Claw-Decompositions and Tutte-Orientations*

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Abstract: We conjecture that, for each tree T, there exists a natural number k_T such that the following holds: If G is a k_T-edge-connected graph such that $|E(T)|$ divides $|E(G)|$, then the edges of G can be divided into parts, each of which is isomorphic to T. We prove that for $T=K_{1,3}$ (the claw), this holds if and only if there exists a (smallest) natural number k_t such that every k_t-edge-connected graph has an orientation for which the indegree of each vertex equals its outdegree modulo 3. Tutte's 3-flow conjecture says that $k_t=4$. We prove the weaker statement that every $4\lceil \log n \rceil$-edge-connected graph with n vertices has an edge-decomposition into claws provided its number of edges is divisible by 3. We also prove that every triangulation of a surface has an edge-decomposition into claws.

Keywords: *claw-decompositions; orientations; triangulations*

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1. INTRODUCTION

Let $\mathcal{H}$ be a collection of graphs. We say that a multigraph G has an *$\mathcal{H}$-decomposition* if the edges of G can be divided into subgraphs each of which is isomorphic to a graph in $\mathcal{H}$. If $\mathcal{H} = \{H\}$, then we speak of an *H-decomposition* of G. The H-decompositions are widely studied when G is a complete graph. If H is the 3-cycle C_3, then they are the well-known Steiner triples. If G is not complete, then it may be hard to find H-decompositions. Indeed, if H has at least three edges, then the problem of deciding if a graph G has an H-decomposition is NP-complete [5].

Igor Pak has kindly informed us that $K_{1,3}$-decompositions were studied already in 1916 in connection with rigidity of polyhedra. Dehn [3] proved (among other things) that every planar triangulation (minus a 3-cycle) has a $K_{1,3}$-decomposition.

Jünger, Reinelt, and Pulleyblank [10] studied H-decompositions, where H has three edges. Among other things, they proved that every 2-edge-connected graph, whose size (that is, the number of edges) is divisible by 3, has a $\{P_4, C_3, K_{1,3}\}$-decomposition, where P_4 is the path with three edges. They proposed the following, still unsolved problem [10]:

Question 1.1. *Is it true that every planar 2-edge-connected bipartite graph, whose size is divisible by 3, has a P_4-decomposition?*

They also asked the following:

Question 1.2. *Suppose that $\mathcal{H}$ is the class of 2-edge-connected graphs with s vertices. Is there an edge-connectivity (depending on s) that guarantees a graph to have an $\mathcal{H}$-decomposition?*

The answer to this question is negative: there is no such edge-connectivity, because there exist graphs of arbitrarily high connectivity and girth. Erdős [7] proved that there are graphs of arbitrarily large chromatic number and girth. Mader [12] proved that any such graph has a subgraph of large connectivity. In [1], it was shown that the subgraph can also be chosen to have large chromatic number. In view of this, we must require that a finite collection $\mathcal{H}$ must contain a forest if we wish to show that large (fixed) edge-connectivity implies the existence of an $\mathcal{H}$-decomposition. This has inspired us to the general conjecture in the Abstract.

A *graph* has no loops or multiple edges. A *multigraph* may have multiple edges. In order to emphasize that some of the results hold only for graphs we shall sometimes call these *simple graphs*.

2. CONNECTIONS BETWEEN DECOMPOSITIONS AND ORIENTATIONS

Conjecture 2.1 below is a special case of the conjecture in the Abstract.

Conjecture 2.1. *There exists a smallest natural number k_c such that every simple k_c-edge-connected graph G, whose size is divisible by 3, has a $K_{1,3}$-decomposition.*

The graph $K_{1,3}$ is also called the *claw*. Claw-decompositions can be expressed in terms of orientations. For, if a graph G has a claw-decomposition, then we can orient the edges of G as follows. Whenever there is a claw of the decomposition with center x and leaves y_1, y_2, y_3, then let the edges be oriented from x towards y_i, for $i = 1, 2, 3$. In the resulting graph, all outdegrees are congruent to 0 modulo 3. Conversely, if G has such an orientation, then it implies the existence of a claw-decomposition of G. Motivated by this connection, we now focus on orientations. If v is a vertex of an oriented graph such that $d^+(v) \equiv d^-(v) \pmod 3$, then we say that the orientation is *balanced* at $v \pmod 3$. An orientation of a graph G is called a *Tutte-orientation*, if each vertex is balanced $(\mathrm{mod}\, 3)$.

If a graph has a nowhere zero 3-flow, then we obtain a Tutte-orientation by reversing the edges of flow value 2. Tutte's 3-flow conjecture states that every multigraph with no 1-edge-cut and no 3-edge-cut has a nowhere zero 3-flow. Equivalently, every 4-edge-connected multigraph has a Tutte-orientation. For more details on Tutte's 3-flow conjecture, see e.g. [2,4,9]. Jaeger proposed the following weaker conjecture.

Conjecture 2.2 (Jaeger [8]). *There exists a smallest natural number k_t such that every k_t-edge-connected multigraph has a Tutte-orientation.*

Thus Tutte conjectured that $k_t = 4$, and this would imply Grötzsch's theorem that every planar triangle-free graph is 3-colorable. (The dual version of Grötzsch's theorem states that every 4-edge-connected planar multigraph has a Tutte-orientation, see e.g. [9].) We believe that the following holds:

Conjecture 2.3. *If G is a planar, 4-edge-connected graph, whose size is divisible by 3, then G has a claw-decomposition.*

A cubic graph G has a claw-decomposition if and only if G is bipartite. For, such a graph G has $2k$ vertices and $3k$ edges. Hence a claw-decomposition of G must consist of k claws, and the centers must form an independent set. So, a 3-edge-connected, planar graph need not have a claw-decomposition, and hence Conjecture 2.3 is sharp.

It would be tempting to extend Conjecture 2.3 to the stronger statement that $k_c = k_t = 4$. But this is false. To see this, consider three copies of K_4, and add two edges between any pair such that we get a 4-regular graph G_0. This graph has 12 vertices and 24 edges. Assume that a claw-decomposition of G_0 exists. It must consist of eight claws. Orient the edges of each claw away from the center. There must be four sinks, that is, vertices of outdegree 0. By the pigeon-hole principle, two of them must be in the same K_4. This is a contradiction. Thus $k_c > 4$, and the planarity condition cannot be dropped in Conjecture 2.3. The construction can be iterated as follows. Take three copies of G_0 and unfold two edges between the K_4's. These altogether six edges can be used to connect each pair of the three copies of G_0 to make the graph 4-edge-connected. Now this graph has no independent set of twelve sinks by the pigeon-hole principle.

Perhaps $k_c = 5$. If so, then $k_t \leq 8$, as we prove in the theorem below.

Theorem 2.4. *If every* 8*-edge-connected simple graph, whose size is divisible by* 3*, has a claw-decomposition, then every* 8*-edge-connected multigraph has a Tutte-orientation. In other words, if* $k_c \leq 8$*, then* $k_t \leq 8$.

Proof. Let us assume that every 8-edge-connected graph has a claw-decomposition. Then we prove that every 8-edge-connected multigraph has a Tutte-orientation. We proceed by induction on the number of vertices. The multigraph with two vertices and eight edges clearly has a Tutte-orientation. So we proceed to the induction step.

We may assume that G is 2-connected since otherwise, we apply the induction hypothesis to each block of G.

If e_1 and e_2 are parallel edges in the multigraph G under consideration, then we contract all edges parallel with e_1. The resulting multigraph is called G'. We use the induction hypothesis for G'. We orient all edges parallel with e_1 and distinct from e_1, e_2 at random. We claim that G also has a Tutte-orientation. It suffices to consider the endvertices x and y of e_1. There are three different possible orientations of the edges e_1 and e_2. Since they contribute to the outdegree of x by 0, 1 or 2, one of them will give a balanced orientation at x (mod 3). Then also y will be balanced (mod 3). We may therefore assume that G has no multiple edges.

Suppose that $v \in V(G)$ is a vertex of even degree. Using a theorem by Mader (namely Theorem 10 in [13]), there exist two edges vx and vy that we can split (that is, replace by a new edge xy) such that the edge-connectivity between any two vertices of $V(G) \setminus \{v\}$ does not change. Since the degree of v is even, we may split all edges incident with v and complete the proof by induction. (Mader's theorem allows multiple edges. That theorem only requires that v is not a cutvertex and that v has degree at least 4 and has at least two distinct neighbors.)

Assume next that $v \in V(G)$ is of odd degree, $2k + 9$ say. If $k > 0$, we split two edges, and use induction for the resulting multigraph. Note that the resulting multigraph is 8-edge-connected because v has degree at least 8 (in fact, at least 9) after the splitting.

There remains only the case in which G is 8-edge-connected and 9-regular, and has no multiple edges. By assumption, G has a claw-decomposition, which corresponds to an orientation with all outdegrees divisible by 3. As all degrees are 9, such an orientation is a Tutte-orientation. ■

In Theorem 2.4, the number 8 may be replaced by any number of the form $8 + 6k$, where k is a natural number. Thus, the existence of k_c implies the existence of k_t.

We now prove the converse, that the existence of k_t implies the existence of k_c. For this, it is convenient to study more general orientations. Let G be a multigraph, and $w: V(G) \to \{0, 1, 2\}$ a prescribed weight function on the vertices such that $\sum_{v \in V(G)} w(v) \equiv |E(G)| \pmod 3$.

If there is an orientation of the edges of G with $d^+(v) \equiv w(v) \pmod 3$ for each $v \in V(G)$, then we say that *G admits the generalized Tutte-orientation prescribed by w*.

If, for every such $w : V(G) \to \{0, 1, 2\}$, there is an orientation of the edges of G with $d^+(v) \equiv w(v) \pmod 3$ for each $v \in V(G)$, then we say that G *admits all generalized Tutte-orientations*.

Conjecture 2.5. *There exists a smallest natural number k_g such that every k_g-edge-connected multigraph admits all generalized Tutte-orientations.*

Clearly $k_t \le k_g$. Also $k_c \le k_g$. (Just consider the generalized Tutte-orientation prescribed by the zero-function.) We show that the three parameters are essentially equal. We shall use the following fundamental result by Nash-Williams [15] and Tutte [16].

Theorem 2.6. *Every $2k$-edge-connected multigraph G has k pairwise edge-disjoint spanning trees.*

Theorem 2.7. *If one of k_c, k_t, k_g exists, then they all exist. In this case, $k_g \le 2k_t + 2$, $k_c \le k_g$, and $k_t \le k_c + 5$.*

Proof. Assume that k_t exists. We shall prove that k_g exists and that $k_g \le 2k_t + 2$. Let G be a multigraph with edge-connectivity at least $2k_t + 2$, and let w be any prescribed weight function. By Theorem 2.6, G has $k_t + 1$ edge-disjoint spanning trees $T_1, \dots, T_{k_t+1}$.

Put $w^*(v) = -d_G(v) - w(v)$ for each vertex v. We orient some edges of T_{k_t+1} such that $d^+_F(v) - d^-_F(v) \equiv w^*(v) \pmod 3$ for each $v \in V(G)$, where F denotes the resulting oriented forest F. It is an easy exercise to show that such a partial orientation of T_{k_t+1} exists.

The unoriented edges of G form a k_t-edge-connected multigraph H, which has a Tutte-orientation by the assumption. That is $d^+_H(v) \equiv d^-_H(v) \pmod 3$ for each vertex v. Hence $d_H(v) = d^+_H(v) + d^-_H(v) \equiv 2d^+_H(v) \equiv -d^+_H(v) \pmod 3$. Similarly, $d_F(v) \equiv d^+_F(v) + d^-_F(v) = 2d^+_F(v) - (d^+_F(v) - d^-_F(v)) \equiv -d^+_F(v) - w^*(v) \pmod 3$ for each vertex v. Hence $d^+_G(v) = d^+_H(v) + d^+_F(v) \equiv -d_H(v) - d_F(v) - w^*(v) = -d_G(v) - w^*(v) = w(v) \pmod 3$ for each vertex v. Hence, k_g exists and $k_g \le 2k_t + 2$.

As noted after Conjecture 2.5, $k_c \le k_g$. The remark after the proof of Theorem 2.4 shows that $k_t \le k_c + 5$. ∎

Corollary 2.8. *If the 3-flow conjecture is true, then every 10-edge-connected multigraph admits all generalized Tutte-orientations. In particular, every 10-edge-connected graph has a claw-decomposition, provided its size is divisible by 3.*

Let us call a graph mod $(2p + 1)$-*orientable* if it has an orientation such that each vertex is balanced mod $(2p + 1)$. Jaeger also proposed the following generalization of Conjecture 2.2.

Conjecture 2.9 (Jaeger [8]). *For each $p \ge 1$, there exists a smallest natural number $k_j(p)$ such that every $k_j(p)$-edge-connected multigraph has a* mod $(2p + 1)$-*orientation. Moreover, $k_j(p) \le 4p$.*

A *generalized* mod $(2p+1)$-*orientation* can be defined in the obvious way.

The methods in the proof of Theorem 2.7 show that for each natural number $p \geq 1$, the following are equivalent:

a. There exists a smallest natural number $k_j(p)$ such that every $k_j(p)$-edge-connected multigraph G has a mod $(2p+1)$-orientation.
b. There exists a smallest natural number $k_c(p)$ such that every $k_c(p)$-edge-connected simple graph G, whose size is divisible by $2p+1$, has a $K_{1,2p+1}$-decomposition.
c. There exists a smallest natural number $k_g(p)$ such that every $k_g(p)$-edge-connected multigraph G admits all generalized mod $(2p+1)$-orientations.

Clearly, (c) implies (a) and (b). The proof of Theorem 2.7 shows that (a) implies (c). We now indicate why also (b) implies (c). Specifically, we prove that $k_g(p) \leq 4p(k_c(p)+2p)$. Let G be a multigraph of edge-connectivity at least $4p(k_c(p)+2p)$, and let w be a function, which we shall show prescribes a generalized mod $(2p+1)$-orientation. If G has a multiple edge consisting of at least $2p$ parallel edges, then we contract them and use induction. So assume there is no such multiple edge. For every multiple edge we orient all its edges, except precisely one, at random. We delete the oriented edges and modify the function w accordingly. The resulting simple graph has edge-connectivity at least $2(k_c(p)+2p)$ and contains therefore $k_c(p)+2p$ edge-disjoint spanning trees. We use $2p$ of these spanning trees to orient some of their edges in such a way that deleting the oriented edges and modifying w accordingly, the modified w becomes the zero function.

The resulting graph has $k_c(p)$ edge-disjoint spanning trees and therefore edge-connectivity at least $k_c(p)$. Then we complete the proof using the assumption of Conjecture (b).

Note that, in this way we do not use Mader's splitting theorem. That could also be avoided in Theorem 2.4, but then the inequalities in Theorem 2.7 would become weaker.

3. CLAW-DECOMPOSITION OF GRAPHS ON A FIXED SURFACE

If Conjecture 2.3 is true, then one may proceed to higher surfaces. The graph with 12 vertices described after Conjecture 2.3 can be drawn on the torus. But, it may be that every 4-connected toroidal graph, whose size is divisible by 3 has a claw-decomposition, provided that the graph is locally planar, that is, the graph has large face-width. This holds at least for the Cartesian product of two cycles. To see this, let $G = C_{3k} \times C_t$. Then G has a claw-decomposition using the following orientation. We first specify the sinks in the first copy of C_{3k} to be every third vertex. In the second copy of C_{3k}, we shift the sinks by one to the left, say. Repeat this t times. If $t \equiv 0, 2 \pmod 3$, then this gives us the set of sinks S. When $t \equiv 1 \pmod 3$, this procedure does not work because the last copy is identical to the first one. (So some sinks would be neighbors which is impossible.) In this case, we instead make the last shift to the right. Now orient the edges towards the sinks. Then $V(G) \setminus S$ induces an Eulerian subgraph if $t \equiv 0 \pmod 3$. If $t \equiv 1$ or $2 \pmod 3$, then there is a matching M between the vertices of degree 1 and 3. We orient these edges towards the vertices of degree 1. After deleting M, the remaining graph is Eulerian. Orient the edges along an Euler walk in each component. This orientation gives us the desired claw-decomposition.

4. TRIANGULATIONS OF SURFACES

If a graph on a surface is 3-colorable, then its dual graph has a 3-flow, and hence a Tutte-orientation. In particular, every triangulation of a surface, other then K_4, has a Tutte-orientation.

An n-vertex triangulation of a surface of Euler genus k has $3n - 6 + 3k$ edges, see e.g. [14]. Hence, it is a natural candidate for having a claw-decomposition. In this section, we prove the stronger result, that every triangulation distinct from K_4 admits all generalized Tutte-orientations.

We shall use four lemmas, some of which may be of independent interest.

Lemma 4.1. *If the edges of a multigraph G can be acyclically oriented such that each vertex, except one, has outdegree at least 2, then G admits all generalized Tutte-orientations.*

Proof. Let w be any weight function on the vertices. The assumptions imply that the vertices of G can be labeled $x_1, \ldots, x_n$ such that all arcs (directed edges) go from right to left. Each vertex has at least two outgoing arcs. In particular, there are at least two edges between x_1 and x_2. Contract these edges, and use induction. Orient the edges between x_1 and x_2 such that $d^+(x_2) \equiv w(x_2) \pmod 3$. The condition on w implies that also $d^+(x_1) \equiv w(x_1) \pmod 3$. ■

No graph satisfies Lemma 4.1, as multiple edges are needed. However, any triangulation with one or more edges added, or one or more edges contracted satisfies the assumption of Lemma 4.1. This follows easily from the following observation. If G is a triangulation, and H is a connected subgraph containing at least two but not all vertices, then G has a vertex v that is not in H but which is joined to at least two vertices in H. (We then orient all edges between v and H from v to H, add v to H, and repeat.)

Lemma 4.2. *Let $k \geq 3$ be a natural number, and let w be a weight function of the k-wheel W_k with center c such that $\sum_{x \in V(W_k)} w(x) \equiv |E(W_k)| \pmod 3$. Then the k-wheel admits the generalized Tutte-orientation prescribed by w, unless k is odd, and $w(x) \equiv 0 \pmod 3$ for all vertices $x \in W_k \setminus \{c\}$.*

Proof. Let $x_1 x_2 \ldots x_k x_1$ be the cycle $W_k \setminus \{c\}$. If k is even, and $w(x_i) \equiv 0 \pmod 3$, for $i = 1, 2, \ldots, k$, then we orient the edges of the wheel such that

$x_1, x_2, \ldots, x_k$ are sources and sinks alternately. So assume that $w(x_1) \not\equiv 0 \pmod 3$. If $w(x_1) = 1$, then we orient the edge x_1x_2 towards x_1. If $w(x_1) = 2$, then we orient it away from x_1. Then we successively orient the two unoriented edges incident with x_2, the two unoriented edges incident with x_3 etc. as prescribed by w. Clearly, it is possible to orient the last edge incident with x_1. The condition $\sum_{x \in V(W_k)} w(x) \equiv |E(W_k)| \pmod 3$ ensures that the center receives the correct prescribed outdegree. ■

Lemma 4.3. *Let $k \geq 3$ be a natural number, and let U_k be a multigraph obtained from the k-wheel W_k by adding one or more edges. Then U_k admits all generalized Tutte-orientations.*

Proof. Let w be any prescribed weight function of U_k. Orient all added edges, except one, at random. The last added edge can be oriented in two ways. For each of these two orientations, we modify w accordingly. At least one of these two modifications of w is not the exceptional weight function in Lemma 4.2. Hence, Lemma 4.2 implies that U_k has the desired orientation. ■

Lemma 4.4. *Let x and y be adjacent vertices in a triangulation G such that at least one of x,y, say y, has degree at least* 4 *and such that $N(x) \cap N(y)$ consists of only two vertices. Let H be the subgraph induced by $N(x) \cup N(y)$. Then H admits all generalized Tutte-orientations.*

Proof. Let w be any weight function, and let $x_1, x_2, \ldots, x_{k-1}, x$ be the neighbors of y in clockwise order. The graph induced by $\{y\} \cup N(y)$ is a wheel W_y and possibly some additional edges that we first orient at random. Now we repeat the procedure from the proof of Lemma 4.2 with a slight modification. We orient the edge x_1x_2 arbitrarily. Then we orient successively the two unoriented edges of W_y incident with $x_2, \ldots, x_{k-2}$ to achieve the prescribed outdegrees at these vertices. The remaining unoriented edges of H form a wheel W_x with center x, and possibly some additional edges that we orient at random. As the orientation of x_1x_2 can be chosen in two ways, we may assume that the exceptional case in Lemma 4.2 does not occur now for W_x. Hence, we can orient the edges of W_x by Lemma 4.2. ■

Theorem 4.5. *Let $G \neq K_3, K_4$ be a triangulation of any surface S. Then G admits all generalized Tutte-orientations.*

Proof. If G contains a non-facial triangle, then let x be one of the vertices of this triangle, and let H be the subgraph of G induced by x and its neighbors. Then H is a wheel with at least one additional edge.

If all triangles of G are facial, then we let x be any vertex. As $G \neq K_3, K_4$, x has a neighbor y of degree at least 4. As all triangles containing x are facial, x and y have only two neighbors in common. In this case, we let H be the subgraph of G induced by x, y and the neighbors of x, y.

Let w be any weight function of G. We contract H into a single vertex, and we modify w accordingly. By the remark after Lemma 4.1, the resulting multigraph has the desired orientation. By Lemma 4.3 or Lemma 4.4, the orientation can be extended to H. ■

Theorem 4.5 shows that every triangulation has a claw-decomposition. The decomposition may possibly be chosen such that (almost) every vertex is the center of a claw. We now prove that this holds for triangulations of surfaces of Euler genus at most 2. For this, we use the following well-known consequence of Edmonds' matroid partition theorem [6]. For completeness, we indicate a short proof.

Theorem 4.6. *Let G be a graph with n vertices, and let $k_1, k_2, \ldots, k_n$ be non-negative integers. Then G has an orientation satisfying $d^+(v_i) \leq k_i$ for $i = 1, \ldots, n$ if and only if $|E(H)| \leq \sum_{i:v_i \in V(H)} k_i$ for any subgraph $H \subseteq G$.*

Proof. The necessity is obvious. For the sufficiency, let M_i be the matroid on $E(G)$ whose independent sets are the sets that consist of at most k_i edges, each of which is adjacent to v_i. Then the matroid partition theorem gives us a partition of $E(G)$ into sets $E_1, \ldots, E_n$ such that E_i is independent in M_i if and only if for any $S \subseteq E(G)$ we have $|S| \leq \sum_{i:S \cap M_i \neq \emptyset} k_i$. Now orient the edges in E_i away from v_i. ■

Theorem 4.7. *Let G be a triangulation of the plane or the projective plane or the torus or the Klein bottle. Then G has an orientation such that all outdegrees are* 3 *or* 0, *except when $G = K_4$ in the plane.*

Proof. We prescribe all outdegrees k_i to be at most 3, except for two independent vertices in the plane and one vertex in the projective plane for which we put $k_i = 0$. Then we apply Theorem 4.6. The required inequalities hold by Euler's formula. Since all outdegrees (except one or two) are at most 3, all of them (except one or two) are precisely 3, again by Euler's formula. ■

Conjecture 4.8. *Let G be a triangulation of a surface of Euler genus $k \geq 2$. Then G has an orientation such that each outdegree is at least* 3, *and divisible by* 3.

Theorem 4.7 shows that Conjecture 4.8 holds for $k = 2$.

5. DENSE GRAPHS

Lai and Zhang [11] proved that every $4\lceil \log n \rceil$-edge-connected multigraph has a nowhere zero 3-flow. We now prove that any such graph admits all generalized Tutte-orientations.

Theorem 5.1. *Every $4\lceil \log n \rceil$-edge-connected multigraph with n vertices admits all generalized Tutte-orientations.*

Proof. Assume for simplicity that $\log n$ is a natural number. Let w be any prescribed weight function on the vertices. We show that G can be oriented as prescribed by w. The idea is to find $\log n$ pairwise edge-disjoint spanning Eulerian subgraphs $G_1, G_2, \ldots, G_{\log n}$. We orient the remaining edges arbitrarily, and modify w accordingly. We use G_1 to give half of the vertices the prescribed outdegree modulo 3. We then use G_2 to take care of half of the remaining vertices, and so on.

We now argue formally. By Theorem 2.6, G has $2\log n$ pairwise edge-disjoint spanning trees $T_1, T_2, \ldots, T_{2\log n}$. It is well known and easy to see that the union of any two of these contains a connected spanning Eulerian subgraph. Therefore G contains $\log n$ pairwise edge-disjoint spanning Eulerian subgraphs $G_1, G_2, \ldots, G_{\log n}$.

We orient all edges not in $E(G_1) \cup E(G_2) \cup \ldots \cup E(G_{\log n})$ at random, and we modify w accordingly. Then we define w^* as in the proof of Theorem 2.7. In other words, we are going to orient $G_1 \cup G_2 \cup \ldots \cup G_{\log n}$ such that for each vertex v, there are $w^*(v)$ outgoing arcs, and the remaining $d(v) - w^*(v)$ arcs incident with v are balanced at v (mod 3). (We assume here that $w^*(v)$ is one of 0,1,2.) We now define the *mode* of a vertex v. Initially the mode of v is $w^*(v)$. If all vertices are in mode 0, then we just orient each G_i, for $1 \leq i \leq \log n$, such that each vertex is balanced. If some vertices are in a mode $\neq 0$, then we orient $G_1, G_2, \ldots, G_{\log n}$ successively such that we use each G_i to turn at least half of the vertices of mode $\neq 0$ into mode 0. We explain how this is done for G_1. The procedure for $G_2, \ldots, G_{\log n}$ is similar. So, we let $v_1, e_1, v_2, e_2, \ldots, e_m, v_1$ be a closed Euler walk of G_1. There may be repetition of vertices. Suppose v_2 is in mode 2. Then we orient both edges e_1, e_2 away from v_2. Suppose we have already oriented $e_1, e_2, \ldots, e_{k-1}$, and that e_{k-1} is directed towards v_k. If v_k is in mode 0, then we direct e_k away from v_k, and we say that v_k is still in mode 0. If v_k is in mode 1, then we direct e_k towards v_k, and we say that v_k is in mode 0. Now v_k is in the required mode and will remain there. Finally, we consider the case in which v_k is in mode 2. In this case, we consider the first vertex v_p in the sequence $v_{k+1}, v_{k+2}, \ldots$ which is not in mode 0. We orient the edges $e_k, e_{k+1}, \ldots, e_p$ such that v_p turns into mode 0, and $v_{k+1}, \ldots, v_{p-1}$ remain in mode 0. Then v_k will be in either mode 1 or mode 2. If there are k vertices in the undesired mode 1 or 2, then we change in this way at least $(k-1)/2$ of these into the desired mode 0. We repeat this argument for $G_2, \ldots, G_{\log n}$. When this procedure terminates, all vertices will be in mode 0. ■

Theorem 5.2. *There exists a constant n_1 such that every graph G with $n \geq n_1$ vertices and minimum degree $\delta(G) \geq n/2$ admits all generalized Tutte-orientations.*

Proof. If the edge-connectivity of G is at least $4\lceil \log n \rceil$, then the claim holds by Theorem 5.1. Otherwise, G has an edge-cut of size smaller than $4\lceil \log n \rceil$. The minimum degree ensures that there are at least $\frac{n}{2} - 4\lceil \log n \rceil$ vertices on both sides of the cut. When n is large enough, both sides are $4\lceil \log n \rceil$-edge-connected, and hence they admit all generalized Tutte-orientations. So, for any prescribed weight function w on $V(G)$, we first orient the edges in the cut and then apply Theorem 5.1 to each side of the cut. We only need to make sure that the modified weight functions satisfy the congruence relation. As the cut has at least two edges, this is always possible. ■

If n is even, then the graph consisting of the union of two copies of $K_{\frac{n}{2}}$ and one edge between them has neither a Tutte-orientation nor a claw-decomposition. The degree condition in Theorem 5.2 is therefore sharp. However, for 2-edge-connected graphs, there is a better bound.

Theorem 5.3. *There exists a constant n_2 such that every 2-edge-connected graph G with $n \geq n_2$ vertices and minimum degree $\delta(G) \geq n/4$ admits all generalized Tutte-orientations.*

The proof of Theorem 5.3 is similar to, but more tedious than that of Theorem 5.2. Theorem 5.3 is best possible in the following sense: If n is divisible by 4, then take the union of four copies of $K_{\frac{n}{4}}$. Add six independent edges such that there is precisely one edge between any two copies of $K_{\frac{n}{4}}$. The resulting graph has minimum degree $\frac{n}{4} - 1$ but has no Tutte-orientation.

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