \sum_{k=0}^{\infty} \eta(1-\eta)^k \varphi_L((1-\eta)^k) & \text{if } \eta \in (0, 1). \end{cases}$$

Combining the first inequalities from (4.13) and (4.14), in the case $\eta = 0$, we get

$$\int_0^L f\left(\frac{1}{x}\right) dx = \int_0^1 f\left(\frac{1}{Lx}\right) dx \leq 0 \leq \int_0^1 f\left(\frac{1}{Kx}\right) dx = \int_0^K f\left(\frac{1}{x}\right) dx,$$

while, for $\eta \in (0, 1)$, we obtain

$$\begin{aligned} \eta F(L^{-1}, 1-\eta) &= \sum_{k=0}^{\infty} \eta(1-\eta)^k f\left(\frac{1}{L(1-\eta)^k}\right) \\ &\leq 0 \leq \sum_{k=0}^{\infty} \eta(1-\eta)^k f\left(\frac{1}{K(1-\eta)^k}\right) = \eta F(K^{-1}, 1-\eta). \end{aligned}$$

If we now take the common limits $K \nearrow c$ and $L \searrow c$, and we use the continuity of F established in Lemma 4.6, we get (4.12). \square

5. APPLICATIONS

Now we are going to present some weighted Hardy constants for quasiarithmetic means. It is well known that for $\pi_p(x) := x^p$ if $p \neq 0$ and $\pi_0(x) := \ln x$ equality $\mathcal{A}_{\pi_p} = \mathcal{P}_p$ holds. Furthermore, the comparability problem within this family can be (under natural smoothness assumptions) boiled down to pointwise comparability of the mapping $f \mapsto \frac{f''}{f'}$ (cf. [14]). More precisely, we have

Proposition 5.1. *Let $I \subset \mathbb{R}$ be an interval, $f, g: I \rightarrow \mathbb{R}$ be twice differentiable functions having nowhere vanishing first derivatives. Then the following two conditions are equivalent*

- (i) $\mathcal{A}_f(x_1, \dots, x_n) \leq \mathcal{A}_g(x_1, \dots, x_n)$ for all $n \in \mathbb{N}$ and vector $(x_1, \dots, x_n) \in I^n$;
- (ii) $\frac{f''(x)}{f'(x)} \leq \frac{g''(x)}{g'(x)}$ for all $x \in I$.

In a special case $I \subseteq \mathbb{R}_+$ condition (ii) can be equivalently written as

$$\chi_f(x) := \frac{xf''(x)}{f'(x)} + 1 \leq \frac{xg''(x)}{g'(x)} + 1 =: \chi_g(x) \quad (x \in I).$$

It is easy to verify that the equality $\chi_{\pi_p} \equiv p$ holds for all $p \in \mathbb{R}$. Therefore, in view of Proposition 5.1, we have

$$\mathcal{P}_q = \mathcal{A}_{\pi_q} \leq \mathcal{A}_f \leq \mathcal{A}_{\pi_p} = \mathcal{P}_p,$$

where $q := \inf_I \chi_f$ and $p := \sup_I \chi_f$, moreover these parameters are sharp. In other words, the operator $\chi_{(\cdot)}$ could be applied to embed quasiarithmetic means into the scale of power means (cf. [25]).

This fact will be used to establish some weighted Hardy constants for quasiarithmetic means. Our main idea is to compare a quasiarithmetic mean with a suitable power mean. As a matter of fact, this is not so restrictive as it seems to be at first glance. Namely, Mulholland [24] proved that a quasiarithmetic mean is Hardy if and only if it is majorized up to a constant number by some power mean with parameter strictly smaller than one. Throughout this section, we will use the already introduced notation $\Lambda_n := \lambda_1 + \dots + \lambda_n$.

Proposition 5.2. *Let $(\lambda_n) \in W_0$ such that $\Lambda_n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \lambda_n / \Lambda_n =: \eta$ exists. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be a twice continuously differentiable function with a nonvanishing first derivative and define*

$$q := \liminf_{x \rightarrow 0^+} \chi_f(x) \leq \limsup_{x \rightarrow 0^+} \chi_f(x) =: p.$$

Assume that $p < 1$. Then, for all $x \in \mathbb{R}_+$,

$$\begin{aligned} (5.1) \quad C(q, \eta) &\leq \liminf_{n \rightarrow \infty} \frac{\Lambda_n}{x} \mathcal{A}_f \left(\left(\frac{x}{\Lambda_1}, \frac{x}{\Lambda_2}, \dots, \frac{x}{\Lambda_n} \right), (\lambda_1, \dots, \lambda_n) \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{\Lambda_n}{x} \mathcal{A}_f \left(\left(\frac{x}{\Lambda_1}, \frac{x}{\Lambda_2}, \dots, \frac{x}{\Lambda_n} \right), (\lambda_1, \dots, \lambda_n) \right) \leq C(p, \eta), \end{aligned}$$

where the function $C : (-\infty, 1) \times [0, 1) \rightarrow \mathbb{R}$ is defined by

$$(5.2) \quad C(r, \eta) := \begin{cases} \left(\frac{\eta}{1 - (1 - \eta)^{1-r}} \right)^{1/r}, & \eta \in (0, 1) \text{ and } r \neq 0; \\ (1 - \eta)^{1-1/\eta}, & \eta \in (0, 1) \text{ and } r = 0; \\ (1 - r)^{-1/r}, & \eta = 0 \text{ and } r \neq 0; \\ e, & \eta = 0 \text{ and } r = 0. \end{cases}$$

Proof. It is elementary to see that C is a continuous function which is strictly increasing in its first variable.

Following the lines of proof of [39, Theorem 3.1] we get that for all $r \in (p, 1) \setminus \{0\}$,

$$\begin{aligned} U &:= \limsup_{n \rightarrow \infty} \frac{\Lambda_n}{x} \cdot \mathcal{A}_f \left(\left(\frac{x}{\Lambda_1}, \frac{x}{\Lambda_2}, \dots, \frac{x}{\Lambda_n} \right), (\lambda_1, \dots, \lambda_n) \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{\Lambda_n}{x} \cdot \mathcal{P}_r \left(\left(\frac{x}{\Lambda_1}, \frac{x}{\Lambda_2}, \dots, \frac{x}{\Lambda_n} \right), (\lambda_1, \dots, \lambda_n) \right). \end{aligned}$$

Therefore, as \mathcal{P}_r is homogeneous, we obtain

$$U \leq \limsup_{n \rightarrow \infty} \mathcal{P}_r \left(\left(\frac{\Lambda_n}{\Lambda_1}, \frac{\Lambda_n}{\Lambda_2}, \dots, \frac{\Lambda_n}{\Lambda_n} \right), (\lambda_1, \dots, \lambda_n) \right) = \limsup_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{\lambda_k}{\Lambda_n} \cdot \left(\frac{\Lambda_k}{\Lambda_n} \right)^{-r} \right)^{1/r}.$$

Thus, due to Corollary 4.5, we obtain $U \leq C(r, \eta)$ for $r \in (p, 1) \setminus \{0\}$. Now, as the function C is continuous, we can pass the limit $r \searrow p$ and obtain $U \leq C(p, \eta)$. The verification of the left hand side inequality in (5.1) is completely analogous. \square

We will now establish some λ -Hardy constants in a family of quasiarithmetic means.

Corollary 5.3. *Let $(\lambda_n) \in W_0$ such that $\Lambda_n \rightarrow \infty$ and $(\frac{\lambda_n}{\Lambda_n})_{n=1}^\infty$ is nonincreasing with a limit $\eta \in [0, 1)$. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a twice continuously differentiable function with a nonvanishing first derivative, such that the limit*

$$p := \lim_{x \rightarrow 0^+} \chi_f(x)$$

exists, is smaller than 1, and $\chi_f(x) \leq p$ for all $x \in \mathbb{R}_+$. Then $\mathcal{H}_\lambda(\mathcal{A}_f) = C(p, \eta)$, where the function C was defined by (5.2).

Proof. By Corollary 3.6 and Proposition 5.2 we have

$$\mathcal{H}_\lambda(\mathcal{A}_f) \geq \sup_{x > 0} \liminf_{n \rightarrow \infty} \frac{\Lambda_n}{x} \mathcal{A}_f \left(\left(\frac{x}{\Lambda_1}, \dots, \frac{x}{\Lambda_n} \right), (\lambda_1, \dots, \lambda_n) \right) = C(p, \eta).$$

Furthermore, by $\chi_f(x) \leq p$ we get $\mathcal{A}_f \leq \mathcal{P}_p$ so

$$\mathcal{H}_\lambda(\mathcal{A}_f) \leq \mathcal{H}_\lambda(\mathcal{P}_p).$$

But \mathcal{P}_p is repetition invariant and concave, thus it is a λ -Kedlaya mean (in the sense of our paper [37]). Thus, by Proposition 3.7 and Proposition 5.2,

$$\mathcal{H}_\lambda(\mathcal{P}_p) \leq \liminf_{n \rightarrow \infty} \frac{\Lambda_n}{x} \mathcal{P}_p \left(\left(\frac{x}{\Lambda_1}, \dots, \frac{x}{\Lambda_n} \right), (\lambda_1, \dots, \lambda_n) \right) = C(p, \eta).$$

Binding all these inequalities, we get

$$C(p, \eta) \leq \mathcal{H}_\lambda(\mathcal{A}_f) \leq \mathcal{H}_\lambda(\mathcal{P}_p) \leq C(p, \eta),$$

which implies $\mathcal{H}_\lambda(\mathcal{A}_f) = C(p, \eta)$. \square

Theorem 5.4. *Let $(\lambda_n) \in W_0$ such that $\Lambda_n \rightarrow \infty$ and $(\frac{\lambda_n}{\Lambda_n})_{n=1}^\infty$ is nonincreasing with limit $\eta \in [0, 1)$. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a concave function such that $\text{sign}(f(x)) = \text{sign}(x - 1)$ holds for all $x \in \mathbb{R}_+$. Then the homogeneous quasideviation mean \mathcal{E}_f is λ -Hardy if and only if function $x \mapsto f(1/x)$ is integrable over $(0, 1]$. In the latter case, $c := \mathcal{H}_\lambda(\mathcal{E}_f)$ is the unique solution of the equation (4.12).*

Proof. Assume that \mathcal{E}_f is λ -Hardy. Then, by Corollary 3.6,

$$\liminf_{n \rightarrow \infty} \mathcal{E}_f^n \left(\frac{\Lambda_n}{\Lambda_k}, \lambda_k \right) = \mathcal{C}_\lambda(\mathcal{E}_f) \leq \mathcal{H}_\lambda(\mathcal{E}_f) < \infty.$$

Then, there exists a strictly increasing sequence of integers (n_i) such that

$$\mathcal{E}_f^{n_i} \left(\frac{\Lambda_{n_i}}{\Lambda_k}, \lambda_k \right) < \mathcal{H}_\lambda(\mathcal{E}_f) + 1 =: K \quad (i \in \mathbb{N}),$$

which is equivalent to the inequality

$$\sum_{k=1}^{n_i} \frac{\lambda_k}{\Lambda_{n_i}} f \left(\frac{\Lambda_{n_i}}{K \Lambda_k} \right) < 0 \quad (i \in \mathbb{N}).$$

Applying now Proposition 4.4 for the nonincreasing function $\varphi(x) := f(\frac{1}{Kx})$, upon taking the limit $i \rightarrow \infty$, in the case when $\eta = 0$, it follows that

$$\int_0^1 f \left(\frac{1}{Kx} \right) dx \leq 0,$$

while in the case $\eta \in (0, 1)$, we get that

$$\sum_{k=0}^{\infty} \eta(1-\eta)^k f \left(\frac{1}{K(1-\eta)^k} \right) \leq 0.$$

The first inequality implies that φ is integrable over $(0, 1]$, hence the mapping $x \mapsto f(1/x)$ is also integrable on $(0, 1]$. In view of Lemma 4.3, the same conclusion is derived from the second inequality.

In the rest of the proof, assume that the mapping $x \mapsto f(1/x)$ is also integrable on $(0, 1]$. Obviously \mathcal{E}_f is a homogeneous, symmetric and continuously weighted mean. Moreover, in view of Lemma 2.3, \mathcal{E}_f is monotone and Jensen concave. Thus, by Proposition 3.7, $\mathcal{H}_\lambda(\mathcal{E}_f) = \mathcal{C}_\lambda(\mathcal{E}_f)$. Consequently, applying Proposition 4.7, one obtains that $c = \mathcal{H}_\lambda(\mathcal{E}_f)$ is a unique and finite solution of equation (4.12), indeed. In particular, this yields, that \mathcal{E}_f is a λ -Hardy mean. \square

An interesting consequence of the previous result is that a homogeneous quasideviation mean \mathcal{E}_f is λ -Hardy (where λ is like above) if and only if it is **1**-Hardy.

One of our main results is stated in the subsequent theorem.

Theorem 5.5. *Let $(\lambda_n) \in W_0$ such that $\Lambda_n \rightarrow \infty$ and $(\frac{\lambda_n}{\Lambda_n})_{n=1}^\infty$ is nonincreasing with limit $\eta \in [0, 1)$. Let $E : I \times I \rightarrow \mathbb{R}$ be a normalizable quasideviation such that E^* is concave. Assume that, for all $x \in \mathbb{R}_+$, $\lim_{t \rightarrow 0} E^*(xt, t) = 0$ and define $h_E : \mathbb{R}_+ \rightarrow \mathbb{R}$ by (2.4). Then the quasideviation mean \mathcal{D}_E is λ -Hardy if and only if the mapping $x \mapsto h_E(1/x)$ is integrable over $(0, 1]$ and in this case, $c := \mathcal{H}_\lambda(\mathcal{D}_E)$ is a unique solution of (4.12) with $f := h_E$.*

Proof. First, by Lemma 2.4 we know that $f := h_E$ is correctly defined. Furthermore it is nondecreasing on $(0, \infty)$, strictly increasing on $(0, 1)$, and admits the sign property $\text{sign}(f(x)) = \text{sign}(x - 1)$ and \mathcal{E}_f is a homogeneous quasideviation mean.

First assume that \mathcal{D}_E is a λ -Hardy mean. Then, by Theorem 3.4, $(\mathcal{D}_E)_\#$ is also a λ -Hardy mean. On the other hand, Lemma 2.4 implies that $(\mathcal{D}_E)_\# = \mathcal{E}_f$, hence, we get that \mathcal{E}_f is a λ -Hardy mean, too. By the previous theorem, this implies that the mapping $x \mapsto f(1/x)$ is integrable over $(0, 1]$.

In the rest of the proof, assume that the mapping $x \mapsto f(1/x)$ is integrable over $(0, 1]$. In view of Proposition 4.7, we have that $c := \mathcal{H}_\lambda(\mathcal{E}_f)$ is a unique solution of (4.12). On the other hand, by Theorem 3.4, we have that $\mathcal{H}_\lambda(\mathcal{D}_E) = \mathcal{H}_\lambda(\mathcal{E}_f)$, which yields that $\mathcal{H}_\lambda(\mathcal{D}_E) = c$. \square

Corollary 5.6. *Let $(\lambda_n) \in W_0$ such that $\Lambda_n \rightarrow \infty$ and $(\frac{\lambda_n}{\Lambda_n})_{n=1}^\infty$ is nonincreasing with a limit $\eta \in [0, 1)$. Let $p, q \in \mathbb{R}$, $\min(p, q) \leq 0 \leq \max(p, q) < 1$. Then*

$$\mathcal{H}_\lambda(\mathcal{G}_{p,q}) = \begin{cases} \left(\frac{1 - (1 - \eta)^{1-q}}{1 - (1 - \eta)^{1-p}} \right)^{\frac{1}{p-q}}, & \eta \in (0, 1) \text{ and } p \neq q; \\ \left(\frac{1 - q}{1 - p} \right)^{\frac{1}{p-q}}, & \eta = 0 \text{ and } p \neq q; \\ (1 - \eta)^{1-1/\eta}, & \eta \in (0, 1) \text{ and } p = q = 0; \\ e, & \eta = 0 \text{ and } p = q = 0. \end{cases}$$

Proof. Fix p, q like above. In the case $p = 0$ (resp. $q = 0$), we have $\mathcal{G}_{p,q} = \mathcal{P}_q$ (resp. $\mathcal{G}_{p,q} = \mathcal{P}_p$) and the assertion is implied by Corollary 5.3. As $\mathcal{G}_{p,q} = \mathcal{G}_{q,p}$ and the right hand side is symmetric, we can assume that $p < 0 < q < 1$.

Observe that Gini means are homogeneous deviation means – more precisely $\mathcal{G}_{p,q} = \mathcal{E}_f$ with $f(x) = \frac{x^p - x^q}{p - q}$. The condition $p < 0 < q < 1$ implies that f is concave, satisfies the sign condition and the mapping $x \mapsto f(1/x)$ is integrable. Therefore, Theorem 5.4 yields that $\mathcal{H}_\lambda(\mathcal{G}_{p,q})$ is the unique solution c of equation (4.12).

Let us now split our considerations into two parts. For $\eta = 0$, we have

$$0 = \int_0^1 f\left(\frac{1}{cx}\right) dx = \int_0^1 \frac{c^{-p}}{p-q} x^{-p} - \frac{c^{-q}}{p-q} x^{-q} dx = \frac{1}{p-q} \cdot \left(\frac{c^{-p}}{1-p} - \frac{c^{-q}}{1-q} \right),$$

which, after an easy transformation, is equivalent to $c = \left(\frac{1-q}{1-p}\right)^{1/(p-q)}$.

For $\eta > 0$, we need to solve the second equation of (4.12), which in our setting states

$$\frac{1}{p-q} \cdot \sum_{k=0}^{\infty} (1-\eta)^k (c^{-p}(1-\eta)^{-kp} - c^{-q}(1-\eta)^{-kq}) = 0.$$

As $\eta \in (0, 1)$, we can calculate the sums of the geometric series to obtain

$$\frac{1}{p-q} \cdot \left(\frac{c^{-p}}{1 - (1-\eta)^{1-p}} - \frac{c^{-q}}{1 - (1-\eta)^{1-q}} \right) = 0.$$

As $p \neq q$ and $c \neq 0$, it implies

$$c^{p-q} = \frac{1 - (1-\eta)^{1-q}}{1 - (1-\eta)^{1-p}},$$

and yields the assertion in the last case. \square

Remark. It is worth mentioning that (except the case $p = q = 0$) we have the equality $\mathcal{H}_\lambda(\mathcal{G}_{p,q}) = C(p, \eta)^{p/(p-q)} C(q, \eta)^{q/(q-p)}$. As a matter of fact, this assertion could be obtained using a similar identity: $\mathcal{G}_{p,q}(x, \lambda) = \mathcal{P}_p(x, \lambda)^{p/(p-q)} \mathcal{P}_q(x, \lambda)^{q/(q-p)}$, which is valid for all $p, q \in \mathbb{R}$, $p \neq q$ and all admissible pairs (x, λ) .

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