THE HOLONOMY GROUP OF PROJECTIVELY FLAT RANDERS TWO-MANIFOLDS OF CONSTANT CURVATURE

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ABSTRACT. In this paper, we investigate the holonomy structure of the most accessible and demonstrative 2-dimensional Finsler surfaces, the Randers surfaces. Randers metrics can be considered as the solutions of the Zermelo navigation problem. We give the classification of the holonomy groups of locally projectively flat Randers two-manifolds of constant curvature. In particular we prove that the holonomy group of a simply connected non-Riemannian projectively flat Finsler two-manifold of constant non-zero flag curvature is maximal and isomorphic to the orientation preserving diffeomorphism group of the circle.

1. INTRODUCTION

The holonomy group of a Riemann or Finsler manifold is a natural geometric object: it is the transformation group generated by parallel translations along loops with respect to the associated canonical (linear, resp. homogeneous) connection. The Riemannian holonomy groups have been extensively studied during the second half of the last century but, maybe because of the computational difficulties, little attention was paid to the Finslerian case. Although there are similarities, the holonomy properties of Finsler manifolds can be very different from the Riemannian ones, as recent results show. Indeed, the fundamental result of Borel and Lichnerowicz [3] from 1952 claims that the holonomy group of a simply connected Riemannian manifold is a (finite dimensional) closed Lie subgroup of the special orthogonal group $SO(n)$. In contrast with this, in [7] it has been proven that the holonomy group of an at least three-dimensional non-Riemannian Finsler manifold of nonzero constant curvature is not a compact Lie group. In [10] it has been shown that there exist large families of projectively flat Finsler manifolds of constant curvature such that their holonomy groups are not finite dimensional Lie groups. In [8] projective Finsler manifolds of constant curvature having infinite dimensional holonomy group have been characterized. The proofs in the above-mentioned papers [7, 8, 10] give estimates for the dimension of tangent Lie algebras of the holonomy group but unfortunately, they do not give direct information on the structure of the holonomy groups.

In contrast to the Riemannian case where the complete classification is already known, Finsler holonomy groups are described only for few, very special classes of metrics: the holonomy of Landsberg manifolds have been investigated in [6] and the holonomy of Berwald manifolds have been characterized in [13]. First examples describing infinite dimensional Finslerian holonomy groups can be found in [9]. To achieve further significant progress, it would be very important to investigate systematically the holonomy structure of different classes of Finsler metrics.

In this article, we consider one of the most accessible and demonstrative examples for non-Riemannian Finsler manifolds, the Randers manifolds [1, 11]. In the Randers case, the Finsler function is a Riemann norm deformed by a 1-form. As it has been proven in [2], Randers metrics describe the Zermelo navigation problem on Riemannian manifolds. This fact may suggest that the holonomy structure of Randers manifolds are similar to that of Riemann manifolds but, on the contrary: they can be very different, as the results of this paper show. We focus our attention on the holonomy properties of simply connected, locally projectively flat Randers two-manifolds

of constant flag curvature λ. This class of manifolds was already considered in [10] where it has been proved that the holonomy group of such a manifold is finite dimensional if $\lambda = 0$ or the metric is Riemannian, and *infinite dimensional* if $\lambda \neq 0$ and the metric is non-Riemannian. The finite dimensional holonomy structures are already well known, but nothing was known about the infinite dimensional case: the results reveal no information about the structure of the holonomy group.

The goal of this paper is to complete the results of [10] by describing the infinite dimensional holonomy structure of projectively flat Randers surfaces of non-zero constant curvature. The main result (Theorem 3.) shows that the holonomy group of a such manifold is maximal and its closure is isomorphic to $Diff_+(\mathbb{S}^1)$, the orientation preserving diffeomorphism group of the circle $\mathbb{S}^1$. This result is surprising because it shows that even in the case when the geodesic structure is simple (the geodesics are straight lines), the holonomy group can still be a very large group. Finally, we obtain the classification of the holonomy groups of projectively flat Randers surfaces (Corollary 4).

2. Preliminaries

Throughout this article, M is a C^∞ smooth manifold, $\mathfrak{X}^\infty(M)$ denotes the Lie algebra of smooth vector fields on M and $\mathsf{Diff}^\infty(M)$ is the group of C^∞-diffeomorphisms of M. The first and the second tangent bundles of M are denoted by (TM, π, M) and (TTM, τ, TM), respectively.

2.1. Finsler manifolds.

A *Finsler manifold* is a pair $(M, \mathcal{F})$, where the norm function $\mathcal{F} \colon TM \to \mathbb{R}_+$ is continuous, smooth on $\hat{T}M := TM \setminus \{0\}$, its restriction $\mathcal{F}_x = \mathcal{F}|_{T_x M}$ is a positively homogeneous function of degree one and the symmetric bilinear form

$$g_{x,y} \colon (u, v) \;\mapsto\; g_{ij}(x, y) u^i v^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}_x^2(y + su + tv)}{\partial s \, \partial t} \Big|_{t=s=0}$$

is positive definite at every $y \in \hat{T}_x M$.

Geodesics of $(M, \mathcal{F})$ are determined by a system of 2nd order ordinary differential equation $\ddot{x}^i + 2G^i(x, \dot{x}) = 0$, $i = 1, \dots, n$ in a usual local coordinate system (x^i, y^i) of TM, where $G^i(x, y)$ are given by

$$(1) \qquad G^i(x, y) := \frac{1}{4} g^{il}(x, y) \Big(2 \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y) \Big) y^j y^k.$$

A vector field $X(t) = X^i(t) \frac{\partial}{\partial x^i}$ along a curve $c(t)$ is said to be parallel with respect to the associated *homogeneous (nonlinear) connection* if it satisfies

$$(2) \qquad D_{\dot{c}} X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t), X(t)) \dot{c}^j(t) \Big) \frac{\partial}{\partial x^i} = 0,$$

where $G^i_j = \frac{\partial G^i}{\partial y^j}$.

The *horizontal Berwald covariant derivative* $\nabla_X \xi$ of $\xi(x, y) = \xi^i(x, y) \frac{\partial}{\partial y^i}$ by the vector field $X(x) = X^i(x) \frac{\partial}{\partial x^i}$ is expressed locally by

$$(3) \qquad \nabla_X \xi = \left(\frac{\partial \xi^i(x, y)}{\partial x^j} - G^k_j(x, y) \frac{\partial \xi^i(x, y)}{\partial y^k} + G^i_{jk}(x, y) \xi^k(x, y) \right) X^j \frac{\partial}{\partial y^i},$$

where we denote $G^i_{jk}(x, y) := \frac{\partial G^i_j(x, y)}{\partial y^k}$.

2.1.1. *Curvatures of Finsler manifolds.*

The *curvature tensor* field $R = R^i_{jk} dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^i}$ has the expression

$$(4) \qquad R^i_{jk} = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y)G^i_{km}(x,y) - G^m_k(x,y)G^i_{jm}(x,y).$$

The *Riemannian curvature tensor* is $R_y := R(\cdot, y)$, its components can be obtained as $R^i_j = R^i_{jk}y^k$. The Ricci curvature $Ric(y)$ is defined to be the trace of R_y, $Ric(y) := R^m_m(x,y)$. For a tangent plane $P = Span\{y, u\} \subset T_xM$, the flag curvature is defined as

$$\mathbf{K}(P, y) = \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}.$$

If a manifold has *constant flag curvature* $\mathbf{K} = \lambda \in \mathbb{R}$, then the Ricci curvature is constant in the sense that $Ric(y) = (n-1)\lambda F^2$ and the local expression of the coefficients of the curvature is

$$R^i_{jk} = \lambda\big(\delta^i_k g_{jm}(x,y)y^m - \delta^i_j g_{km}(x,y)y^m\big),$$

where δ^i_j is the Kronecker delta symbol. Assume that the Finsler manifold $(M, \mathcal{F})$ is locally projectively flat. Then for every point $x \in M$ there exists an *adapted* local coordinate system, that is a mapping $(x^1, \ldots, x^n)$ on a neighbourhood U of x into the Euclidean space $\mathbb{R}^n$, such that the straight lines of $\mathbb{R}^n$ correspond to the geodesics of $(M, \mathcal{F})$. Then the *geodesic coefficients* are of the form

$$(5) \qquad G^i = \mathcal{P}y^i, \quad G^i_k = \frac{\partial \mathcal{P}}{\partial y^k}y^i + \mathcal{P}\delta^i_k, \quad G^i_{kl} = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l}y^i + \frac{\partial \mathcal{P}}{\partial y^k}\delta^i_l + \frac{\partial \mathcal{P}}{\partial y^l}\delta^i_k$$

where $\mathcal{P}(x, y)$ is a 1-homogeneous function in y, called the *projective factor* of $(M, \mathcal{F})$. According to Lemma 8.2.1 in [4, p.155], if $(M \subset \mathbb{R}^n, \mathcal{F})$ is a projectively flat manifold, then its projective factor can be computed using the formula

$$(6) \qquad \mathcal{P}(x, y) = \frac{1}{2\mathcal{F}}\frac{\partial \mathcal{F}}{\partial x^i}y^i.$$

2.1.2. *Projectively flat Randers manifolds with constant curvature.*

Projectively flat Randers manifolds with constant flag curvature were classified by Z. Shen in [12]. He proved that any projectively flat Randers manifold $(M, \mathcal{F})$ with non-zero constant flag curvature has negative curvature. These metrics can be normalized by a constant factor so that the curvature is $\lambda = -1/4$. In this case $(M, \mathcal{F})$ is isometric to the Finsler manifold defined by the Finsler function

$$(7) \qquad \mathcal{F}_a(x, y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} + \epsilon\left(\frac{\langle x, y\rangle}{1 - |x|^2} + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\right)$$

on the unit ball $\mathbb{D}^n \subset \mathbb{R}^n$, where $a \in \mathbb{R}^n$ is any constant vector with $|a| < 1$ and $\epsilon = \pm 1$ ([12, Theorem 1.1]). We note that the restriction of any orthogonal transformation $\phi \in \mathbb{O}(n, \mathbb{R}^n)$ on $\mathbb{D}^n$ does not change the Finsler function (7), therefore one can assume that $a \in \mathbb{R}^n$ has the form $a = (a_1, 0, \ldots, 0)$. We can consider $(\mathbb{D}^n, \mathcal{F}_a)$ as the *standard model* of projectively flat Randers manifolds with non-zero constant flag curvature.

We remark that the computation of the coefficients of the associated connection is relatively easy: according to Lemma 8.2.1 of [4, p.155], the projective factor $\mathcal{P}(x, y)$ can be computed by the formula (6) which gives in the case (7)

$$(8) \qquad \mathcal{P}(x, y) = \frac{1}{2}\left(\frac{\epsilon\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2} - \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\right).$$

The geodesic coefficients and the connection coefficients can be computed from (8) by using (5).

2.2. **Holonomy group and its tangent Lie algebras.**

The *holonomy group* $\mathsf{Hol}_x(M)$ of the Finsler manifold $(M, \mathcal{F})$ at a point $x \in M$ is the subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(T_x M)$ generated by parallel translations along piece-wise differentiable closed curves initiated and ended at the point $x \in M$. Since the canonical parallel translation is homogeneous and preserves the Finsler function, the holonomy group can be considered on the *indicatrix* $\mathcal{I}_x := \{\, y \in T_x M \mid \mathcal{F}(y) = 1 \,\}$. That way $\mathsf{Hol}_x(M)$ is a subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(\mathcal{I}_x)$. The topological closure $\overline{\mathsf{Hol}_x(M)}$ of the holonomy group in the Fréchet topology of $\mathsf{Diff}^\infty(\mathcal{I}_x)$ is called the *closed holonomy group*.

2.2.1. *The curvature algebra.*

A vector field $\xi \in \mathfrak{X}(TM)$ is called a *curvature vector field* of $(M, \mathcal{F})$ if it is in the image of the curvature tensor, that is $\xi = R(X, Y)$ for some $X, Y \in \mathfrak{X}(M)$. The *curvature algebra* $\mathfrak{R}(M)$ is the Lie algebra generated by curvature vector fields. Its restriction

$$\mathfrak{R}_x(M) := \{\, \xi|_{\mathcal{I}_x} \mid \xi \in \mathfrak{R}(M) \,\}$$

is the curvature algebra at the point $x \in M$. $\mathfrak{R}_x(M)$ is a Lie subalgebra of $\mathfrak{X}^\infty(\mathcal{I}_x)$.

2.2.2. *The infinitesimal holonomy algebra.*

The *infinitesimal holonomy algebra*, denoted by $\mathfrak{hol}^*(M)$ is the smallest Lie algebra generated by curvature vector fields and by the horizontal Berwald covariant derivation. More precisely, $\mathfrak{hol}^*(M)$ is the smallest Lie algebra of vector fields on TM satisfying the following two conditions:

  (1) for any curvature vector field ξ we have $\xi \in \mathfrak{hol}^*(M)$,

  (2) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$ then $\nabla_X \xi \in \mathfrak{hol}^*(M)$.

By considering the restriction of $\mathfrak{hol}^*(M)$ on the indicatrix $\mathcal{I}_x$ we obtain the infinitesimal holonomy algebra at the point $x \in M$:

$$\mathfrak{hol}^*_x(M) := \{\, \xi|_{\mathcal{I}_x} \mid \xi \in \mathfrak{hol}^*(M) \,\}.$$

We remark that if the manifold is two-dimensional, then the curvature algebra $\mathfrak{R}_x(M)$ is at most one-dimensional, but the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ can be higher, even infinite dimensional: $\mathfrak{hol}^*_x(M)$ is generated by the restriction of the curvature vector field and its covariant derivatives. Using the notation $\xi_0 = R(\partial_{x_1}, \partial_{x_2})\big|_{\mathcal{I}_x}$ and $\nabla_{i_1, \ldots, i_k} \xi_0 := (\nabla_{\partial_{x_{i_1}}} \ldots \nabla_{\partial_{x_{i_k}}} \xi)\big|_{\mathcal{I}_x}$ we get

$$(9) \qquad \mathfrak{hol}^*_x(M) = \left\langle \xi_0, \nabla_1 \xi_0, \nabla_2 \xi_0, \nabla_{11} \xi_0, \ldots \right\rangle_{\mathcal{L}ie}$$

Property (Theorem 4, [7]): At any point $x \in M$ the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$.

3. Holonomy of projectively flat Randers two-manifolds of non-zero constant curvature

Our aim is to describe the holonomy structure of projectively flat non-Riemannian Randers two-manifolds with non-zero constant flag curvature. As a first step, we investigate the holonomy of the standard model described in Subsection 2.1.2.

Let $(\mathbb{D}^2, \mathcal{F}_a)$ be the Finsler two-manifold where $\mathbb{D}^2$ is the unit ball in $\mathbb{R}^2$ and $\mathcal{F}_a$ is the Finsler function given by (7) where $a = (a_1, 0) \in \mathbb{R}^2$ is a nonzero constant vector with $|a_1| < 1$. We have the following

Proposition 1. *The holonomy group of $(\mathbb{D}^2, \mathcal{F}_a)$ is maximal and $\overline{\mathsf{Hol}_x(M)}$ is diffeomorphic to* $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$.

Proof. We consider the case when $\epsilon = +1$ in the expression (7) of $\mathcal{F}_a$. The computation when $\epsilon = -1$ is analogous. The projective factor $\mathcal{P}$ and the geodesic coefficients G_j^i can be easily computed by formula (8) and (5). The expression of the curvature vector field $\xi = R(\partial_{x_1}, \partial_{x_2})$ at the point $0 \in \mathbb{R}^2$ is

$$(10) \qquad \xi = R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = \frac{1}{4}\frac{y_2(a_1 y_1 + \|y\|)}{\|y\|}\frac{\partial}{\partial y_1} - \frac{1}{4}\left(y_1 + y_1 a_1^2 + 2a_1\|y\|\right)\frac{\partial}{\partial y_2}.$$

Since the Minkowski norm at $0 \in \mathbb{D}^2$ is $\mathcal{F}_a(0, y) = \|y\| + \langle a, y \rangle$, the indicatrix $\mathcal{I}_0 \subset T_0 M$ at 0 is defined by the equation $\sqrt{y_1^2 + y_2^2} + a_1 y_1 = 1$. Using polar coordinates (r, t) on $T_0 \mathbb{R}^2$, the equation of the indicatrix $\mathcal{I}_0$ is $r(1 + a_1 \cos t) = 1$. A parametrization of $\mathcal{I}_0$ is given by

$$(11) \qquad \phi(t) = \big((y_1(t), y_2(t))\big) = \left(\frac{\cos t}{1 + a_1 \cos t}, \frac{\sin t}{1 + a_1 \cos t}\right),$$

in terms of the parameter t. Using this parametrisation the coordinate expression of the restriction $\xi_0 := \xi\big|_{\mathcal{I}_0}$ of the curvature vector field (10) on the indicatrix $\mathcal{I}_0$ is

$$(12) \qquad \xi_0 := \omega(t)\frac{d}{dt}$$

where

$$(13) \qquad \omega(t) := -\tfrac{1}{4}(1 + a_1 \cos t)^2.$$

Let us introduce the notation

$$(14) \qquad \Sigma_n := \operatorname{Span}_{\mathbb{R}}\left\{\xi_0^{l,m} \mid l + m \le n\right\},$$

where

$$(15) \qquad \xi_0^{l,m} := (\sin^l t \cos^m t) \cdot \xi_0 \in \mathfrak{X}(\mathcal{I}_0)$$

are vector fields on the indicatrix $\mathcal{I}_0$ defined as functional multiples of (12). We have the following

Lemma 1. *For any $n \in \mathbb{N}$ we have $\Sigma_n \subset \mathfrak{hol}_0^*(\mathbb{D}^2)$.*

Remark 2. Using the the Pythagorean trigonometric identity $\sin^2 t + \cos^2 t = 1$, every elements of Σ_n can be expressed as a linear combination of the elements $\xi_0^{0,m} = \cos^m t\,\xi_0$ and $\xi_0^{1,m-1} = \sin t \cos^{m-1} t\,\xi_0$, $0 \le m \le n$, that is

$$(16) \qquad \Sigma_n = \operatorname{Span}_{\mathbb{R}}\left\{\xi_0^{0,m},\ \xi_0^{1,m-1} \mid 0 \le m \le n\right\}.$$

Proof of the Lemma. Taking into account Remark 2 we prove the lemma by mathematical induction by showing that the generating elements (16) of Σ_n are elements of $\mathfrak{hol}_0^*(\mathbb{D}^2)$.

    • *First step.* From the definition of the infinitesimal holonomy algebra (see subsection 2.2.2) we know that $\xi_0^{0,0} = \xi_0$ given by (12) is an element of $\mathfrak{hol}_0^*(\mathbb{D}^2)$. Moreover, as (9) shows, the restriction of successive covariant derivatives of the curvature vector field (10) on $\mathcal{I}_0$ are also elements of $\mathfrak{hol}_0^*(\mathbb{D}^2)$. They can be expressed in terms of multiples of (12). Computing the first covariant derivatives we find that

$$(17a) \qquad (\nabla_1 \xi)\big|_{\mathcal{I}_0} = -\tfrac{3}{2}(a_1 - \cos t)\,\xi_0,$$

$$(17b) \qquad (\nabla_2 \xi)\big|_{\mathcal{I}_0} = \tfrac{3}{2}\sin t\,\xi_0,$$

Using a linear combination of (17a) and (17b) we get that $\xi_0^{1,0} = \sin t\,\xi_0$ and $\xi_0^{0,1} = \cos t\,\xi_0$ are element of $\mathfrak{hol}_0^*(\mathbb{D}^2)$. Therefore we have

$$(18) \qquad \Sigma_1 = \{\xi_0,\ \sin t\,\xi_0,\ \cos t\,\xi_0\} \subset \mathfrak{hol}_{x_0}^*(\mathbb{D}^2),$$

that is the statement of the Lemma is correct for $n = 1$.

• *Second step.* By definition, the second covariant derivatives of the curvature vector field are also elements of the infinitesimal holonomy algebra. Computing them we can find that

(19a) $$\left.(\nabla_1\nabla_1\xi)\right|_{\mathcal{I}_0} = \tfrac{3}{4}\left(5a_1^2 - a_1\cos^3 t - 5a_1\cos t + 3\cos^2 t + 1\right)\xi_0,$$

(19b) $$\left.(\nabla_1\nabla_2\xi)\right|_{\mathcal{I}_0} = -\tfrac{3}{4}\left(a_1\cos^2 t - 3a_1 - 4\cos t\right)\sin t\,\xi_0,$$

(19c) $$\left.(\nabla_2\nabla_2\xi)\right|_{\mathcal{I}_0} = \tfrac{3}{4}\left(a_1\cos^3 t + 5 - 4\cos^2 t\right)\xi_0.$$

Using linear combinations of the elements (18) of Σ_1 and (19a)–(19c) we get that $\left\{\cos^2 t\,\xi_0,\ \sin t\cos t\,\xi_0\right\} \subset \mathfrak{hol}^*_{x_0}(\mathbb{D}^2)$. Completing this set with the elements of (18) we get that

$$\Sigma_2 \subset \mathfrak{hol}^*_0(\mathbb{D}^2).$$

• *Third step.* Let us suppose that the statement of the lemma is true for some $n \in \mathbb{N}$, that is $\Sigma_n \subset \mathfrak{hol}^*_0(\mathbb{D}^2)$. We will show that $\Sigma_{n+1} \subset \mathfrak{hol}^*_0(\mathbb{D}^2)$ too. According to the Remark, Σ_{n+1} is generated by the elements

(20) $$\left\{\xi_0^{0,m},\ \xi_0^{1,m-1}\mid 0 \leq m \leq n\right\} \cup \left\{\xi_0^{0,n+1},\ \xi_0^{1,n}\right\}$$

One can observe that the vector fields of the first set are elements of Σ_n, and by the inductive hypotheses, they are elements of $\mathfrak{hol}^*_0(\mathbb{D}^2)$. Hence, to prove the lemma we have to show that $\xi_0^{0,n+1}$ and $\xi_0^{1,n}$ are elements of $\mathfrak{hol}^*_0(\mathbb{D}^2)$. We have

$$\begin{aligned}
\left[\xi_0, \xi_0^{0,n-1}\right] = \left[\xi_0, \cos^{n-1}t\,\xi_0\right] &= \left(\mathcal{L}_{\xi_0}(\cos^{n-1}t)\right)\xi_0\\
&= -(n-1)\omega(t)(\sin t\cos^{n-2}t)\,\xi_0 \overset{(13)}{=}\\
&= \tfrac{n-1}{4}(1 + 2a_1\cos t + a_1^2\cos^2 t)(\sin t\cos^{n-2}t)\,\xi_0\\
&= \tfrac{n-1}{4}\xi_0^{1,n-2} + \tfrac{a_1(n-1)}{2}\xi_0^{1,n-1} + \tfrac{a_1^2(n-1)}{4}\xi_0^{1,n}.
\end{aligned}$$

By the inductive hypothesis, the Lie bracket on the left hand side and the first two terms in the last line are elements of $\mathfrak{hol}^*_0(\mathbb{D}^2)$. Consequently the last one must be also an element of $\mathfrak{hol}^*_0(\mathbb{D}^2)$. Moreover, the coefficients of $\xi_0^{1,n}$ is a nonzero constant, therefore we get that $\xi_0^{1,n} \in \mathfrak{hol}^*_0(\mathbb{D}^2)$.

Similarly, computing the Lie bracket of the elements ξ_0 and $\xi_0^{1,n-2}$ of the Lie algebra $\mathfrak{hol}^*_0(\mathbb{D}^2)$ we get

$$\begin{aligned}
\left[\xi_0, \xi_0^{1,n-2}\right] = \left[\xi_0, \sin t\cos^{n-2}t\,\xi_0\right] &= \mathcal{L}_{\xi_0}(\sin t\cos^{n-2}t)\,\xi_0\\
&= \omega(t)(\cos^2 t\cos^{n-2}t - (n-2)\sin^2 t\cos^{n-3}t)\,\xi_0 \overset{(13)}{=}\\
&= \tfrac{n-3}{4}\xi_0^{0,n-3} + \tfrac{a_1(n-2)}{2}\xi_0^{0,n-2} + \tfrac{a_1^2(n-2)-(n-1)}{4}\xi_0^{0,n-1} - \tfrac{a_1(n-1)}{2}\xi_0^{0,n-1} - \tfrac{a_1^2(n-1)}{4}\xi_0^{0,n+1}
\end{aligned}$$

From the inductive hypothesis we know that the Lie bracket on the left hand side and the first four terms in the last line on the right hand side are elements of $\mathfrak{hol}^*_0(\mathbb{D}^2)$, therefore the last one must be also an element of $\mathfrak{hol}^*_0(\mathbb{D}^2)$. Since the coefficient of $\xi_0^{0,n+1}$ is nonzero we get that $\xi_0^{0,n+1} \in \mathfrak{hol}^*_0(\mathbb{D}^2)$. Consequently, the vector fields (20) generating Σ_{n+1} are all elements of $\mathfrak{hol}^*_0(\mathbb{D}^2)$ and $\Sigma_{n+1} \subset \mathfrak{hol}^*_0(\mathbb{D}^2)$. $\qquad\square$

Proof of Proposition 1. From the multiple-angle formulas of the sine and cosine functions

$$\sin nt = \sum_{k=0}^{n}\binom{n}{k}\cos^k t\sin^{n-k}t\sin\left(\tfrac{n-k}{2}\pi\right), \qquad \cos nt = \sum_{k=0}^{n}\binom{n}{k}\cos^k t\sin^{n-k}t\cos\left(\tfrac{n-k}{2}\pi\right),$$

we get that the vector fields $\sin nt\,\xi_0, \cos nt\,\xi_0 \in \mathfrak{X}(\mathcal{I}_0)$ can be expressed as a linear combination of elements of Σ_n:

(21a) $$\sin nt\,\xi_0 = \sum_{k=0}^{n}\binom{n}{k}\sin\left(\tfrac{n-k}{2}\pi\right)\xi_0^{k,n-k},$$

(21b) $$\cos nt\,\xi_0 = \sum_{k=0}^{n}\binom{n}{k}\cos\left(\tfrac{n-k}{2}\pi\right)\xi_0^{k,n-k}.$$

On the other side, Lemma 1 shows that the vector fields of Σ_n are elements of the holonomy algebra $\mathfrak{hol}_0^*(\mathbb{D}^2)$. Therefore, the vector fields (21a) and (21b), $n \in \mathbb{N}$, are element of $\mathfrak{hol}_0^*(\mathbb{D}^2)$ and

$$(22) \qquad \mathrm{Span}\left\{ \sin nt\, \xi_0,\ \cos nt\, \xi_0 \mid n = 0, 1, \dots \right\} \subset \mathfrak{hol}_0^*(\mathbb{D}^2).$$

Moreover, any 2π periodic smooth function can be approximated uniformly by the arithmetical means of the partial sums of its Fourier series [5, Theorem 2.12]. In particular the functions $(\sin nt)/\omega(t)$ and $(\cos nt)/\omega(t)$ can be approximated uniformly by their Fourier sums. Hence we get

$$(23) \qquad \left\{ \tfrac{d}{dt},\ \cos nt \tfrac{d}{dt},\ \sin nt \tfrac{d}{dt} \right\}_{n \in \mathbb{N}} \subset \overline{\left\{ \mathrm{Span}\left\{ \xi_0,\ \cos nt\, \xi_0,\ \sin nt\, \xi_0 \right\}_{n \in \mathbb{N}} \right\}} \subset \mathfrak{hol}_0^*(\mathbb{D}^2).$$

The Lie algebra generated by the left hand side of (23) is diffeomorphic to the *Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ on $\mathbb{S}^1$. Therefore from (23) we get that the infinitesimal holonomy algebra contains a Lie algebra diffeomorphic to the *Fourier algebra* $\mathsf{F}(\mathbb{S}^1)$ on $\mathbb{S}^1$. Hence from Proposition 5.1 of [9] we get that the holonomy group $\mathsf{Hol}_0(\mathbb{D}^2)$ is maximal and $\overline{\mathsf{Hol}_0(\mathbb{D}^2)}$ is diffeomorphic to $\mathsf{Diff}_+^\infty(\mathbb{S}^1)$. $\qquad \square$

Using Z. Shen's classification theorem of Randers manifolds we can get the following

Theorem 3. *The holonomy group of a simply connected non-Riemannian projectively flat Finsler two-manifold of constant non-zero flag curvature is maximal and $\overline{\mathsf{Hol}(M)}$ is diffeomorphic to the orientation preserving diffeomorphism group of $\mathbb{S}^1$, that is*

$$\overline{\mathsf{Hol}(M)} \cong \mathsf{Diff}_+^\infty(\mathbb{S}^1).$$

Proof. Let $(M, \mathcal{F})$ be a simply connected non-Riemannian projectively flat Finsler two-manifold of constant non-zero flag curvature and $x_0 \in M$. Since rescaling the metric by a constant factor does not change the connection and the parallel translation, it does not change the holonomy group either. Hence we can suppose that the metric is normalized so that the curvature is $\lambda = -\frac{1}{4}$. Using Shen's results, $\mathcal{F}$ can be locally expressed in the form $\mathcal{F}_a$ given in (7) where $a = (a_1, 0) \in \mathbb{R}^2$ is a nonzero constant vector with $|a_1| < 1$. From Proposition 1 we get, that the closed holonomy group of $(\mathbb{D}^2, \mathcal{F}_a)$ is maximal and diffeomorphic to $\mathsf{Diff}_+^\infty(\mathbb{S}^1)$, therefore the same is true for the closed holonomy group $\overline{\mathsf{Hol}_x(M)}$ of $(M, \mathcal{F})$ at $x_0 \in M$. $\qquad \square$

We can obtain the following classification:

Corollary 4. *The closure of the holonomy group $\mathsf{Hol}(M)$ of a simply connected, locally projectively flat Randers two-manifold of constant flag curvature λ is*

(1) *the trivial group $\{id\}$, when $\lambda = 0$;*

(2) *the rotation group $SO(2)$, when $\lambda \neq 0$ and the metric is Riemannian;*

(3) *the orientation preserving diffeomorphism group of the circle $\mathsf{Diff}_+^\infty(\mathbb{S}^1)$, when $\lambda \neq 0$ and the metric is non-Riemannian.*

Proof. The holonomy structure of projectively flat Finsler manifolds was investigated in [10]: It has been proved that the holonomy group of projectively flat Finsler manifold is *a)* finite dimensional if $\lambda = 0$ or the metric is Riemannian, and *b)* infinite dimensional if $\lambda \neq 0$ and the metric is non-Riemannian. It is clear that the holonomy structures listed in *(1)* and *(2)* correspond to the (already well known) finite dimensional holonomy cases. Moreover, when $\lambda \neq 0$ and the metric is non-Riemannian we get *(3)* from Theorem 3. $\qquad \square$

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