

On the finite volume expectation values of local operators in the sine-Gordon model

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Abstract

In this paper we present sets of linear integral equations which make possible to compute the finite volume expectation values of the trace of the stress energy tensor (Θ) and the $U(1)$ current (J_μ) in any eigenstate of the Hamiltonian of the sine-Gordon model. The solution of these equations in the large volume limit allows one to get exact analytical formulas for the expectation values in the Bethe-Yang limit. These analytical formulas are used to test an earlier conjecture for the Bethe-Yang limit of expectation values in non-diagonally scattering theories. The analytical tests have been carried out upto three particle states and gave agreement with the conjectured formula, provided the definition of polarized symmetric diagonal form-factors is modified appropriately. Nevertheless, we point out that our results provide only a partial confirmation of the conjecture and further investigations are necessary to fully determine its validity. The most important missing piece in the confirmation is the mathematical proof of the finiteness of the symmetric diagonal limit of form-factors in a non-diagonally scattering theory.

1 Introduction

Finite volume form-factors of integrable quantum field theories play an important role in the AdS_5/CFT_4 correspondence [1, 2] and in condensed matter applications [3], as well. In AdS/CFT, their knowledge is indispensable for the computation of the string field theory vertex [1] and of the heavy-heavy-light 3-point functions [2] of the theory. In condensed matter systems finite volume form-factors can be used to compute quantum correlation-functions describing quasi 1-dimensional quantum-magnets, Mott insulators and carbon nanotubes [3].

The systematic study of finite volume form-factors of integrable quantum field theories was initiated in [4, 5], where the finite volume matrix-elements of local operators are sought in the form of a systematic large volume series. From the investigation of finite volume 2-point functions it turned out, that upto exponentially small finite volume corrections, but including all corrections in the inverse of the volume, the non-diagonal finite volume form-factors are equal to their infinite volume counterparts taken at the positions of the solutions of the Bethe-Yang equations and normalized by the square roots of the densities of the sandwiching states.

As a consequence of the Dirac-delta contact terms in the crossing relations of the form-factor axioms, the diagonal form-factors cannot be obtained from the non-diagonal ones by taking their straightforward diagonal limit. Thus, diagonal form-factors are related to the infinite volume form-factors in a bit more indirect way. According to the conjectures [5], they can be represented as density weighted linear combinations of the so-called "connected" or "symmetric" diagonal limits of the infinite volume form-factors of the theory. In [8], it has been shown, that the conjecture [5] being valid for purely elastic scattering-theories, can be derived from the leading order formula for the non-diagonal finite volume form-factors by considering such non-diagonal matrix elements, in which there is one particle more in the "bra" sandwiching state and the rapidity of this additional particle is taken to infinity appropriately.

The conjectures for purely elastic scattering theories [5] went through extensive analytical and numerical tests [12] providing convincing amount of evidence for their validity. So far the conjecture for the more subtle non-diagonally scattering theories [10] has not gone through convincing amount of tests. It has been tested in the sine-Gordon model, where it was checked analytically in the whole pure soliton sector against exact [19, 20] and numerical [9] results and numerically against TCSA data for mixed soliton-antisoliton two particles states [10]. Thus, analytical tests of this conjecture is still missing in the soliton-antisoliton mixed sector. In this paper we would like to fill this gap and we check the conjecture of [10] upto 3-particle soliton-antisoliton mixed states.

Beyond the leading polynomial in the inverse of the volume terms, the exponentially small in volume corrections are also necessary. Their determination is still an open problem in general. Nevertheless, some progress has been reached in this direction, as well. For the non-diagonal form-factors in purely elastic scattering-theories there is some knowledge about the leading order exponentially small in volume corrections termed the Lüscher corrections. The so-called μ -term Lüscher corrections were determined in [6] and the F-term corrections for vacuum-1-particle form factors has been determined in [7]. Unfortunately,

the Lüscher corrections to form-factors in non-diagonally scattering theories and higher order exponentially small in volume corrections in any integrable quantum field theory are presently out of reach.

Nevertheless, much is known about the exact finite volume behavior of the diagonal form-factors both in purely elastic and non-diagonally scattering theories.

In [11, 12] a LeClair-Mussardo type [18] series representation was conjectured to describe exactly the finite volume dependence of diagonal matrix elements of local operators in purely elastic scattering theories. In non-diagonally scattering theories the description of finite volume diagonal matrix elements is less complete. So far only the sine-Gordon model has been studied in this class of theories. There, based on computations done in the framework of its integrable lattice regularization [22], a LeClair-Mussardo type series representation was proposed to describe the finite volume dependence of the expectation values of local operators in pure soliton states [19]. Nevertheless, soliton-antisoliton mixed states have not been investigated so far. In this paper we partly fill this gap and derive integral equations to get any finite volume diagonal matrix elements of two important operators of the theory. These are the trace of the stress energy tensor (Θ) and the $U(1)$ current (J_μ). Our formulas are valid to any value of the volume and to any eigenstate of the Hamiltonian of the model.

In the paper the Bethe-Yang limit of the diagonal form-factors will play an important role. In our terminology this limit means, that the exponentially small in volume corrections are neglected from the large volume expansion of the exact result¹.

In the repulsive regime, where there are no breathers in the spectrum, we solve our equations in the Bethe-Yang limit and give exact formulas for the expectation values of our operators in this limit. The formulas depend on the rapidities of the physical particles and on the magnonic Bethe-roots of the Bethe-Yang equations.

With the help of these exact formulas we check the conjecture of [10] for the Bethe-Yang limit of the diagonal matrix elements of local operators in non-diagonally scattering theories. The conjectured formula in [10] contains the symmetric diagonal limit of the infinite volume form-factors. The determination of these symmetric diagonal form-factors becomes more and more complicated as the number of particles increases. This is why we complete the test upto three particle states. Upto 3-particle states our exact formulas give perfect agreement with the conjectured formula of [10] for the operators Θ and J_μ , provided the sandwiching color wave function Ψ is replaced by its complex conjugate in the original formulas of [10].

However despite the success of our checks, the details of the computations shed light on some subtle points of the conjecture, which require further work to be confirmed. The most important of them is to prove that the symmetric diagonal limit of form-factors is finite for a generic sandwiching state in a non-diagonally scattering theory. Though, this statement looks intuitively quite trivial, in section 9, where we comment the computation of the symmetric diagonal limit of form-factors, we argue that this statement is not trivial at all.

The outline of the paper is as follows.

¹Only the terms being polynomials in the inverse of the volume remain.

In section 2. we summarize the most important facts about the models and about the operators of our interest. In section 3. the equations governing the exact finite volume dependence of diagonal matrix elements of the operators Θ and J_μ are derived. The solution of the equations in the large volume limit is given in section 4.

The basic ingredients of the form-factor bootstrap program for the sine-Gordon model can be found in section 5. In section 6. we summarize the conjecture of [10] for the Bethe-Yang limit of the diagonal matrix elements of local operators in non-diagonally scattering theories. In section 7. we compute the symmetric diagonal form-factors of the operators under consideration upto 3-particle states. In section 8. we perform the analytical checks of the conjecture [10] upto 3-particle states. In section 9. we comment on some subtle points of the conjecture of [10]. The body of the paper is closed by our summary and conclusions in section 10.

The paper contains two appendices, as well. Appendix A contains the detailed form of the linear integral equations governing the finite volume dependence of the expectation values of the operators Θ and J_μ . In appendix B the diagonalization of the soliton transfer-matrix is performed by means of algebraic Bethe-Ansatz. This appendix contains the classification of Bethe-roots, as well.

2 The models and operators

In this paper we investigate the sine-Gordon and the massive Thirring models. They are given by the well known Lagrangians:

$$\mathcal{L}_{SG} = \frac{1}{2} \partial_\nu \Phi \partial^\nu \Phi + \alpha_0 (\cos(\beta \Phi) - 1), \quad 0 < \beta^2 < 8\pi, \quad (2.1)$$

and

$$\mathcal{L}_{MT} = \bar{\Psi}(i\gamma_\nu \partial^\nu - m_0)\Psi - \frac{g}{2} \bar{\Psi} \gamma^\nu \Psi \bar{\Psi} \gamma_\nu \Psi, \quad (2.2)$$

where m_0 and g denote the bare mass and the coupling constant of the massive Thirring model, respectively. In (2.2) γ_μ stand for the γ -matrices, which satisfy the algebraic relations: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = \text{diag}(1, -1)$.

The two models are equivalent in their even $U(1)$ charge sector [33, 34], provided the coupling constants of the two theories are related by the formula:

$$1 + \frac{g}{4\pi} = \frac{4\pi}{\beta^2}. \quad (2.3)$$

In the sequel we will prefer the following parameterization of the coupling constant β :

$$\frac{\beta^2}{4\pi} = \frac{2p}{p+1}, \quad 0 < p \in \mathbb{R}. \quad (2.4)$$

The ranges $0 < p < 1$ and $1 < p$ correspond to the attractive and repulsive regimes of the theory respectively.

The fundamental particles in the theory are the soliton (+) and the antisoliton (−) of mass \mathcal{M} . Their exact S-matrix is well known [39] and in terms of the coupling constant p it can be written in the form as follows:

$$\mathcal{S}_{ab}^{cd}(\theta) = S_0(\theta) S_{ab}^{cd}(\theta), \quad a, b, c, d, \in \{\pm\}, \quad (2.5)$$

where θ is the relative rapidity of the scattering particles, $S_0(\theta)$ is the soliton-soliton scattering amplitude:

$$S_0(\theta) = -e^{i\chi(\theta)}, \quad \chi(\theta) = \int_0^\infty d\omega \frac{\sin(\omega\theta)}{\omega} \frac{\sinh(\frac{(p-1)\pi\omega}{2})}{\cosh(\frac{\pi\omega}{2}) \sinh(\frac{p\pi\omega}{2})}. \quad (2.6)$$

The nonzero matrix elements of $\mathcal{S}_{ab}^{cd}(\theta)$ in (2.5) can be expressed in terms of elementary functions as follows:

$$\begin{aligned} S_{++}^{++}(\theta) &= S_{--}^{--}(\theta) = 1, \\ S_{+-}^{+-}(\theta) &= S_{-+}^{-+}(\theta) = B_0(\theta), \\ S_{+-}^{+ -}(\theta) &= S_{-+}^{- +}(\theta) = C_0(\theta), \end{aligned} \quad (2.7)$$

where

$$B_0(\theta) = \frac{\sinh \frac{\theta}{p}}{\sinh \frac{i\pi - \theta}{p}}, \quad (2.8)$$

$$C_0(\theta) = \frac{\sinh \frac{i\pi}{p}}{\sinh \frac{i\pi - \theta}{p}}. \quad (2.9)$$

The S-matrix (2.5) obeys the Yang-Baxter equation²:

$$\mathcal{S}_{k_2 k_3}^{j_2 j_3}(\theta_{23}) \mathcal{S}_{k_1 i_3}^{j_1 k_3}(\theta_{13}) \mathcal{S}_{i_1 i_2}^{k_1 k_2}(\theta_{12}) = \mathcal{S}_{k_1 k_2}^{j_1 j_2}(\theta_{12}) \mathcal{S}_{i_1 k_3}^{k_1 j_3}(\theta_{13}) \mathcal{S}_{i_2 i_3}^{k_2 k_3}(\theta_{23}), \quad (2.10)$$

with $\theta_{ij} = \theta_i - \theta_j$, for $i, j \in \{1, 2, 3\}$, and it satisfies the properties as follows:

- Parity-symmetry: $\mathcal{S}_{ab}^{cd}(\theta) = \mathcal{S}_{ba}^{dc}(\theta), \quad (2.11)$

- Time-reversal symmetry: $\mathcal{S}_{ab}^{cd}(\theta) = \mathcal{S}_{cd}^{ab}(\theta), \quad (2.12)$

- Crossing-symmetry: $\mathcal{S}_{ab}^{cd}(\theta) = \mathcal{S}_{a\bar{d}}^{c\bar{b}}(i\pi - \theta), \quad (2.13)$

- Unitarity: $\mathcal{S}_{ab}^{ef}(\theta) \mathcal{S}_{ef}^{cd}(-\theta) = \delta_a^c \delta_b^d, \quad (2.14)$

- Real analyticity: $\mathcal{S}_{ab}^{cd}(\theta)^* = \mathcal{S}_{ab}^{cd}(-\theta^*), \quad (2.15)$

where for any index a , \bar{a} denotes the charge conjugated particle ($\bar{a} = -a$). The charge conjugate of a soliton is an antisoliton and vice versa, thus the charge conjugation matrix

²The matrix part $\mathcal{S}_{ab}^{cd}(\theta)$ of the S-matrix also satisfies the Yang-Baxter equation.

acting on the two dimensional vector space spanned by the soliton and the antisoliton, is equal to the first Pauli-matrix:

$$C = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{or equivalently: } C_{ab} = \delta_{a\bar{b}}. \quad (2.16)$$

In this paper we determine the finite volume expectation values of the operators as follows; the trace of the stress energy tensor:

$$\Theta = 2\alpha_0 \left(1 - \frac{\beta^2}{8\pi}\right) \cos(\beta\Phi). \quad (2.17)$$

and the $U(1)$ current of the theory:

$$J_\mu = \frac{\beta}{2\pi} \epsilon_{\mu\nu} \partial^\nu \Phi, \quad \mu = 0, 1, \quad (2.18)$$

where $\epsilon_{\mu\nu}$ denotes the antisymmetric matrix with nonzero entries: $\epsilon_{10} = -\epsilon_{01} = 1$.

Both operators correspond to some conserved quantity of the theory and in the subsequent sections their finite volume expectation values will be expressed in terms of the counting-function governing the finite volume spectrum of the theory.

The two operators have different parities under charge conjugation; Θ is positive, while J_μ is negative. This property proves to be an important difference between the two operators, when the symmetric diagonal limit of their form-factors are computed.

3 Finite volume expectation values of Θ and J_μ

In this section we give the equations, which govern the finite volume dependence of all diagonal form-factors of the trace of the stress energy tensor (Θ) and of the $U(1)$ current (J_μ) of the sine-Gordon theory. The equations for pure solitonic expectation values have been derived in [19, 20]. The derivations were based on an integrable lattice regularization of the model, on the so-called light-cone lattice regularization [22]. In this section we extend the results of [19, 20] from the pure soliton sector to all excited states of the model. To keep the paper within reasonable size, instead of repeating the lattice regularization based derivations we will derive the equations in a more pragmatic way. From [41] it is well known, that the expectation values of the trace of the stress energy tensor can be computed from the finite volume dependence of the energy of the sandwiching state by the formula as follows:

$$\langle \Theta \rangle_L = \mathcal{M} \left(\frac{E(\ell)}{\ell} + \frac{d}{d\ell} E(\ell) \right), \quad (3.1)$$

where $\ell = \mathcal{M}L$ with \mathcal{M} and L being the soliton mass and the finite volume respectively. This implies that the diagonal form-factors of Θ can be expressed in terms of certain derivatives of the counting-function of the model [20]. In the case of Θ , the derivatives entering the equations are the derivative with respect to the spectral parameter and the derivative with respect to the dimensionless volume of the model. The computations

achieved in the pure soliton sector [19] imply, that the same derivatives describe the finite volume dependence of the expectation values of the $U(1)$ current, too.

In order to formulate the equations describing the finite volume diagonal form-factors of our interest, we have to recall how the finite volume spectrum of the theory is described in terms of the nonlinear integral equations (NLIE) [24, 25] satisfied by the counting-function. Since we know that the expectation values of our interest can be expressed in terms of certain derivatives of the counting function, we can skip the intermediate lattice versions of the equations, and we can formulate the problem directly in the continuum limit.

3.1 Nonlinear integral equations for the counting-function

The counting-function $Z(\theta)$ is a periodic function on the complex plane with period $i\pi(1+p)$. To describe general excited states of the model one needs to know how to determine $Z(\theta)$ for any θ lying in the whole strip $[-i\frac{\pi}{2}(1+p), i\frac{\pi}{2}(1+p)]$. The counting-function satisfies different equations in the different domains of the periodicity strip. In the fundamental domain defined by the strip $|\text{Im}\theta| \leq \min(p\pi, \pi)$ the continuum limit of the counting-function satisfies the nonlinear-integral equations as follows [28, 29, 30, 31]:

$$\begin{aligned} Z(\theta) = & \ell \sinh \theta + \sum_{k=1}^{m_H} \chi(\theta - h_k) - \sum_{k=1}^{m_C} \chi(\theta - c_k) - \sum_{k=1}^{m_S} (\chi(\theta - y_k + i\eta) + \chi(\theta - y_k - i\eta)) \\ & - \sum_{k=1}^{m_W} \chi_{II}(\theta - w_k) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi i} G(\theta - \theta' - i\eta) L_+(\theta' + i\eta) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi i} G(\theta - \theta' + i\eta) L_-(\theta' - i\eta), \end{aligned} \quad (3.2)$$

where

$$L_{\pm}(\theta) = \ln \left(1 + (-1)^{\delta} e^{\pm i Z(\theta)} \right), \quad (3.3)$$

such that the parameter δ can take values 0 or 1. Its value affect the quantization equations of the objects entering the source terms of the integral equation. In (3.2), $\chi(\theta)$ is the soliton-soliton scattering phase given by (2.6) and $G(\theta)$ denotes its derivative. It can be given by the Fourier-integral as follows:

$$G(\theta) = \frac{d}{d\theta} \chi(\theta) = \int_{-\infty}^{\infty} d\omega e^{-i\omega\theta} \frac{\sinh(\frac{(p-1)\pi\omega}{2})}{2 \cosh(\frac{\pi\omega}{2}) \sinh(\frac{p\pi\omega}{2})}. \quad (3.4)$$

The equations contain the so-called second determination [28] of $\chi(\theta)$, as well. For any function f , the definition of second determination is different in the attractive ($0 < p < 1$) and repulsive ($1 < p$) regimes of the model:

$$f_{II}(\theta) = \begin{cases} f(\theta) + f(\theta - i\pi \text{sign}(\text{Im}\theta)), & 1 < p, \\ f(\theta) - f(\theta - i\pi p \text{sign}(\text{Im}\theta)), & 0 < p < 1. \end{cases} \quad (3.5)$$

For the function $\chi(\theta)$ we provide the concrete functional forms as well [32]:

$$\chi_{II}(\theta) = \begin{cases} i \operatorname{sign}(\operatorname{Im} \theta) \left(\log \sinh \frac{\theta}{p} - \log \sinh \frac{\theta - i\pi \operatorname{sign} \operatorname{Im} \theta}{p} \right), & 1 < p, \\ i \operatorname{sign}(\operatorname{Im} \theta) \left(\log \left(-\tanh \frac{\theta}{p} \right) + \log \tanh \frac{\theta - i\pi p \operatorname{sign} \operatorname{Im} \theta}{p} \right), & 0 < p < 1. \end{cases} \quad (3.6)$$

In (3.2), η is an arbitrary positive contour-deformation parameter, which should be in the range $[0, \min(p\pi, \pi, |\operatorname{Im} c_j|)]$. As we have already mentioned, ℓ denotes the dimensionless volume made out of the the soliton mass \mathcal{M} and of the volume L of the theory by the formula $\ell = \mathcal{M}L$. All objects entering the source terms in (3.2) satisfy the equation:

$$1 + (-1)^\delta e^{iZ(\mathfrak{D})} = 0, \quad \mathfrak{D} \in \{h_k\}_{k=1}^{m_H} \cup \{c_k\}_{k=1}^{m_C} \cup \{w_k\}_{k=1}^{m_W} \cup \{y_k\}_{k=1}^{m_S}. \quad (3.7)$$

It is useful to classify them as follows [28]:

- holes: $h_k \in \mathbb{R}$, $k = 1, \dots, m_H$
- close roots: c_k $k = 1, \dots, m_C$, with $|\operatorname{Im} c_k| \leq \min(\pi, p\pi)$,
- wide roots: w_k $k = 1, \dots, m_W$, with $\min(\pi, p\pi) < |\operatorname{Im} w_k| \leq \frac{\pi}{2}(1+p)$,
- special objects³: $y_k \in \mathbb{R}$, $k = 1, \dots, m_S$ defined by the equations $1 + (-1)^\delta e^{iZ(y_k)} = 0$ with $Z'(y_k) < 0$.

Their numbers determine the topological charge Q of the state by the so-called counting-equation:

$$Q = m_H - 2m_S - m_C - 2H(p-1)m_W, \quad (3.8)$$

where here $H(x)$ denotes the Heaviside-function. As a consequence of (3.7) the source objects satisfy the quantization equations as follows:

$$\bullet \text{ holes: } Z(h_k) = 2\pi I_{h_k}, \quad I_{h_k} \in \mathbb{Z} + \frac{1+\delta}{2}, \quad k = 1, \dots, m_H, \quad (3.9)$$

$$\bullet \text{ close roots: } Z(c_k) = 2\pi I_{c_k}, \quad I_{c_k} \in \mathbb{Z} + \frac{1+\delta}{2}, \quad k = 1, \dots, m_C, \quad (3.10)$$

$$\bullet \text{ wide roots: } Z(w_k) = 2\pi I_{w_k}, \quad I_{w_k} \in \mathbb{Z} + \frac{1+\delta}{2}, \quad k = 1, \dots, m_W, \quad (3.11)$$

$$\bullet \text{ special objects: } Z(y_k) = 2\pi I_{y_k}, \quad I_{y_k} \in \mathbb{Z} + \frac{1+\delta}{2}, \quad k = 1, \dots, m_S. \quad (3.12)$$

From this list one can see that the actual value of the parameter $\delta \in \{0, 1\}$ determines whether the source objects are quantized by integer or half integer quantum numbers. It was shown in [29, 30, 31], that not all choices of δ are possible to describe properly the states of the sine-Gordon or of the Massive Thirring model. To describe the proper states

³With this interpretation of special objects the contour deformation parameter η should be considered to be a positive infinitesimal number.

of these quantum field theories the following selection rules have to be satisfied by the parameter δ :

- $\frac{Q + \delta + M_{sc}}{2} \in \mathbb{Z}$, sine-Gordon, (3.13)

- $\frac{\delta + M_{sc}}{2} \in \mathbb{Z}$, massive Thirring, (3.14)

where here M_{sc} stands for the number of self-conjugate roots, which are such wide roots, whose imaginary parts are fixed by the periodicity of $Z(\theta)$ to $i \frac{\pi}{2}(1+p)$.

In order to be able to impose the quantization equations (3.11) for the wide roots, the integral representation of $Z(\theta)$ must be known in the strip $\min(p\pi, \pi) < \text{Im } \theta \leq \frac{\pi}{2}(1+p)$, as well. In this "wide-root domain" $Z(\theta)$ is given by the equations as follows [28, 29, 30, 31]:

$$Z(\theta) = \ell \sinh_{II}(\theta) + \mathcal{D}_{II}(\theta) + \sum_{\alpha=\pm} \alpha \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi i} G_{II}(\theta - \theta' - i\alpha\eta) L_{\alpha}(\theta' + i\alpha\eta), \quad (3.15)$$

where $\mathcal{D}_{II}(\theta)$ is the second determination (3.5) of the source term function of (3.2):

$$\mathcal{D}(\theta) = \sum_{k=1}^{m_H} \chi(\theta - h_k) - \sum_{k=1}^{m_C} \chi(\theta - c_k) - \sum_{k=1}^{m_S} (\chi(\theta - y_k + i\eta) + \chi(\theta - y_k - i\eta)) - \sum_{k=1}^{m_W} \chi_{II}(\theta - w_k). \quad (3.16)$$

The energy and momentum of the model can be expressed in terms of the solution of the nonlinear integral equations by the following formulas [28, 29, 30, 31]:

$$E(L) = \mathcal{M} \left(\sum_{k=1}^{m_H} \cosh(h_k) - \sum_{k=1}^{m_C} \cosh(c_k) - \sum_{k=1}^{m_S} (\cosh(y_k + i\eta) + \cosh(y_k - i\eta)) - \sum_{k=1}^{m_W} \cosh_{II}(w_k) - \int_{-\infty}^{\infty} \frac{d\theta}{2\pi i} \sinh(\theta + i\eta) L_+(\theta + i\eta) + \int_{-\infty}^{\infty} \frac{d\theta}{2\pi i} \sinh(\theta - i\eta) L_-(\theta - i\eta) \right), \quad (3.17)$$

$$P(L) = \mathcal{M} \left(\sum_{k=1}^{m_H} \sinh(h_k) - \sum_{k=1}^{m_C} \sinh(c_k) - \sum_{k=1}^{m_S} (\sinh(y_k + i\eta) + \sinh(y_k - i\eta)) - \sum_{k=1}^{m_W} \sinh_{II}(w_k) - \int_{-\infty}^{\infty} \frac{d\theta}{2\pi i} \cosh(\theta + i\eta) L_+(\theta + i\eta) + \int_{-\infty}^{\infty} \frac{d\theta}{2\pi i} \cosh(\theta - i\eta) L_-(\theta - i\eta) \right). \quad (3.18)$$

3.2 Expectation values of Θ

The computations of finite volume expectation values of the trace of the stress energy tensor goes analogously to the former computations done in purely elastic scattering theories

[12]. Formula (3.1) implies that the finite volume expectation values of the trace of the stress energy tensor Θ can be expressed in terms of the θ and ℓ derivatives of $Z(\theta)$. By differentiating the equations (3.2)-(3.15) it is easy to show that these derivatives satisfy linear integral equations with kernels containing the counting-equation itself [19, 20].

We introduce two functions with related sets of discrete variables by the definitions as follows:

$$\begin{aligned}
\mathcal{G}_d(\theta) &= Z'(\theta), \\
X_{d,k}^{(h)} &= \frac{\mathcal{G}_d(h_k)}{Z'(h_k)} = 1, \quad k = 1, \dots, m_H, \\
X_{d,k}^{(c)} &= \frac{\mathcal{G}_d(c_k)}{Z'(c_k)} = 1, \quad k = 1, \dots, m_C, \\
X_{d,k}^{(y)} &= \frac{\mathcal{G}_d(y_k)}{Z'(y_k)} = 1, \quad k = 1, \dots, m_S, \\
X_{d,k}^{(w)} &= \frac{\mathcal{G}_d(w_k)}{Z'(w_k)} = 1, \quad k = 1, \dots, m_W.
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
\mathcal{G}_\ell(\theta) &= \frac{d}{d\ell} Z(\theta|\ell), \\
X_{\ell,k}^{(h)} &= \frac{\mathcal{G}_\ell(h_k)}{Z'(h_k)} = -h'_k(\ell), \quad k = 1, \dots, m_H, \\
X_{\ell,k}^{(c)} &= \frac{\mathcal{G}_\ell(c_k)}{Z'(c_k)} = -c'_k(\ell), \quad k = 1, \dots, m_C, \\
X_{\ell,k}^{(y)} &= \frac{\mathcal{G}_\ell(y_k)}{Z'(y_k)} = -y'_k(\ell), \quad k = 1, \dots, m_S, \\
X_{\ell,k}^{(w)} &= \frac{\mathcal{G}_\ell(w_k)}{Z'(w_k)} = -w'_k(\ell), \quad k = 1, \dots, m_W.
\end{aligned} \tag{3.20}$$

Taking the appropriate derivatives of the NLIE (3.2)-(3.15) it can be shown, that the variables in (3.19) and in (3.20) satisfy sets of linear integral equations. We relegated these equations to appendix A, where their explicit form is given by the formulas (A.1)-(A.11).

With the help of (3.1) it can be shown, that the finite volume expectation value of Θ in a state described by the NLIE (3.2)-(3.15) can be expressed in terms of the variables (3.19) and (3.20) by the following formula:

$$\langle \Theta \rangle_L = \langle \Theta \rangle_\infty + \mathcal{M}^2 \Theta_{rest}(\ell), \tag{3.21}$$

where $\langle \Theta \rangle_\infty$ stands for the infinite volume "bulk" vacuum expectation value [26, 27]:

$$\langle \Theta \rangle_\infty = -\frac{\mathcal{M}^2}{4} \tan\left(\frac{p\pi}{4}\right), \tag{3.22}$$

and $\Theta_{rest}(\ell)$ denotes the dimensionless part of the rest of the expectation value. It is given by the formula:

$$\begin{aligned} \Theta_{rest}(\ell) = & \sum_{k=1}^{m_H} \left(\cosh h_k \frac{X_{d,k}^{(h)}}{\ell} - \sinh h_k X_{\ell,k}^{(h)} \right) - \sum_{k=1}^{m_C} \left(\cosh c_k \frac{X_{d,k}^{(c)}}{\ell} - \sinh c_k X_{\ell,k}^{(c)} \right) - \\ & - \sum_{k=1}^{m_S} \left((\cosh(y_k + i\eta) + \cosh(y_k - i\eta)) \frac{X_{d,k}^{(y)}}{\ell} - (\sinh(y_k + i\eta) + \sinh(y_k - i\eta)) X_{\ell,k}^{(y)} \right) \\ & - \sum_{k=1}^{m_W} \left(\cosh_{II}(w_k) \frac{X_{d,k}^{(w)}}{\ell} - \sinh_{II}(w_k) X_{\ell,k}^{(w)} \right) + \\ & \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \left[\cosh(\theta + i\alpha\eta) \frac{\mathcal{G}_d(\theta + i\alpha\eta)}{\ell} - \sinh(\theta + i\alpha\eta) \mathcal{G}_\ell(\theta + i\alpha\eta) \right] \mathcal{F}_\alpha(\theta + i\alpha\eta), \end{aligned} \quad (3.23)$$

where $\mathcal{F}_\pm(\theta)$ stands for the nonlinear combinations:

$$\mathcal{F}_\pm(\theta) = \frac{(-1)^\delta e^{\pm i Z(\theta)}}{1 + (-1)^\delta e^{\pm i Z(\theta)}}. \quad (3.24)$$

3.3 Expectation values of J_μ

The finite volume expectation values of the $U(1)$ current can be derived from the light-cone lattice regularization [22] of the model. In this way the expectation values of J_μ between pure soliton states have been determined in [19]. Nevertheless, the computations of [19] can be easily extended to all excited states of the model. Here, we skip the lengthy, but quite straightforward computations and present only the final result. As the pure soliton results of [19] suggest, the expectation values of J_0 and J_1 can be expressed in terms of the set of variables of (3.19) and of (3.20), respectively:

$$\begin{aligned} \langle J_0 \rangle_L = & \frac{1}{L} \left\{ \sum_{j=1}^{m_H} X_{d,j}^{(h)} - 2 \sum_{j=1}^{m_S} X_{d,j}^{(y)} - \sum_{j=1}^{m_C} X_{d,j}^{(c)} - 2 H(p-1) \sum_{j=1}^{m_W} X_{d,j}^{(w)} - \right. \\ & \left. \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \mathcal{G}_d(\theta + i\alpha\eta) \mathcal{F}_\alpha(\theta + i\alpha\eta) \right\}, \end{aligned} \quad (3.25)$$

$$\begin{aligned} \langle J_1 \rangle_L = & \mathcal{M} \left\{ \sum_{j=1}^{m_H} X_{\ell,j}^{(h)} - 2 \sum_{j=1}^{m_S} X_{\ell,j}^{(y)} - \sum_{j=1}^{m_C} X_{\ell,j}^{(c)} - 2 H(p-1) \sum_{j=1}^{m_W} X_{\ell,j}^{(w)} - \right. \\ & \left. \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \mathcal{G}_\ell(\theta + i\alpha\eta) \mathcal{F}_\alpha(\theta + i\alpha\eta) \right\}. \end{aligned} \quad (3.26)$$

Here $H(x)$ is the Heaviside function. Using the definitions (3.19) and the counting equation (3.8), it is easy to show that formula (3.25) gives the correct result $\langle J_0 \rangle_L = \frac{Q}{L}$ for the finite volume expectation values of J_0 in sandwiching states with topological charge Q .

4 Large volume solution

In this section we provide exact formulas for the Bethe-Yang limit of the expectation values of the trace of the stress energy tensor and of the $U(1)$ current in the repulsive ($1 < p$) regime of the sine-Gordon model⁴. The reason why we restrict ourselves to the repulsive regime is that in this regime the correspondence between the Bethe-roots entering the NLIE (3.2) and magnonic Bethe-roots of the Bethe-Yang equations (B.12) is quite direct. In the attractive regime the correspondence is much more complicated and more indirect.

The first step to make a correspondence between the source objects or the Bethe-roots of the NLIE and the roots of (B.12), is to find the relation between the holes of the NLIE (3.2) and the rapidity of physical particles entering the magnonic part of the Bethe-Yang equations (B.12). It is well known in the literature [28] that the holes in the NLIE description describe the rapidities of the physical particles in the large volume limit ($\{h_j\} = \{\theta_j\}$). Then one has to know what kind of complexes the roots of the NLIE fall into, when the infinite volume limit is taken. In the repulsive regime these complexes are as follows [21]:

- 2-strings: $s_j^{(2)} = s_j \pm i \frac{\pi}{2}$, with: $s_j \in \mathbb{R}$, $j = 1, \dots, n_2$,
- quartets: $q_j = \{q_j^{(\pm)} \pm i \frac{\pi}{2}\}$, with: $q_j^{(+)} = (q_j^{(-)})^*$, $|\text{Im } q_j^{(\pm)}| \leq \frac{\pi}{2}$, $j = 1, \dots, n_4$,
- wide-roots: w_j , with: $\pi < |\text{Im } w_j| \leq \frac{(1+p)\pi}{2}$, $j = 1, \dots, m_w$, (4.1)

such that wide-roots either form complex conjugate pairs or they are self-conjugate roots with fixed imaginary part: $\text{Im } w_j^{(sc)} = \frac{(1+p)\pi}{2}$. From this classification, one can see that only the close-roots fall into special configurations in the infinite volume limit ($m_C = 2n_2 + 4n_4$). Namely, they form either quartets or 2-strings, where the latter can be thought of as degenerate quartets. Here we note that in the $\ell \rightarrow \infty$ limit there are no special objects, so they do not enter the expressions in this limit.

The counting equation (3.8) tells us how these root configurations act on the topological charge of a state:

- a 2-string decreases the charge by 2,
- a quartet decreases the charge by 4,
- a wide-root decreases the charge by 2.

On the other hand each root of the magnonic part of the Bethe-Yang equations decrease the topological charge of a state by 2 units and as it was mentioned in appendix B.2 the

⁴Our formulas are valid for the massive Thirring model, as well. Only the value of the parameter δ should be set in accordance with (3.14).

roots of these equations form conjugate pairs with respect to the line $\text{Im } z = \frac{\pi}{2}$. These suggest the following identification between the $\ell \rightarrow \infty$ complexes in (4.1) and the different types of roots of the magnonic part of the Bethe-Yang equations given in (B.36):

- Real-roots of (B.36) correspond to 2-strings in (4.1), such that the real part of a real-root is equal to the center of the corresponding 2-string: $\lambda_j - i\frac{\pi}{2} = s_j$.
- a close-pair $\lambda_j^{(\pm)}$ in (B.36) corresponds to a quartet in (4.1), such that the positions of the close-pair are given by complex conjugate pair describing the quartet: $\{\lambda_j^{(\pm)} - i\frac{\pi}{2}\} = \{q_j^{(\pm)}\}$.
- wide-roots in (B.36) correspond to wide roots of (4.1), such that: $\lambda_j^{wide} - i\frac{\pi}{2} = w_j - i \text{sign}(\text{Im } w_j) \frac{\pi}{2}$.

With this correspondence the $\ell \rightarrow \infty$ limit of the NLIE can be mapped to the equations (B.12). This proves to be important in finding an exact formula expressed in terms of the roots of (B.12) for the Bethe-Yang limit of the diagonal form-factors of Θ and J_μ .

To find the leading order large volume solution of the diagonal form-factors of Θ and J_μ , one should recognize that the integral terms in (3.21), (3.23), (3.25) and (3.26) are exponentially small in the volume, and so negligible at leading order. Consequently, the only task is to determine the Bethe-Yang limit of the discrete variables $X_{d,j}$ and $X_{\ell,j}$. They are solutions of the equations (A.1)-(A.7). For the first sight, it does not seem to be easy to find the general solutions of these equations in the large volume limit, but the relations of these X -variables to the θ and ℓ derivatives of the counting-function given in (3.19) and in (3.20), makes it quite easy to find the required solutions.

First let us consider the variables related to the ℓ derivative of $Z(\theta)$ in (3.20). Then using (3.20), for a complex root⁵ u_j the corresponding X -variable can be written as:

$$X_{\ell,j}^{(u)} = -u_j'(\ell) = -\sum_{k=1}^{m_H} \frac{\partial u_j}{\partial h_k} h_k'(\ell) \stackrel{\ell \rightarrow \infty}{\approx} \sum_{k=1}^{m_H} \frac{\partial \lambda_j}{\partial h_k} X_{\ell,j}^{(h)}, \quad (4.2)$$

where we used (3.20) and exploited the large volume correspondence between the Bethe-roots of the NLIE (3.2) and magnonic the Bethe-roots of (B.12). The formula (4.2) expresses the complex root's X -variables in terms of those of the holes in the large volume limit. Then the large ℓ solution goes as follows: One should insert (4.2) into (A.5) taken at $\nu = \ell$, such that the integral terms are neglected because they are exponentially small in volume. This way one gets a closed discrete set of linear equations for the variables $X_{\ell,j}^{(h)}$.

The equations through (4.2) contain the derivative matrix $\frac{\partial \lambda_j}{\partial h_k}$, which can be computed by differentiating the logarithm of the equations (B.12). The final result takes the form:

$$\frac{\partial \lambda_j}{\partial h_k} = \sum_{s=1}^r \psi_{js}^{-1} V_{sk}, \quad V_{sk} = (\ln B_0)'(\lambda_s - h_k), \quad k = 1, \dots, m_H, \quad s = 1, \dots, r, \quad (4.3)$$

⁵For more precise notation see (A.1) in appendix A.

where we exploited the infinite volume correspondence between the holes and the rapidities of the physical particles $\{h_j\}_{j=1}^{m_H} \leftrightarrow \{\theta_j\}_{j=1}^n$ and we introduced ψ , a symmetric $r \times r$ matrix with the definition as follows:

$$\psi_{jk} = z'(\lambda_j) \delta_{jk} + (\ln E_0)'(\lambda_j - \lambda_k), \quad j, k = 1, \dots, r, \quad (4.4)$$

with

$$z(\lambda) = \sum_{k=1}^{m_H} \ln B_0(\lambda - h_k) - \sum_{k=1}^r \ln E_0(\lambda - \lambda_k), \quad E_0(\lambda) = \frac{B_0(\lambda)}{B_0(-\lambda)}. \quad (4.5)$$

Then equations (A.5) with $\nu = \ell$ for $X_{\ell,j}^{(h)}$ can be written in the repulsive regime as follows:

$$\sum_{k=1}^{m_H} \Phi_{jk} X_{\ell,k}^{(h)} = \sinh h_j, \quad j = 1, \dots, m_H, \quad (4.6)$$

where Φ_{jk} is the Gaudin-matrix of the physical particles in the state described by the magnonic roots $\{\lambda_j\}_{j=1}^r$:

$$\Phi_{jk} = \begin{cases} \ell \cosh h_j + \sum_{\substack{s=1 \\ s \neq j}}^{m_H} \tilde{G}_{js}, & j = k, \\ -\tilde{G}_{jk}, & j \neq k, \end{cases} \quad (4.7)$$

$$\tilde{G}_{jk} = G_{jk} + \frac{1}{i} \sum_{s,q=1}^r V_{sj} \psi_{sq}^{-1} V_{qk}, \quad j, k = 1, \dots, m_H. \quad (4.8)$$

Here V_{jk} is defined in (4.3) and we introduced the short notation: $G_{jk} = G(h_j - h_k)$. Now it is easy to solve (4.6) for $X_{\ell,j}^{(h)}$:

$$X_{\ell,j}^{(h)} = \sum_{k=1}^{m_H} \Phi_{jk}^{-1} \sinh h_k, \quad j = 1, \dots, m_H. \quad (4.9)$$

Then using (4.2) and (4.3), the X -variables of the complex roots can also be obtained from (4.9):

$$X_{\ell,j}^{(u)} = \sum_{k=1}^{m_H} \sum_{s=1}^r \psi_{js}^{-1} V_{sk} \sum_{k'=1}^{m_H} \Phi_{kk'}^{-1} \sinh h_{k'}, \quad j = 1, \dots, r. \quad (4.10)$$

As for the d -type of X -variables, from (3.19) we know exactly that the value of each of them is exactly 1. Nevertheless, with the help of the large ℓ solution of (A.5), this value can be expressed in a more complicated way, as well:

$$X_{d,j}^{(h)} = \sum_{k=1}^{m_H} \Phi_{jk}^{-1} \ell \cosh h_k, \quad j = 1, \dots, m_H, \quad (4.11)$$

$$X_{d,j}^{(u)} = \sum_{k=1}^{m_H} \sum_{s=1}^r \psi_{js}^{-1} V_{sk} \sum_{k'=1}^{m_H} \Phi_{kk'}^{-1} \ell \cosh h_{k'}, \quad j = 1, \dots, r. \quad (4.12)$$

For the derivation of the second expression the following discrete set of equations should have been used as well:

$$\sum_{j=1}^r \sum_{k=1}^{m_H} V_{jk} \sum_{q=1}^r \psi_{jq}^{-1} V_{qs} = \sum_{j=1}^r V_{js}, \quad s = 1, \dots, m_H, \quad (4.13)$$

which can be derived from the logarithmic derivative of the (B.12). The point in making the simple to complicated is that in this way the solutions for both subscript $\nu = \ell, d$ can be written on equal footing:

$$X_{\nu,j}^{(h)} = \sum_{k=1}^{m_H} \Phi_{jk}^{-1} f_{\nu}(h_k), \quad \nu \in \{d, \ell\}, \quad j = 1, \dots, m_H, \quad (4.14)$$

$$X_{d,j}^{(u)} = \sum_{k=1}^{m_H} \sum_{s=1}^r \psi_{js}^{-1} V_{sk} \sum_{k'=1}^{m_H} \Phi_{kk'}^{-1} f_{\nu}(h_{k'}), \quad \nu \in \{d, \ell\}, \quad j = 1, \dots, r, \quad (4.15)$$

where $f_{\nu}(\theta)$ is the source term of the linear problem (A.4). It is given in (A.8) and (A.9). Inserting the large volume solutions (4.14), (4.15) into the expectation value formulas: (3.23), (3.25) and (3.26), one ends up with the large volume solutions as follows:

$$\Theta_{rest}(\ell)_{BY} = \sum_{j,k=1}^{m_H} \left(\cosh h_j \Phi_{jk}^{-1} \cosh h_k - \sinh h_j \Phi_{jk}^{-1} \sinh h_k \right), \quad (4.16)$$

$$\langle J_0 \rangle_{BY} = \mathcal{M} \sum_{k,s=1}^{m_H} \Phi_{sk}^{-1} \cosh h_k \left(1 - 2 \sum_{j,q=1}^r \psi_{jq}^{-1} V_{qs} \right), \quad (4.17)$$

$$\langle J_1 \rangle_{BY} = \mathcal{M} \sum_{k,s=1}^{m_H} \Phi_{sk}^{-1} \sinh h_k \left(1 - 2 \sum_{j,q=1}^r \psi_{jq}^{-1} V_{qs} \right). \quad (4.18)$$

In the computations we have done so far, it was not necessary to impose the quantization equations (3.9) for the holes. Thus the formulas above can be considered as analytical functions of the m_H pieces of holes (rapidities). Nevertheless, if one would like to get the Bethe-Yang limit of the expectation values, formulas (4.16), (4.17) and (4.18) must be taken at the solutions of Bethe-Yang limit of the quantization equations (3.9), which takes the well-known form:

$$e^{i\ell \sinh \tilde{h}_j} \Lambda(\tilde{h}_j | \vec{h}) = 1, \quad j = 1, \dots, m_H, \quad (4.19)$$

where \tilde{h}_j denotes the solutions of the Bethe-Yang equations and $\Lambda(\theta | \vec{h})$ denotes that eigenvalue (B.14) of the soliton transfer matrix (B.4), which corresponds to the sandwiching state.

5 Form-factors in the sine-Gordon theory

Having the exact formulas (4.16), (4.17) and (4.18), for the Bethe-Yang limit of the expectation values of our operators, we would like to check analytically the conjecture of [10] for the Bethe-Yang limit of the diagonal matrix elements of local operators in a non-diagonally scattering theory. To do so, we need the infinite volume form-factors of the theory. There are several ways to determine these form-factors. The earliest construction is written in the seminal work of Smirnov [13]. Later other constructions arose in the literature, like Lukyanov's free-field representation [14, 15] and the off-shell Bethe-Ansatz based method of [16, 17]. In this section we summarize the axiomatic equations satisfied by the form-factors of local operators in an integrable quantum field theory.

Let $\mathcal{O}(x, t)$ a local operator of the theory. Then its matrix elements between asymptotic multiparticle states is given by [13]:

$$\begin{aligned} {}^{(in)}\langle \gamma_1, b_1; \dots; \gamma_m, b_m | \mathcal{O}(x, t) | \beta_1, a_1; \dots; \beta_n, a_n \rangle^{(in)} &= e^{i t (E_\gamma - E_\beta) - i x (P_\gamma - P_\beta)} \times \\ F_{\bar{b}_m \dots \bar{b}_1 a_1 \dots a_n}^{\mathcal{O}}(\gamma_m + i\pi - i\epsilon_m, \dots, \gamma_1 + i\pi - i\epsilon_1, \beta_1, \dots, \beta_n) &+ \text{Dirac-delta terms}, \end{aligned} \quad (5.1)$$

where the form-factors of the local operator \mathcal{O} is denoted by $F^{\mathcal{O}}$, ϵ_j s are positive infinitesimal numbers and the orderings $\beta_n < \dots < \beta_2 < \beta_1$, $\gamma_1 < \gamma_2 < \dots < \gamma_m$ are meant in the *in* states. The Latin and Greek letters denote the polarizations and rapidities of the sandwiching multisoliton states, respectively. Thus $a_j, b_j \in \{\pm\}$, and $E_\gamma, E_\beta, P_\gamma, P_\beta$ denote the energies and the momenta of the corresponding states:

$$\begin{aligned} E_\gamma &= \sum_{j=1}^m \mathcal{M} \cosh \gamma_j, & E_\beta &= \sum_{j=1}^n \mathcal{M} \cosh \beta_j, \\ P_\gamma &= \sum_{j=1}^m \mathcal{M} \sinh \gamma_j, & P_\beta &= \sum_{j=1}^n \mathcal{M} \sinh \beta_j. \end{aligned} \quad (5.2)$$

We choose the normalization for the scalar product of states in infinite volume as follows:

$${}^{(in)}\langle \gamma_1, b_1; \dots; \gamma_n, b_n | \beta_1, a_1; \dots; \beta_n, a_n \rangle^{(in)} = (2\pi)^n \prod_{j=1}^n \delta_{b_j a_j} \delta(\gamma_j - \beta_j). \quad (5.3)$$

In this convention the form-factors $F^{\mathcal{O}}$ of the operator $\mathcal{O}(x, t)$ satisfy the the following axioms [13]:

I. Lorentz-invariance:

$$F_{a_1 \dots a_n}^{\mathcal{O}}(\theta_1 + \theta, \dots, \theta_n + \theta) = e^{s(\mathcal{O})\theta} F_{a_1 \dots a_n}^{\mathcal{O}}(\theta_1, \dots, \theta_n), \quad (5.4)$$

where $s(\mathcal{O})$ is the Lorentz-spin of \mathcal{O} .

II. Exchange:

$$F_{\dots a_j a_{j+1} \dots}^{\mathcal{O}}(\dots, \theta_j, \theta_{j+1}, \dots) = \mathcal{S}_{a_j a_{j+1}}^{b_j b_{j+1}}(\theta_j - \theta_{j+1}) F_{\dots b_j b_{j+1} \dots}^{\mathcal{O}}(\dots, \theta_{j+1}, \theta_j, \dots), \quad (5.5)$$

III. Cyclic permutation:

$$F_{a_1 a_2 \dots a_n}^{\mathcal{O}}(\theta_1 + 2\pi i, \dots, \theta_n) = e^{2\pi i \omega(\mathcal{O})} F_{a_2 \dots a_n a_1}^{\mathcal{O}}(\theta_2, \dots, \theta_n, \theta_1), \quad (5.6)$$

where $\omega(\mathcal{O})$ denotes the mutual locality index between \mathcal{O} and the asymptotic field which creates the solitons.

IV. Kinematical singularity:

$$F_{abu_1 \dots u_n}^{\mathcal{O}}(\theta + i\pi + \epsilon, \theta, \theta_1, \dots, \theta_n) \stackrel{\epsilon \rightarrow 0}{\simeq} \frac{i}{\epsilon} \left\{ C_{ab} F_{u_1 \dots u_n}^{\mathcal{O}}(\theta_1, \dots, \theta_n) - e^{2\pi i \omega(\mathcal{O})} \sum_{v_1, \dots, v_n = \pm} \mathcal{T}_b^{\bar{a}}(\theta | \theta_1, \dots, \theta_n)_{u_1 \dots u_n}^{v_1 \dots v_n} F_{v_1 \dots v_n}^{\mathcal{O}}(\theta_1, \dots, \theta_n) \right\}, \quad (5.7)$$

where \mathcal{T} denotes the soliton monodromy matrix defined in (B.1), and C_{ab} is the charge conjugation matrix (2.16). In this paper we will focus on the repulsive regime of the sine-Gordon theory, where there are no soliton-antisoliton bound states in the spectrum. This is why we skipped to present the dynamical singularity axiom, which relates the form-factors of bound states to those of its constituents.

We just remark that for the operators of our interest the mutual locality index is zero: $\omega(\Theta) = \omega(J_\mu) = 0$.

6 The Pálmai-Takács conjecture

In this section we summarize the conjecture of Pálmai and Takács [10] for the Bethe-Yang limit of the diagonal matrix elements of local operators in a non-diagonally scattering theory. By Bethe-Yang limit we mean those terms of the large volume expansion which are polynomials in the inverse of the volume. Namely, the exponentially small in volume corrections are neglected from the exact result.

In finite volume the particle rapidities become quantized. The quantization equations which account for the polynomial in the inverse of the large volume corrections are called the Bethe-Yang equations. In a non-diagonally scattering theory at large volume, the amplitudes describing the "color part" of the multisoliton eigenstates of the Hamiltonian are eigenvectors of the soliton transfer matrix (B.4). They form a complete normalized basis on the space of "color" degrees of freedom of the wave function. On an n -particle state it can be formulated as follows:

$$\tau(\theta | \vec{\theta})_{a_1 \dots a_n}^{b_1 \dots b_n} \Psi^{(t)}(\vec{\theta})_{b_1 \dots b_n} = \Lambda^{(t)}(\theta | \vec{\theta}) \Psi^{(t)}(\vec{\theta})_{a_1 \dots a_n} \quad t = 1, \dots, 2^n, \quad (6.1)$$

$$\sum_{a_1, \dots, a_n = \pm} \Psi^{(s)}(\vec{\theta})_{a_1 \dots a_n} \Psi^{(t)*}(\vec{\theta})_{a_1 \dots a_n} = \delta_{st}, \quad (6.2)$$

$$\sum_{t=1}^{2^n} \Psi^{(t)}(\vec{\theta})_{a_1 \dots a_n} \Psi^{(t)*}(\vec{\theta})_{b_1 \dots b_n} = \prod_{j=1}^n \delta_{a_j b_j},$$

where for short we use the notation $\vec{\theta} = \{\theta_1, \dots, \theta_n\}$ and $\Lambda^{(t)}(\theta|\vec{\theta})$ stands for the eigenvalue of the t th eigenstate. This eigenvalue can be obtained by the Algebraic Bethe method [43] summarized in appendix B. Its expression in terms of the magnonic Bethe-roots is given by (B.14).

In the language of the soliton transfer matrix (B.4), for an n -particle state the Bethe-Yang quantization equations take the form:

$$e^{i\ell \sinh \tilde{\theta}_j} \Lambda^{(t)}(\tilde{\theta}_j|\vec{\tilde{\theta}}) = 1, \quad j = 1, \dots, n, \quad (6.3)$$

where we introduced the notation that the set $\{\theta_j\}_{j=1}^n$, means an arbitrary unquantized set of rapidities, while the set with tilde $\{\tilde{\theta}_j\}_{j=1}^n$, denotes the set of rapidities satisfying the Bethe-Yang equations (6.3). It is more common to rephrase (6.3) in its logarithmic form. To do so, first one has to define the function:

$$Q^{(t)}(\theta|\theta_1, \dots, \theta_n) = \ell \sinh \theta + \frac{1}{i} \Lambda^{(t)}(\theta|\{\theta_1, \dots, \theta_n\}). \quad (6.4)$$

Then the logarithmic form of the Bethe-Yang equations take the form:

$$Q^{(t)}(\tilde{\theta}_j|\tilde{\theta}_1, \dots, \tilde{\theta}_n) = 2\pi I_j^{(t)}, \quad j = 1, \dots, n, \quad t = 1, \dots, 2^n, \quad (6.5)$$

where $I_j^{(t)} \in \mathbb{Z}$ are the quantum numbers characterizing the individual rapidities of the t th eigenstate. The function $Q^{(t)}$ in (6.4) allows one to define the density of states in the t th eigenstate $\Psi^{(t)}$ by the Jacobi determinant as follows:

$$\rho^{(t)}(\theta_1, \dots, \theta_n) = \det \left\{ \frac{\partial Q^{(t)}(\theta_j|\theta_1, \dots, \theta_n)}{\partial \theta_k} \right\}_{j,k=1, \dots, n} \quad t = 1, \dots, 2^n. \quad (6.6)$$

With the help of the basis (6.1), (6.2) one can define form-factors being polarized with respect to the eigenvectors (6.1). In [10] this quantity was defined by the formula as follows:

$$\begin{aligned} F_{(s,t)}^{\mathcal{O}}(\theta'_m, \dots, \theta'_1|\theta_1, \dots, \theta_n) &= \sum_{b_1, \dots, b_m = \pm} \sum_{a_1, \dots, a_n = \pm} \Psi_{b_1 \dots b_m}^{(s)*}(\theta'_1, \dots, \theta'_m) \times \\ F_{\bar{b}_m \dots \bar{b}_1 a_1 \dots a_n}^{\mathcal{O}}(\theta'_m + i\pi, \dots, \theta'_1 + i\pi, \theta_1, \dots, \theta_n) &\Psi_{a_1 \dots a_n}^{(t)}(\theta_1, \dots, \theta_n), \quad s = 1, \dots, 2^m, \quad t = 1, \dots, 2^n. \end{aligned} \quad (6.7)$$

Based on our computations described in the forthcoming sections, we suggest the following slightly modified definition:

$$\begin{aligned} F_{(s,t)}^{\mathcal{O}}(\theta'_m, \dots, \theta'_1|\theta_1, \dots, \theta_n) &= \sum_{b_1, \dots, b_m = \pm} \sum_{a_1, \dots, a_n = \pm} \Psi_{b_1 \dots b_m}^{(s)}(\theta'_1, \dots, \theta'_m) \times \\ F_{\bar{b}_m \dots \bar{b}_1 a_1 \dots a_n}^{\mathcal{O}}(\theta'_m + i\pi, \dots, \theta'_1 + i\pi, \theta_1, \dots, \theta_n) &\Psi_{a_1 \dots a_n}^{(t)*}(\theta_1, \dots, \theta_n), \quad s = 1, \dots, 2^m, \quad t = 1, \dots, 2^n. \end{aligned} \quad (6.8)$$

The only difference between the two definitions is a $\Psi \rightarrow \Psi^*$ exchange. Or equivalently, as a consequence of the hermiticity property of the soliton transfer matrix (B.18), one can maintain the original form (6.7) for the definition of polarized form-factors, but in this case the vector Ψ should not be considered as a right eigenvector of the $\tau(\theta|\vec{\theta})$, but it should be considered as a left eigenvector of the soliton transfer matrix (B.4). In the rest of the paper we will keep the form of the original definition (6.7), but we will consider Ψ as a left eigenvector of $\tau(\theta|\vec{\theta})$.

Now we are in the position to formulate the conjecture of Pálmai and Takács for the expectation values of local operators in non-diagonally scattering theories. Let

$$|\bar{\theta}_1, \dots, \bar{\theta}_n\rangle_L^{(s)}, \quad (6.9)$$

that eigenstate of the Hamiltonian defined in finite volume L of the system, which is described by the eigenstate $\Psi^{(s)}$ of the soliton transfer matrix in the large volume limit. Here $\{\theta\}_{j=1}^n$ denote the exact finite volume rapidities, which become $\{\tilde{\theta}\}_{j=1}^n$ if the exponentially small in volume corrections are neglected in the large volume limit. Then the conjecture of [10] states that the finite volume expectation value of a local operator in an n -particle state can be written as follows:

$${}^{(s)}\langle \bar{\theta}_1, \dots, \bar{\theta}_n | \mathcal{O}(0,0) | \bar{\theta}_1, \dots, \bar{\theta}_n \rangle_L^{(s)} = F_n^{\mathcal{O},(s)}(\tilde{\theta}_1, \dots, \tilde{\theta}_n) + O(e^{-\ell}), \quad s = 1, \dots, 2^n, \quad (6.10)$$

where according to the conjecture, the function $F_n^{\mathcal{O},(s)}$, which should be taken at the positions of the roots of the Bethe-Yang equations (6.5) can be constructed from the infinite volume form-factors of the theory by the following formula:

$$F_n^{\mathcal{O},(s)}(\theta_1, \dots, \theta_n) = \frac{1}{\rho_n^{(s)}(1, \dots, n)} \sum_{A \subset \{1, \dots, n\}} \sum_{q,t} |C_{qt}^{(s)}(\{\theta_k\}|A)|^2 F_{2|A|, \text{symm}}^{\mathcal{O},(q)}(A) \rho_{|\bar{A}|}^{(t)}(\bar{A}), \quad (6.11)$$

where the first sum runs for all bipartite partitions of the set of indexes $A^{(n)} = \{1, 2, \dots, n\}$. Namely, $A \cup \bar{A} = A^{(n)}$. The number of elements of A is denoted by $|A|$, then the number of elements of \bar{A} is $|\bar{A}| = n - |A|$. In the sequel we denote the elements of the sets A and \bar{A} as follows⁶:

$$\begin{aligned} A &= \{A_1, A_2, \dots, A_{|A|}\}, \\ \bar{A} &= \{\bar{A}_1, \bar{A}_2, \dots, \bar{A}_{|\bar{A}|}\}. \end{aligned} \quad (6.12)$$

The second sum in (6.11) runs for all decompositions of the n -particle color wave function with respect to the normalized eigenvectors⁷ of the transfer matrices acting only on the index sets A and \bar{A} :

$$\Psi_{a_1 \dots a_n}^{(t)}(\theta_1, \dots, \theta_n) = \sum_{q=1}^{2^{|A|}} \sum_{s=1}^{2^{|\bar{A}|}} C_{qs}^{(t)}(\{\theta_k\}|A) \Psi_{a_{A_1} \dots a_{A_{|A|}}}^{(q)}(\theta_{A_1}, \dots, \theta_{A_{|A|}}) \Psi_{a_{\bar{A}_1} \dots a_{\bar{A}_{|\bar{A}|}}}^{(s)}(\theta_{\bar{A}_1}, \dots, \theta_{\bar{A}_{|\bar{A}|}}), \quad (6.13)$$

⁶Though it was not specified clearly in [10], we assume the following orderings within these sets: $A_i < A_j$ and $\bar{A}_i < \bar{A}_j$ if $i < j$.

⁷Normalized eigenvectors mean that they fulfill the conditions (6.1) and (6.2).

where as a consequence of (6.2) the branching coefficients $C_{qs}^{(t)}(\{\theta_k\}|A)$ satisfy the normalization condition:

$$\sum_{q,s} |C_{qs}^{(t)}(\{\theta_k\}|A)|^2 = 1. \quad (6.14)$$

Here we note, that the earlier discussed $\Psi \rightarrow \Psi^*$ exchange in the formulation of the problem, doesnot cause problem in the determination of these branching coefficients, since it corresponds to a simple complex conjugation. This is irrelevant from the conjecture's point of view, since the final formula depends only on the absolute value square of these branching coefficients.

Now we have two further missing definitions in (6.11). In accordance with [10] we introduced some more compact notations for the densities:

$$\begin{aligned} \rho_n^{(s)}(1, 2, \dots, n) &= \rho^{(s)}(\theta_1, \theta_2, \dots, \theta_n), \\ \rho_{|\bar{A}|}^{(t)}(\bar{A}) &= \rho^{(t)}(\theta_{\bar{A}_1}, \theta_{\bar{A}_2}, \dots, \theta_{\bar{A}_{|\bar{A}|}}), \end{aligned} \quad (6.15)$$

with $\rho^{(s)}$ functions in the right hand side given by (6.6). The last so far undefined object in (6.11) is $F_{2|A|,symm}^{\mathcal{O},(q)}(A)$. It is defined as the uniform diagonal limit of a (q, q) polarized form-factor of \mathcal{O} , such that the indexes of the rapidities of the sandwiching states run the set A :

$$F_{2|A|,symm}^{\mathcal{O},(q)}(A) = \lim_{\epsilon \rightarrow 0} F_{(q,q)}^{\mathcal{O}}(\theta_{A_{|A|} + \epsilon}, \dots, \theta_{A_1 + \epsilon} | \theta_{A_1}, \dots, \theta_{A_{|A|}}), \quad (6.16)$$

with $F_{(q,q)}^{\mathcal{O}}$ defined in (6.7). In analogy with the terminology in purely elastic scattering theories the function $F_{2n,symm}^{\mathcal{O},(q)}$ is called the the $2n$ -particle q -polarized symmetric diagonal form-factor of the operator \mathcal{O} .

For the operators Θ and J_μ the functions $F_n^{\mathcal{O},(s)}(\theta_1, \dots, \theta_n)$ were computed in the previous sections. Their form taken at the positions of the holes $\{h_j\}_{j=1}^{m_H}$ are given by the formulas (4.16), (4.17) and (4.18). In the rest of the paper we will compare these formulas with the conjecture (6.10), (6.11) applied to the operators Θ and J_μ . In the forthcoming sections we will do the comparison upto 3-particle states. The only missing piece to this comparison is the knowledge of the symmetric diagonal form-factors. Thus our next task is to compute them upto the required particle numbers.

7 Symmetric diagonal form-factors for Θ and J_μ

Both the trace of the stress energy tensor and the $U(1)$ current are related to some conserved quantities of the theory. In purely elastic scattering theories the symmetric diagonal form-factors of such operators can be computed in a simple way [18, 40]. The key point in the computation is that by exploiting of the corresponding conservation law, it is not necessary to find the explicit solutions of the axioms (5.4)-(5.7).

In this paper we use the same method to compute the symmetric diagonal form-factors upto 3-particle states. It turns out that this simple method allows one to compute the

symmetric diagonal form-factors for any number of particles in the pure soliton sector, but for soliton-antisoliton mixed states it works only upto 3-particle states. For higher number of particles the explicit solution of the axioms (5.4)-(5.7) is required.

The form-factor axioms allow one to compute form-factors of higher number of particles from those of lower number of particles. Thus, we should start with the computation of the 2-particle symmetric diagonal form-factors of the operators of our interest.

7.1 2-particle symmetric diagonal form-factors

The case of Θ :

The stress energy tensor $T_{\mu\nu}$ is a conserved quantity, which implies that it can be written as appropriate derivative of some Lorentz scalar field ϕ :

$$T_{\mu\nu} = (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^\tau \partial_\tau) \phi, \quad (7.1)$$

where $\eta_{\mu\nu}$ is the 2-dimensional Minkowski metric. In this representation the trace of the stress energy tensor take the form:

$$\Theta = T^\mu_\mu = (\partial_1^2 - \partial_0^2) \phi. \quad (7.2)$$

It can be shown [44], that the Lorentz scalar field ϕ is not a local quantum field. Consequently, not all of its form-factors satisfy the axioms (5.4)-(5.7). To be more precise from the representation (7.1), it can be shown, that the 3- or more particle form-factors of ϕ satisfy the axioms (5.4)-(5.7), but the 2-particle ones become more singular, than it is expected from (5.7). (See (7.6).)

Using the space-time structure of the form-factors (5.1), the form-factors of Θ being close to the diagonal limit can be written as follows:

$$F^\Theta(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) = -\mathcal{M}^2 \left[\sum_{j,k=1}^n \epsilon_j \epsilon_k \cosh(\theta_j - \theta_k) + O(\epsilon^3) \right] F^\phi(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n), \quad (7.3)$$

where F^ϕ denotes the form-factors of the scalar operator ϕ in (7.2) and we introduced the notation $\hat{\theta}_j = \theta_j + i\pi + \epsilon_j$ for all values of the index j . In (7.3) the symbol $O(\epsilon^3)$ means at least cubic in ϵ terms when the uniform $\epsilon_1 = \dots = \epsilon_n = \epsilon \rightarrow 0$ limit is taken. For the sake of simplicity we did not write out the subscripts of the form-factors.

The basic idea of computing the 2-particle form-factors near their diagonal limit is that the near diagonal matrix elements of the Hamiltonian $\mathcal{H} = \int dx T_{00}$ can be computed in two different ways. First, it can be computed directly by acting with \mathcal{H} on the eigenstates:

$$\langle \theta + \epsilon, a | \mathcal{H} | \theta, b \rangle = 2\pi \mathcal{M} \cosh \theta \delta_{ab} \delta(\epsilon), \quad a, b \in \{\pm\}. \quad (7.4)$$

Second, it can be computed by using the representation $\int dx T_{00}$ for the Hamiltonian, and the matrix element is computed by integrating the space-time dependence of the

corresponding form-factor:

$$\langle \theta + \epsilon, a | \mathcal{H} | \theta, b \rangle = \int dx \langle \theta + \epsilon, a | T_{00}(x, 0) | \theta, b \rangle = -2\pi(\epsilon^2 + O(\epsilon^3)) \mathcal{M} \cosh \theta \delta(\epsilon) F_{ab}^\phi(\theta + i\pi + \epsilon, \theta), \quad (7.5)$$

where we used (5.1) and (7.3). Comparing the results (7.4) and (7.5) of the two different computations allows one to compute the near diagonal limit of the scalarized form-factor:

$$F_{ab}^\phi(\theta + i\pi + \epsilon, \theta) = -\frac{1}{\epsilon^2} \delta_{ab} + O(\frac{1}{\epsilon}), \quad a, b \in \{\pm\}. \quad (7.6)$$

Combining (7.6) with (7.3), the symmetric diagonal 2-particle form-factor of Θ can also be determined:

$$F_{ab}^\Theta(\theta + i\pi, \theta) = \mathcal{M}^2 \delta_{ab}, \quad a, b \in \{\pm\}. \quad (7.7)$$

The matrix structure δ_{ab} in (7.6) and (7.7) accounts for the charge conjugation invariance of the operator Θ .

The J_μ case:

The computation of the near diagonal limit of the 2-particle form-factors of the $U(1)$ current goes analogously to that of the operator Θ . The conservation law for the current implies the following representation:

$$J_0 = -i \partial_1 \psi, \quad J_1 = -i \partial_0 \psi, \quad (7.8)$$

with ψ being a (non-local) Lorentz scalar operator. The form-factors of ψ satisfy the same form-factor axioms as the form-factors of ϕ do. This together with (5.1) gives the following representation for the near diagonal form-factors:

$$\begin{aligned} F^{J_0}(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) &= -\mathcal{M} \left[\sum_{j=1}^n \cosh \theta_j \epsilon_j + O(\epsilon^2) \right] F^\psi(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n), \\ F^{J_1}(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) &= \mathcal{M} \left[\sum_{j=1}^n \sinh \theta_j \epsilon_j + O(\epsilon^2) \right] F^\psi(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n). \end{aligned} \quad (7.9)$$

The topological charge $Q = \int dx J_0$ acts on one-particle states as follows:

$$Q|\theta, a\rangle = \sum_{b=\pm} q_{ab} |\theta, b\rangle, \quad a = \pm, \quad \text{with} \quad q_{++} = 1, \quad q_{--} = -1, \quad q_{+-} = q_{-+} = 0. \quad (7.10)$$

Using this action and the scalar product formula (5.3) the near diagonal limit of the matrix elements of the charge can be computed directly:

$$\langle \theta + \epsilon, a | Q | \theta, b \rangle = 2\pi q_{ba} \delta(\epsilon). \quad (7.11)$$

On the other hand this matrix element can also be computed by integrating the space-time dependence of the form-factor of J_0 :

$$\langle \theta + \epsilon, a | Q | \theta, b \rangle = \int dx \langle \theta + \epsilon, a | J_0 | \theta, b \rangle = \frac{2\pi}{\mathcal{M} \cosh \theta} \delta(\epsilon) F_{ab}^{J_0}(\theta + i\pi + \epsilon, \theta). \quad (7.12)$$

Comparing the results of the two different computations one obtains the symmetric diagonal limit of the 2-particle form-factors of J_0 :

$$F_{ab}^{J_0}(\theta + i\pi, \theta) = \mathcal{M} \cosh \theta q_{\bar{b}a}, \quad (7.13)$$

with q_{ab} given in (7.10). Formula (7.13) and (7.9) allows one to compute the near diagonal limit of the 2-particle scalarized form-factor F_{ab}^ψ :

$$F_{ab}^\psi(\theta + i\pi + \epsilon, \theta) = \frac{1}{\epsilon} q_{\bar{b}a} + O(\epsilon), \quad (7.14)$$

which together with (7.9) gives the 2-particle symmetric diagonal form-factor of J_1 as well:

$$F_{ab}^{J_1}(\theta + i\pi, \theta) = \mathcal{M} \sinh \theta q_{\bar{b}a}. \quad (7.15)$$

We note that the pure comparison of (7.13) and (7.9) would imply that in (7.14) there are $O(1)$ terms in ϵ as well. However, the Lorentz invariance (5.4), the cyclic axiom (5.6) and the charge conjugation negativity of J_μ , implies that the form-factor $F_{ab}^\psi(\theta + i\pi + \epsilon, \theta)$ is independent of θ and is an odd function of ϵ . This oddity forbids the appearance of constant in ϵ terms in the right hand side of (7.14).

7.2 4-particle symmetric diagonal form-factors

The next step in solving the form-factor axioms (5.4)-(5.7) in the near diagonal limit is the determination of the 4-particle form-factors. To obtain them we need to determine the singular-parts of the near diagonal 4-particle form-factors of the scalar fields ϕ and ψ of (7.1) and (7.8).

To analyse the near diagonal limit of 4-particle form-factors, the following two useful formulas can be derived from the appropriate combination of the axioms (5.5)-(5.7):

$$F_{a_2 a_1 b_1 b_2}(\hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2) = \frac{i}{\epsilon_1} \left\{ C_{a_1 b_1} F_{a_2 b_2}(\hat{\theta}_2, \theta_2) - \mathcal{T}_{b_1}^{\bar{a}_1}(\theta_1 | \theta_2, \tilde{\theta}_2')_{b_2 a_2}^{v_1 v_2} F_{v_2 v_1}(\hat{\theta}_2, \theta_2) \right\} + O(1)_{\epsilon_1}, \quad (7.16)$$

$$F_{a_2 a_1 b_1 b_2}(\hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2) = -\frac{i}{\epsilon_2} \left\{ C_{b_2 a_2} F_{a_1 b_1}(\hat{\theta}_1, \theta_1) - \mathcal{T}_{a_2}^{\bar{b}_2}(\hat{\theta}_2 | \hat{\theta}_1, \theta_1)_{a_1 b_1}^{v_1 v_2} F_{v_1 v_2}(\hat{\theta}_2, \theta_2) \right\} + O(1)_{\epsilon_2}, \quad (7.17)$$

where we introduced the short notation $\tilde{\theta}_j' = \theta_j - i\pi + \epsilon_j$ for any value of the index j . The symbol $O(1)_{\epsilon_1}$ denotes terms which are of order one in ϵ_1 .

The application of formulas (7.16) and (7.17) to the 4-particle form factors of the scalar field ϕ , one obtains the result as follows:

$$F_{\alpha\beta\gamma\delta}^\phi(\hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2) = \frac{1}{\epsilon_1 \epsilon_2} a_{\alpha\beta\gamma\delta}^\phi(\theta_1, \theta_2) + O\left(\frac{1}{\epsilon}\right), \quad \alpha, \beta, \gamma, \delta = \pm, \quad (7.18)$$

where the nonzero elements of the tensor $a^\phi(\theta_1, \theta_2)$ are as follows:

$$a_{--++}^\phi(\theta_1, \theta_2) = a_{++--}^\phi(\theta_1, \theta_2) = -G(\theta_1 - \theta_2), \quad (7.19)$$

$$a_{+-+-}^\phi(\theta_1, \theta_2) = a_{-+-+}^\phi(\theta_1, \theta_2) = -\varphi(\theta_1 - \theta_2), \quad (7.20)$$

$$a_{+--+}^\phi(\theta_1, \theta_2) = a_{-++-}^\phi(\theta_1, \theta_2) = -\Omega(\theta_1 - \theta_2). \quad (7.21)$$

The functions φ and Ω are given by the formulas:

$$\begin{aligned} \varphi(\theta) &= -i(C_0(\theta) B_0'(-\theta) + B_0(\theta) C_0'(-\theta)), \\ \Omega(\theta) &= -i(C_0(\theta) C_0'(-\theta) + B_0(\theta) B_0'(-\theta)) + G(\theta), \end{aligned} \quad (7.22)$$

where G , B_0 and C_0 are defined in (3.4), (2.8) and (2.9) respectively. As a consequence of the unitarity of the S-matrix (2.14), all the functions of (7.22) are even in θ . Inserting (7.18) with (7.19), (7.20) and (7.21) into (7.3) and taking the uniform $\epsilon_1 = \epsilon_2 = \epsilon \rightarrow 0$ limit, one obtains the symmetric diagonal 4-particle form-factors of Θ :

$$F_{--++}^{\Theta, \text{symm}}(\theta_1, \theta_2) = F_{++--}^{\Theta, \text{symm}}(\theta_1, \theta_2) = 2\mathcal{M}^2 (1 + \cosh(\theta_1 - \theta_2)) G(\theta_1 - \theta_2), \quad (7.23)$$

$$F_{+-+-}^{\Theta, \text{symm}}(\theta_1, \theta_2) = F_{-+-+}^{\Theta, \text{symm}}(\theta_1, \theta_2) = 2\mathcal{M}^2 (1 + \cosh(\theta_1 - \theta_2)) \Omega(\theta_1 - \theta_2), \quad (7.24)$$

$$F_{+--+}^{\Theta, \text{symm}}(\theta_1, \theta_2) = F_{-++-}^{\Theta, \text{symm}}(\theta_1, \theta_2) = 2\mathcal{M}^2 (1 + \cosh(\theta_1 - \theta_2)) \varphi(\theta_1 - \theta_2). \quad (7.25)$$

All functions entering these formulas are even, thus these form-factors are really symmetric with respect to the exchange of the two rapidities $\theta_1 \leftrightarrow \theta_2$.

The very same procedure can be repeated for the $U(1)$ current and for the scalar operator ψ associated to it by (7.8). We just write down the final results below. In the near diagonal limit the 4-particle form factors of the scalar ψ take the form:

$$F_{--++}^\psi(\hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2) = -F_{++--}^\psi(\hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2) = -2\pi\sigma(\theta_{12}) \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \right) + O(1)_\epsilon, \quad (7.26)$$

$$F_{+-+-}^\psi(\hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2) = -F_{-+-+}^\psi(\hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2) = \frac{\mathcal{G}_0(\theta_{12})}{\epsilon_1 \epsilon_2} + \frac{\mathcal{G}_1(\theta_{12})}{\epsilon_1} + \frac{\mathcal{G}_2(\theta_{12})}{\epsilon_2} + O(1)_\epsilon, \quad (7.27)$$

$$F_{+--+}^\psi(\hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2) = -F_{-++-}^\psi(\hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2) = \frac{\mathcal{H}_0(\theta_{12})}{\epsilon_1 \epsilon_2} + \frac{\mathcal{H}_1(\theta_{12})}{\epsilon_1} + \frac{\mathcal{H}_2(\theta_{12})}{\epsilon_2} + O(1)_\epsilon, \quad (7.28)$$

where $\theta_{12} = \theta_1 - \theta_2$ and

$$\mathcal{G}_0(\theta) = -i(B_0(\theta) C_0(-\theta) - C_0(\theta) B_0(-\theta)), \quad (7.29)$$

$$\mathcal{H}_0(\theta) = -i (1 + C_0(\theta) C_0(-\theta) - B_0(\theta) B_0(-\theta)), \quad (7.30)$$

$$\mathcal{G}_j(\theta) = g_j(\theta) + G(\theta) \hat{g}_j(\theta), \quad \mathcal{H}_j(\theta) = h_j(\theta) + G(\theta) \hat{h}_j(\theta), \quad j = 1, 2, \quad (7.31)$$

with

$$\begin{aligned} g_1(\theta) &= -i (B_0(\theta) C'_0(-\theta) - C_0(\theta) B'_0(-\theta)), & g_2(\theta) &= -g_1(\theta), \\ \hat{g}_1(\theta) &= B_0(\theta) C_0(-\theta) - C_0(\theta) B_0(-\theta), & \hat{g}_2(\theta) &= -\hat{g}_1(\theta), \end{aligned} \quad (7.32)$$

$$\begin{aligned} h_1(\theta) &= -i (C_0(\theta) C'_0(-\theta) - B_0(\theta) B'_0(-\theta)), & h_2(\theta) &= -h_1(\theta), \\ \hat{h}_1(\theta) &= C_0(\theta) C_0(-\theta) - B_0(\theta) B_0(-\theta), & \hat{h}_2(\theta) &= -\hat{h}_1(\theta). \end{aligned} \quad (7.33)$$

Then using (7.9) the symmetric diagonal 4-particle form-factors of J_μ can be computed. It turns out that only the ones which correspond to the expectation values in pure soliton or pure antisoliton states have finite uniform $\epsilon_1 = \epsilon_2 = \epsilon \rightarrow 0$ limit:

$$\begin{aligned} F_{--++}^{J_0, \text{symm}}(\theta_1, \theta_2) &= -F_{++--}^{J_0, \text{symm}}(\theta_1, \theta_2) = 2 \mathcal{M} (\cosh \theta_1 + \cosh \theta_2) G(\theta_{12}), \\ F_{--++}^{J_1, \text{symm}}(\theta_1, \theta_2) &= -F_{++--}^{J_1, \text{symm}}(\theta_1, \theta_2) = 2 \mathcal{M} (\sinh \theta_1 + \sinh \theta_2) G(\theta_{12}). \end{aligned} \quad (7.34)$$

The other form-factors will diverge as $\frac{1}{\epsilon}$ when the symmetric diagonal limit is taken. Nevertheless it can be shown, that these divergences cancel, when according to (6.7) the symmetric diagonal⁸ limit is taken between Bethe eigenvectors of the soliton transfer matrix. Simple application of the charge conjugation negativity of J_μ shows that these non-pure solitonic 4-particle symmetric diagonal form-factors are actually zero.

7.3 6-particle symmetric diagonal form-factors

If one would like to compute the symmetric diagonal limit of the 6-particle form-factors of the operators of our interest, after some computations it becomes obvious, that with fixed subscripts in general this diagonal limit does not exist. Namely, the $\epsilon \rightarrow 0$ limit becomes divergent. Nevertheless, in the Pálmai-Takács conjecture summarized in section 6, the symmetric diagonal limit of form-factors polarized with respect to eigenvectors of the soliton transfer matrix (6.7) should be determined. To do this computation, first we rewrite the necessary form-factor axioms in the language of the eigenvectors of the soliton transfer matrix (B.4). For our computations we need the appropriate versions of two axioms, the exchange (5.5) and the kinematical singularity (5.7) ones.

The kinematical pole axiom for a near diagonal settings of the rapidities can be written as follows:

$$\begin{aligned} F_{a_n \dots a_1 b_1 \dots b_n}(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) &= \frac{i}{\epsilon_1} \left\{ \delta_{b_1}^{\bar{a}_1} \prod_{k=2}^n \delta_{a_k}^{\beta_k} \delta_{b_k}^{\alpha_k} - \tau(\theta_1 | \vec{\theta})_{b_1 b_2 \dots b_n}^{l \alpha_2 \dots \alpha_n} \tau^{-1}(\theta_1^\epsilon | \vec{\theta}^\epsilon)_{l \bar{\beta}_2 \dots \bar{\beta}_n}^{\bar{a}_1 \bar{a}_2 \dots \bar{a}_n} \right\} \times \\ &F_{\beta_n \dots \beta_2 \alpha_2 \dots \alpha_n}(\hat{\theta}_n, \dots, \hat{\theta}_2, \theta_2, \dots, \theta_n) + O(1)_{\epsilon_1}, \end{aligned} \quad (7.35)$$

⁸Here the word diagonal means diagonality in the Bethe eigenstates, as well.

where we introduced the notations $\theta_j^\epsilon = \theta_j + \epsilon_j$ and $\vec{\theta}^\epsilon = \{\theta_1^\epsilon, \dots, \theta_n^\epsilon\}$. Now, analogously to the definition (6.7), one can sandwich this axiom with two color wave functions Ψ and $\Psi^{(\epsilon)*}$, such that they become complex conjugate to each other in the $\epsilon \rightarrow 0$ diagonal limit:

$$F_\Psi(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) = \sum_{i_1, \dots, i_n = \pm} \sum_{j_1, \dots, j_n = \pm} \Psi_{j_1 \dots j_n}^{(\epsilon)*} F_{j_n \dots j_1 i_1 \dots i_n}(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) \Psi^{i_1 \dots i_n}. \quad (7.36)$$

Then this form-factor satisfies the kinematical pole equation as follows:

$$F_\Psi(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) = \frac{i}{\epsilon_1} \left\{ \Psi_{k\bar{\beta}_2 \dots \bar{\beta}_n}^{(\epsilon)*} \Psi^{k\alpha_2 \dots \alpha_n} - \Psi^{i_1 \dots i_n} \tau(\theta_1 | \vec{\theta})_{i_1 i_2 \dots i_n}^{l\alpha_2 \dots \alpha_n} \tau^{-1}(\theta_1^\epsilon | \vec{\theta}^\epsilon)_{l\bar{\beta}_2 \dots \bar{\beta}_n}^{j_1 j_2 \dots j_n} \Psi_{j_1 \dots j_n}^{(\epsilon)*} \right\} \times \\ F_{\beta_n \dots \beta_2 \alpha_2 \dots \alpha_n}(\hat{\theta}_n, \dots, \hat{\theta}_2, \theta_2, \dots, \theta_n) + O(1)_{\epsilon_1}. \quad (7.37)$$

It follows, that this equation can be diagonalized, if Ψ is chosen to be a left eigenvector of $\tau(\theta_1 | \vec{\theta})$ and $\Psi^{(\epsilon)*}$ is chosen to be a right eigenvector of $\tau(\theta_1^\epsilon | \vec{\theta}^\epsilon)$:

$$\Psi^{i_1 \dots i_n} \tau(\theta_1 | \vec{\theta})_{i_1 i_2 \dots i_n}^{l\alpha_2 \dots \alpha_n} = \Lambda(\theta_1 | \vec{\theta}) \Psi^{l\alpha_2 \dots \alpha_n}, \\ \tau(\theta_1^\epsilon | \vec{\theta}^\epsilon)_{l\bar{\beta}_2 \dots \bar{\beta}_n}^{j_1 j_2 \dots j_n} \Psi_{j_1 \dots j_n}^{(\epsilon)*} = \Lambda(\theta_1^\epsilon | \vec{\theta}^\epsilon) \Psi_{l\bar{\beta}_2 \dots \bar{\beta}_n}^{(\epsilon)*}. \quad (7.38)$$

With such sandwiching states the kinematical singularity axiom in the near diagonal limit takes the form:

$$F_\Psi(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) = \frac{i}{\epsilon_1} \left(1 - \frac{\Lambda(\theta_1 | \vec{\theta})}{\Lambda(\theta_1^\epsilon | \vec{\theta}^\epsilon)} \right) \Psi_{k\bar{\beta}_2 \dots \bar{\beta}_n}^{(\epsilon)*} \Psi^{k\alpha_2 \dots \alpha_n} F_{\beta_n \dots \beta_2 \alpha_2 \dots \alpha_n}(\hat{\theta}_n, \dots, \hat{\theta}_2, \theta_2, \dots, \theta_n) \\ + O(1)_{\epsilon_1} \quad (7.39)$$

A few important comments are in order. First, we pay the attention that $\Psi^{(\epsilon)*}$ is not the complex conjugate vector of Ψ , because it is an eigenvector of a transfer matrix whose inhomogeneities are shifted with ϵ s with respect to those of τ . They form a conjugate pair only in the $\epsilon_j \rightarrow 0$ limit. On the other hand in [10] the symmetric diagonal form-factors are defined by a sandwich (6.7), where Ψ must be a **right** eigenvector of τ (B.4). Nevertheless, the near diagonal limit formulation of the kinematical singularity axiom (7.39) suggest, that the diagonal limit, should be taken such that in (6.7) the vector Ψ must be the **left** eigenvector of the transfer matrix (B.4). Actually this was the reason why we redefined the original definition of polarized form-factors (6.7) by the formula (6.8). Nevertheless, in the sequel we keep the defining formula (6.7), but based on the implications of formulas (7.38) and (7.39), we require Ψ to be a left eigenvector of $\tau(\theta_1 | \vec{\theta})$ and $\Psi^{(\epsilon)*}$ to be right eigenvector of $\tau(\theta_1^\epsilon | \vec{\theta}^\epsilon)$.

Now an important remark is in order. It is worth to analyse, what the form-factor equation (7.39) tells about the symmetric diagonal limit, when ϵ_j tends to zero uniformly. The term $\frac{i}{\epsilon_1} \left(1 - \frac{\Lambda(\theta_1 | \vec{\theta})}{\Lambda(\theta_1^\epsilon | \vec{\theta}^\epsilon)} \right)$ on the right hand side have a finite limiting value. The sum

$\Psi_{k\bar{\beta}_2\ldots\bar{\beta}_n}^* \Psi^{k\alpha_2\ldots\alpha_n} F_{\beta_n\ldots\beta_2\alpha_2\ldots\alpha_n}(\hat{\theta}_n, \ldots, \hat{\theta}_2, \theta_2, \ldots, \theta_n)$ contains the sum of near diagonal form-factors with all possible indexes. In the previous section we saw, that not all of them have finite $\epsilon \rightarrow 0$ limit. This implies that the existence of the symmetric diagonal limit of a form-factor is not obvious, and if eventually it exists, it must be a consequence of non-trivial cancellations between divergent terms. We will discuss this point in more detail in section 9.

We continue with writing the exchange axiom (5.5) applied to the near diagonal limit in terms of the eigenvectors of the transfer matrix. These eigenvectors can be given as actions of the off diagonal elements of the monodromy matrix (B.1) on the trivial vacuum (B.10). Using the representations (B.26) and (B.27) for the Bethe-eigenvectors:

$$\begin{aligned} \Psi^{a_1\ldots a_n} &\equiv \Psi^{a_1\ldots a_n}(\{\lambda_j\}|\vec{\theta}) \sim {}^{a_1\ldots a_n}(\langle 0| \prod_{j=1}^r \mathcal{C}(\lambda_j|\vec{\theta})), \\ \Psi_{b_1\ldots b_n}^{(\epsilon)*} &\equiv \Psi(\{\lambda_j^\epsilon\}|\vec{\theta}^\epsilon)_{b_1\ldots b_n}^* \sim (\prod_{j=1}^r \mathcal{B}(\lambda_j^\epsilon|\vec{\theta}^\epsilon)|0\rangle)_{b_1\ldots b_n}, \end{aligned} \quad (7.40)$$

the exchange axiom in the near diagonal limit can be written as follows:

$$\begin{aligned} \Psi(\{\lambda_j^\epsilon\}|\vec{\theta}^\epsilon)_{b_1\ldots b_n}^* F_{\bar{b}_n\ldots\bar{b}_1 a_1\ldots a_n}(\ldots, \hat{\theta}_{s+1}, \hat{\theta}_s, \ldots, \theta_s, \theta_{s+1}, \ldots) \Psi^{a_1\ldots a_n}(\{\lambda_j\}|\vec{\theta}) = S_0(\theta_{s+1}^\epsilon - \theta_s^\epsilon) \times \\ S_0(\theta_s - \theta_{s+1}) \Psi(\{\lambda_j^\epsilon\}|\vec{\theta}_{ex}^\epsilon)_{b_1\ldots b_n}^* F_{\bar{b}_n\ldots\bar{b}_1 a_1\ldots a_n}(\ldots, \hat{\theta}_s, \hat{\theta}_{s+1}, \ldots, \theta_{s+1}, \theta_s, \ldots) \Psi^{a_1\ldots a_n}(\{\lambda_j\}|\vec{\theta}_{ex}), \end{aligned} \quad (7.41)$$

where the set $\{\lambda_j\}_{j=1}^r$ is the solution of the Bethe-equations (B.12) and the set $\{\lambda_j^\epsilon\}_{j=1}^r$ also solves (B.12) but with $\theta_j \rightarrow \theta_j^\epsilon = \theta_j + \epsilon_j$ replacement⁹. The most important details of the formula are the vectors $\vec{\theta}$ and $\vec{\theta}_{ex}$. In these vectors the order of the rapidity matters! The difference between them is the order of the exchanged rapidities θ_s and θ_{s+1} . Namely,

$$\begin{aligned} \vec{\theta} &= \{\theta_1, \ldots, \theta_s, \theta_{s+1}, \ldots, \theta_n\}, & \vec{\theta}^\epsilon &= \{\theta_1 + \epsilon_1, \ldots, \theta_s + \epsilon_s, \theta_{s+1} + \epsilon_{s+1}, \ldots, \theta_n + \epsilon_n\}, \\ \vec{\theta}_{ex} &= \{\theta_1, \ldots, \theta_{s+1}, \theta_s, \ldots, \theta_n\}, & \vec{\theta}_{ex}^\epsilon &= \{\theta_1 + \epsilon_1, \ldots, \theta_{s+1} + \epsilon_{s+1}, \theta_s + \epsilon_s, \ldots, \theta_n + \epsilon_n\}. \end{aligned} \quad (7.42)$$

This means that the Bethe-vectors on the left and right hand sides of the equation (7.41) are different, since they are eigenvectors of different transfer matrices! We would like to explain this in a bit more detail. The rapidities θ_j are inhomogeneities of the transfer matrix. The transfer matrix is not invariant under the permutation of the inhomogeneities among the n lattice sites. Nevertheless, the Bethe-equations (B.12) and the eigenvalue expression are also invariant under the permutation of the rapidities. Thus the transfer matrices $\tau(\theta|\vec{\theta})$ and $\tau(\theta|\vec{\theta}_{ex})$ are only isospectral, but have different eigenvectors connected by a unitary transformation. This recognition has also some implication on the Pálmai-Takács conjecture (section 6), since there in the wave-function decomposition (6.13) the orders of rapidities in the arguments of the wave functions matter!

⁹We just note that (7.41) remains valid if the sets $\{\lambda_j\}_{j=1}^r$ and $\{\lambda_j^\epsilon\}_{j=1}^r$ are not solutions of the Bethe-equations, but are arbitrary sets. Here we require them to be solutions of (B.12) for later convenience.

7.3.1 Solitonic 6-particle symmetric diagonal form-factors

If one starts to analyse the 3-particle Bethe-equations (B.12), it becomes immediately obvious that the relevant solutions are the zero and 1-root solutions, since they account for all states in the $Q=3$ and $Q=1$ sectors. The missing $Q=-3$ and $Q=-1$ sectors can be obtained from the previous ones by the charge conjugation symmetry. The $Q=3$ sector is the pure soliton sector with no Bethe-root in (B.12). In this case the complicated sum in the right hand side of the kinematical pole equation (7.39) applied to the scalar operators ϕ and ψ will contain only a single term, which includes only the pure solitonic near diagonal form-factors (7.19) and (7.26). Since in this limit the diagonal pure solitonic matrix elements does not mix with other states, the computation of their symmetric and connected limits can be computed in exactly the same way as in a purely elastic scattering theory [40]. Their explicit form for the operators Θ and J_μ can be found in references [20] and [19], respectively. In these papers analytical formulas describing the Bethe-Yang limit of pure solitonic expectation values of the operators J_μ and Θ can also be found. This made it possible to verify the conjecture of [10] in this sector for any number of solitons. The pure solitonic sector is very similar to the case of a purely elastic scattering theory. As a consequence the actual form of conjecture of [10] goes through remarkable simplifications in this sector and becomes identical with the formula conjectured for diagonally scattering theories [11, 12]. Our purpose is to check the general form of the conjecture of [10]. Thus we will test it in a sector, where there is mixing between the states with different polarizations. This simplest such nontrivial sector is the $Q = 1$ sector of the 3-particle space. In the language of the Bethe-equations (B.12) it is described by a single Bethe-root.

7.3.2 6-particle symmetric diagonal form-factors in the $Q = 1$ sector

The first step to compute the symmetric diagonal form-factors of the operators of our interest in the $Q = 1$ sector, is to write down the actual form of the wave functions which should sandwich our form-factors according to (6.7). Here we denote their matrix elements as follows:

$$\begin{aligned}\Psi^{i_1 i_2 i_3} &= \frac{C^{i_1 i_2 i_3}}{N_\Psi}, \\ \Psi_{i_1 i_2 i_3}^{(\epsilon)*} &= \frac{B_{i_1 i_2 i_3}^\epsilon}{N_\Psi},\end{aligned}\tag{7.43}$$

where the nonzero coefficients in the $Q = 1$ sector can be read off from the formulas (B.26), (B.27) coming from the Algebraic Bethe-Ansatz diagonalization of the soliton-transfer matrix:

$$\begin{aligned}C^{+--} &= C_1, & C^{-+-} &= B_1 C_2, & C^{--+} &= B_1 B_2 C_3, \\ B_{+--}^\epsilon &= C_1^\epsilon B_2^\epsilon B_3^\epsilon, & B_{-+-}^\epsilon &= C_2^\epsilon B_3^\epsilon, & B_{--+}^\epsilon &= C_3^\epsilon,\end{aligned}\tag{7.44}$$

where for later convenience we introduced the short notations as follows:

$$\begin{aligned}B_j &= B_0(\lambda_1 - \theta_j), & C_j &= C_0(\lambda_1 - \theta_j), & B_j^\epsilon &= B_0(\lambda_1^\epsilon - \theta_j^\epsilon), & C_j^\epsilon &= C_0(\lambda_1^\epsilon - \theta_j^\epsilon), \\ \text{with } \theta_j^\epsilon &= \theta_j + \epsilon_j, & \text{for } j &= 1, 2, 3,\end{aligned}\tag{7.45}$$

such that the single Bethe-roots λ_1 and λ_1^ϵ are solutions of the Bethe-equations (B.29):

$$B_1 B_2 B_3 = 1, \quad B_1^\epsilon B_2^\epsilon B_3^\epsilon = 1. \quad (7.46)$$

The normalization factor N_Ψ is chosen to be the Gaudin-norm (B.31) of the vector¹⁰ Ψ . We note that this normalization factor is invariant under any permutations of the three rapidities $\{\theta_j\}_{j=1}^3$.

The case of Θ :

Now we are in the position to compute the 6-particle symmetric diagonal form-factors of Θ in the $Q = 1$ subsector. This subsector is characterized by a single Bethe-root solving the equation (7.46).

Looking at the formula (7.3) it turns out that to get the required limit of our 6-particle form-factor one needs to know the $\frac{1}{\epsilon^2}$ order part of the Ψ -sandwiched matrix element of the scalar operator ϕ defined in (7.1). To compute this part, one needs to use only the equations (7.39) and (7.41). These equations together with the concrete forms (7.18)-(7.21) of the near diagonal 4-particle form factors imply the following small ϵ series for the required form-factor of ϕ :

$$W^\phi(\theta_1, \epsilon_1; \theta_2, \epsilon_2; \theta_3, \epsilon_3) = \frac{1}{N_\Psi^2} B_{j_1 j_2 j_3}^\epsilon F_{j_3 \bar{j}_2 \bar{j}_1 i_1 i_2 i_3}^\phi(\hat{\theta}_3, \hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2, \theta_3) C^{i_1 i_2 i_3}, \quad (7.47)$$

$$W^\phi(\theta_1, \epsilon_1; \theta_2, \epsilon_2; \theta_3, \epsilon_3) = \frac{A_{12}(\theta_1, \theta_2, \theta_3)}{\epsilon_1 \epsilon_2} + \frac{A_{13}(\theta_1, \theta_2, \theta_3)}{\epsilon_1 \epsilon_3} + \frac{A_{23}(\theta_1, \theta_2, \theta_3)}{\epsilon_2 \epsilon_3} + O(\frac{1}{\epsilon}). \quad (7.48)$$

Then equation (7.41) tells us how W^ϕ of (7.47) changes when exchanging the pairs $(\theta_j, \epsilon_j) \leftrightarrow (\theta_k, \epsilon_k)$ in the argument. This gives the following relations among the A_{ij} functions in (7.48):

$$A_{13}(\theta_1, \theta_2, \theta_3) = A_{12}(\theta_3, \theta_1, \theta_2), \quad A_{23}(\theta_1, \theta_2, \theta_3) = A_{13}(\theta_3, \theta_1, \theta_2), \quad (7.49)$$

and in addition A_{ij} is invariant under the exchange of its i th and j th arguments.

The functions A_{12} and A_{13} can be directly computed from the $\frac{1}{\epsilon_1}$ pole given by equation (7.39). Then A_{23} can be determined from them by using (7.49). Straightforward application of (7.39) leads to the following expressions for A_{12} and A_{13} :

$$\begin{aligned} A_{12}(\theta_1, \theta_2, \theta_3) &= i \partial_3 \ln \Lambda(\theta_1 | \vec{\theta}) T^\phi(\theta_1, \theta_2, \theta_3), \\ A_{13}(\theta_1, \theta_2, \theta_3) &= i \partial_2 \ln \Lambda(\theta_1 | \vec{\theta}) T^\phi(\theta_1, \theta_2, \theta_3), \end{aligned} \quad (7.50)$$

¹⁰Namely, the $\epsilon \rightarrow 0$ limit of the scalar product $C^{i_1 i_2 i_3} B_{i_1 i_2 i_3}^\epsilon$.

where T^ϕ is the singularity eliminated tensorial sum part of (7.39):

$$T^\phi(\theta_1, \theta_2, \theta_3) = \lim_{\epsilon \rightarrow 0} \frac{1}{N_\Psi^2} B_{k\bar{\beta}_2\bar{\beta}_3}^\epsilon C^{k\alpha_2\alpha_3} a_{\beta_3\beta_2\alpha_2\alpha_3}^\phi(\theta_2, \theta_3) =$$

$$- \frac{1}{N_\Psi^2} \left[\frac{C_1^2}{B_1} G(\theta_{23}) + \left(\frac{C_2^2}{B_2} + \frac{C_3^2}{B_3} \right) \Omega(\theta_{23}) + (1 + B_1) C_2 C_3 \varphi(\theta_{23}) \right], \quad (7.51)$$

with the constituent functions given in (7.22) and (7.45). Having the explicit expression for A_{12} and A_{13} , with the help of the exchange relation (7.49) A_{23} can also be obtained from them. Finally using (7.3) and (7.47), the symmetric diagonal limit of the form-factors of Θ in a 3-particle state described by the Bethe-root λ_1 can be given by the formula as follows:

$$F_{6,symm}^{\Theta,(\Psi)}(\theta_1, \theta_2, \theta_3) = -\frac{\mathcal{M}^2}{N_\Psi^2} [A_{12}(\theta_1, \theta_2, \theta_3) + A_{13}(\theta_1, \theta_2, \theta_3) + A_{23}(\theta_1, \theta_2, \theta_3)] \times$$

$$[3 + 2 \cosh(\theta_{12}) + 2 \cosh(\theta_{13}) + 2 \cosh(\theta_{23})]. \quad (7.52)$$

We note that the λ_1 dependence is implicit in this expression. It is hidden in the expression of T^Φ in (7.51) and in the derivative of the eigenvalue in (7.50). A useful formula for the latter is given in (B.32).

The case of J_μ :

The computation of the 6-particle symmetric diagonal form-factors of the $U(1)$ current is a bit more subtle than that of the trace of the stress energy tensor. The method described in the previous paragraphs is the same, but one should be much more careful in the small ϵ_j expansion. In this case the linear in ϵ_j terms of the Bethe-vector $B_{i_1 i_2 i_3}^\epsilon$ (7.44) will also give relevant contributions to the symmetric form-factors. The first step is to compute the 6-particle form-factor of the scalar field ψ in the near diagonal limit. Thus the quantity we compute is defined by:

$$W^\psi(\lambda_1^\epsilon, \lambda_1 | 1^\epsilon, 2^\epsilon, 3^\epsilon) = \frac{1}{N_\Psi^2} B_{j_1 j_2 j_3}^\epsilon F_{\bar{j}_3 \bar{j}_2 \bar{j}_1 i_1 i_2 i_3}^\psi(\hat{\theta}_3, \hat{\theta}_2, \hat{\theta}_1, \theta_1, \theta_2, \theta_3) C^{i_1 i_2 i_3}, \quad (7.53)$$

where for short we introduced the symbolic notation for a pair: $\theta_j, \epsilon_j \rightarrow j^\epsilon$, and we also indicated in the list of arguments the Bethe-root dependence of this form-factor. Using the kinematical pole equation for the $\frac{1}{\epsilon_1}$ singularity, the terms proportional to $\frac{1}{\epsilon_1}$ in the small ϵ expansion of W^ψ can be computed. To facilitate this task first we do the computations in some smaller building blocks of W^ψ . Let Y denote the eigenvalue part of (7.39):

$$Y(\lambda_1^\epsilon, \lambda_1 | 1^\epsilon, 2^\epsilon, 3^\epsilon) = 1 - \frac{\Lambda(\theta_1 | \vec{\theta})}{\Lambda(\theta_1^\epsilon | \vec{\theta}^\epsilon)} = \sum_{j=1}^3 \epsilon_j \partial_j \ln \Lambda(\theta_1 | \vec{\theta}) + O(\epsilon^2), \quad (7.54)$$

and let denote T^ψ the tensorial sum part of (7.39):

$$T^\psi(\lambda_1^\epsilon, \lambda|1^\epsilon, 2^\epsilon, 3^\epsilon) = B_{k\bar{\beta}_2\bar{\beta}_3}^\epsilon C^{k\alpha_2\alpha_3} F_{\beta_3\beta_2\alpha_2\alpha_3}^\psi(\hat{\theta}_3, \hat{\theta}_2, \theta_2, \theta_3). \quad (7.55)$$

Taking the near diagonal limit of the 4-particle form-factors of ψ given in (7.26)-(7.33), one obtains the following small ϵ expansion for T^ψ :

$$T^\psi(\lambda_1^\epsilon, \lambda|1^\epsilon, 2^\epsilon, 3^\epsilon) = \frac{T_{23}(\vec{\theta})}{\epsilon_2\epsilon_3} + \frac{T_2(\vec{\theta})}{\epsilon_2} + \frac{T_3(\vec{\theta})}{\epsilon_3} + O(1), \quad (7.56)$$

where the functions $T_{23}(\vec{\theta})$, $T_2(\vec{\theta})$, $T_3(\vec{\theta})$ take the form:

$$T_{23}(\vec{\theta}) = \mathcal{H}_0(\theta_{23}) \left(\frac{C_3^2}{B_3} - \frac{C_2^2}{B_2} \right) + \mathcal{G}_0(\theta_{23}) C_2 C_3 (B_1 - 1), \quad (7.57)$$

$$\begin{aligned} T_2(\vec{\theta}) = & \mathcal{G}_0(\theta_{23}) [B_3^{(3)}(\vec{\theta}) C^{(2)}(\vec{\theta}) - B_3^{(2)}(\vec{\theta}) C^{(3)}(\vec{\theta})] + \mathcal{G}_1(\theta_{23}) [B_0^{(3)}(\vec{\theta}) C^{(2)}(\vec{\theta}) - B_0^{(2)}(\vec{\theta}) C^{(3)}(\vec{\theta})] + \\ & \mathcal{H}_0(\theta_{23}) [B_3^{(3)}(\vec{\theta}) C^{(3)}(\vec{\theta}) - B_3^{(2)}(\vec{\theta}) C^{(2)}(\vec{\theta})] + \mathcal{H}_1(\theta_{23}) [B_0^{(3)}(\vec{\theta}) C^{(3)}(\vec{\theta}) - B_0^{(2)}(\vec{\theta}) C^{(2)}(\vec{\theta})] + \\ & G(\theta_{23}) B_0^{(1)}(\vec{\theta}) C^{(1)}(\vec{\theta}), \end{aligned} \quad (7.58)$$

$$\begin{aligned} T_3(\vec{\theta}) = & \mathcal{G}_0(\theta_{23}) [B_2^{(3)}(\vec{\theta}) C^{(2)}(\vec{\theta}) - B_2^{(2)}(\vec{\theta}) C^{(3)}(\vec{\theta})] + \mathcal{G}_2(\theta_{23}) [B_0^{(3)}(\vec{\theta}) C^{(2)}(\vec{\theta}) - B_0^{(2)}(\vec{\theta}) C^{(3)}(\vec{\theta})] + \\ & \mathcal{H}_0(\theta_{23}) [B_2^{(3)}(\vec{\theta}) C^{(3)}(\vec{\theta}) - B_2^{(2)}(\vec{\theta}) C^{(2)}(\vec{\theta})] + \mathcal{H}_2(\theta_{23}) [B_0^{(3)}(\vec{\theta}) C^{(3)}(\vec{\theta}) - B_0^{(2)}(\vec{\theta}) C^{(2)}(\vec{\theta})] + \\ & G(\theta_{23}) B_0^{(1)}(\vec{\theta}) C^{(1)}(\vec{\theta}), \end{aligned} \quad (7.59)$$

where the functions $B_k^{(j)}$ and $C^{(k)}$ are coming from the small ϵ expansion of the components of the Bethe-eigenvectors (7.44) in the following way:

$$\begin{aligned} B_{+-}^\epsilon &= B_0^{(1)}(\vec{\theta}) + \sum_{j=1}^3 B_j^{(1)}(\theta) \epsilon_j + O(\epsilon^2), & C_{+-} &= C^{(1)}(\vec{\theta}), \\ B_{-+-}^\epsilon &= B_0^{(2)}(\vec{\theta}) + \sum_{j=1}^3 B_j^{(2)}(\theta) \epsilon_j + O(\epsilon^2), & C_{-+-} &= C^{(2)}(\vec{\theta}), \\ B_{--+}^\epsilon &= B_0^{(3)}(\vec{\theta}) + \sum_{j=1}^3 B_j^{(3)}(\theta) \epsilon_j + O(\epsilon^2), & C_{--+} &= C^{(3)}(\vec{\theta}). \end{aligned} \quad (7.60)$$

Their actual form can be computed from (7.44) and (7.45). Here we give only the ones entering (7.58) and (7.59):

$$\begin{aligned} B_0^{(1)}(\theta) &= C_1 B_2 B_3, & C^{(1)}(\vec{\theta}) &= C_1, \\ B_0^{(2)}(\theta) &= C_2 B_3, & C^{(2)}(\vec{\theta}) &= B_1 C_2, \\ B_0^{(3)}(\theta) &= C_3, & C^{(3)}(\vec{\theta}) &= B_1 B_2 C_3, \end{aligned} \quad (7.61)$$

$$B_2^{(1)}(\vec{\theta}) = \partial_{\lambda_1} B_0^{(1)}(\vec{\theta}) \cdot \frac{\partial \lambda_1}{\partial \theta_2} - C_1 B_2' B_3, \quad (7.62)$$

$$B_3^{(1)}(\vec{\theta}) = \partial_{\lambda_1} B_0^{(1)}(\vec{\theta}) \cdot \frac{\partial \lambda_1}{\partial \theta_3} - C_1 B_2 B_3',$$

$$B_2^{(2)}(\vec{\theta}) = \partial_{\lambda_1} B_0^{(2)}(\vec{\theta}) \cdot \frac{\partial \lambda_1}{\partial \theta_2} - C_2' B_3, \quad (7.63)$$

$$B_3^{(2)}(\vec{\theta}) = \partial_{\lambda_1} B_0^{(2)}(\vec{\theta}) \cdot \frac{\partial \lambda_1}{\partial \theta_3} - C_2 B_3',$$

$$B_2^{(3)}(\vec{\theta}) = \partial_{\lambda_1} B_0^{(3)}(\vec{\theta}) \cdot \frac{\partial \lambda_1}{\partial \theta_2}, \quad (7.64)$$

$$B_3^{(3)}(\vec{\theta}) = \partial_{\lambda_1} B_0^{(3)}(\vec{\theta}) \cdot \frac{\partial \lambda_1}{\partial \theta_3} - C_3',$$

where introduced the notations:

$$B_j' = B_0'(\lambda_1 - \theta_j), \quad C_j' = C_0'(\lambda_1 - \theta_j), \quad j = 1, 2, 3. \quad (7.65)$$

In the above formulas we did not write down explicitly the λ_1 dependence of the functions. Nevertheless, it is important because of the ∂_{λ_1} partial derivatives. Here the λ_1 dependence is simply meant by the λ_1 dependence of the objects B_j and C_j given by (7.45).

Now we have all ingredients to compute the 6-particle symmetric diagonal form-factors of J_μ in the $Q = 1$ sector of the 3-particle subspace. Looking at the formula (7.9) one can see that the symmetric diagonal limit is finite only if F^ψ or equivalently T^ψ has only $\frac{1}{\epsilon_j}$ order divergences. However, the order $\frac{1}{\epsilon^2}$ term in the expansion (7.56) of T^ψ implies, that the symmetric diagonal limit is divergent in this case, provided the coefficient function T_{23} is nonzero. Looking at its explicit form (7.57) it does not seem to be zero. Nevertheless, with some work, exploiting the Yang-Baxter equations (2.10) and the Bethe-equations (7.46) for λ_1 one can show that:

$$T_{23}(\vec{\theta}) = 0. \quad (7.66)$$

This nontrivial for the first sight result ensures, that the symmetric diagonal limit of the 6-particle form-factors of J_μ in the $Q = 1$ sector will be well defined. Nevertheless this computation sheds light on the fact that the higher and higher $\frac{1}{\epsilon}$ divergences of the non-diagonal form-factors could make the symmetric diagonal limit divergent¹¹, too. On the other hand this computation might also imply that the special properties of integrability might ensure the cancellation of these (would be?) divergences.

Due to the cancellation of the $\frac{1}{\epsilon^2}$ divergent term in T^ψ , W^ψ (7.53) admits the following small ϵ expansion:

$$W^\psi(\lambda_1^\epsilon, \lambda_1 | 1^\epsilon, 2^\epsilon, 3^\epsilon) = \frac{W_1(\vec{\theta})}{\epsilon_1} + \frac{W_2(\vec{\theta})}{\epsilon_2} + \frac{W_3(\vec{\theta})}{\epsilon_3} + W^{(1)}(\vec{\theta}) \frac{\epsilon_1}{\epsilon_2 \epsilon_3} + W^{(2)}(\vec{\theta}) \frac{\epsilon_2}{\epsilon_1 \epsilon_3} + W^{(3)}(\vec{\theta}) \frac{\epsilon_3}{\epsilon_1 \epsilon_2} + O(1), \quad (7.67)$$

¹¹Since their sum enter the right hand of the kinematical pole axiom (7.39). See section 9 for a more detailed discussion.

such that the coefficient functions W_1 , $W^{(2)}$ and $W^{(3)}$ can be computed from the kinematical pole equation (7.39) by using the formulas (7.56) and (7.54):

$$\begin{aligned} W_1(\vec{\theta}) &= \frac{i}{N_\Psi^2} \left[T_2(\vec{\theta}) \partial_2 \ln \Lambda(\theta_1 | \vec{\theta}) + T_3(\vec{\theta}) \partial_3 \ln \Lambda(\theta_1 | \vec{\theta}) \right], \\ W^{(2)}(\vec{\theta}) &= \frac{i}{N_\Psi^2} T_3(\vec{\theta}) \partial_2 \ln \Lambda(\theta_1 | \vec{\theta}), \\ W^{(3)}(\vec{\theta}) &= \frac{i}{N_\Psi^2} T_2(\vec{\theta}) \partial_3 \ln \Lambda(\theta_1 | \vec{\theta}). \end{aligned} \quad (7.68)$$

The exchange equation (7.41) allows one to compute from (7.68) the other still unknown W -functions of the expansion (7.67), since (7.41) implies that they are related by argument exchanges:

$$\begin{aligned} W_2(\theta_1, \theta_2, \theta_3) &= W_1(\theta_2, \theta_1, \theta_3), \\ W_3(\theta_1, \theta_2, \theta_3) &= W_1(\theta_3, \theta_2, \theta_1), \\ W^{(1)}(\theta_1, \theta_2, \theta_3) &= W^{(2)}(\theta_2, \theta_1, \theta_3). \end{aligned} \quad (7.69)$$

Nevertheless, (7.41) gives further relations among these functions, which can be used to test the obtained result. These are as follows. The functions $W_j(\theta_1, \theta_2, \theta_3)$ and $W^{(j)}(\theta_1, \theta_2, \theta_3)$ are symmetric with respect to the exchange of the rapidities θ_s and θ_q with $s, q \neq j$. According to (7.41), $W^{(2)}$ and $W^{(3)}$ are also not independent:

$$W^{(2)}(\theta_1, \theta_2, \theta_3) = W^{(3)}(\theta_3, \theta_1, \theta_2). \quad (7.70)$$

It can be checked that our formulas in (7.68) satisfy this requirement.

With the help of (7.9) the symmetric diagonal 6-particle form-factors of the current can be expressed in terms of the previously computed W -functions as follows:

$$F_{6, \text{symm}}^{J_\mu, (\Psi)}(\theta_1, \theta_2, \theta_3) = (-1)^{\mu+1} \mathcal{M} \left(\sum_{j=1}^3 v_j^{(\mu)} \right) \sum_{j=1}^3 \left[W_j(\vec{\theta}) + W^{(j)}(\vec{\theta}) \right], \quad (7.71)$$

with the vector

$$v_j^{(\mu)} = \begin{cases} \cosh(\theta_j), & \text{for } \mu = 0, \\ \sinh(\theta_j), & \text{for } \mu = 1. \end{cases} \quad (7.72)$$

The formula (7.71) can also be rephrased in an equivalent way, which reflects manifestly the invariance of the symmetric form-factor with respect to the permutations of the rapidities:

$$\begin{aligned} F_{6, \text{symm}}^{J_\mu, (\Psi)}(\theta_1, \theta_2, \theta_3) &= (-1)^{\mu+1} \frac{\mathcal{M}}{2} \left(\sum_{j=1}^3 v_j^{(\mu)} \right) \times \\ &\sum_{\sigma \in S^3} \left[W_1(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \theta_{\sigma(3)}) + W^{(2)}(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \theta_{\sigma(3)}) \right], \end{aligned} \quad (7.73)$$

where the second sum runs for the six possible permutations of the indexes $\{1, 2, 3\}$.

8 Checking the Pálmai-Takács conjecture

In the previous sections we computed the symmetric diagonal form-factors of the operators Θ and J_μ upto 6-particles. This makes it possible to check the conjecture of Pálmai and Takács for the diagonal matrix elements of local operators [10] (summarized in section 6.) against the exact results given in (4.16)-(4.18) upto 3-particle expectation values. In the pure soliton sector¹² the validity of this conjecture have been already verified for the operators Θ and J_μ in references [20] and [19], respectively. This is why in our work we will only focus on states in which soliton and antisoliton states are mixed.

As implied by (6.1), in the conjecture the eigenvectors of the multisoliton transfer matrix (B.4), play an important role. To test the conjecture upto 3-particle states, one needs the complete Bethe-basis on the space of 1- and 2-particle states and one also needs the Bethe-eigenvector corresponding to the sandwiching 3-particle state. Thus, as a first step we write down these Bethe-eigenvectors.

For the one particle states the eigenvectors are simple:

$$\varphi_{i_1}^{(a)} = \delta_{i_1, a}, \quad a, i_1 = \pm. \quad (8.1)$$

In the space of 1-particle states the basis is two dimensional corresponding to the soliton and the antisoliton. The index a distinguishes the two basis vectors of this space and i_1 is the index of the vector. Here we pay the attention to two trivial, but for later considerations important properties of this basis. First of all the vector components are independent of the particle's rapidities. Second of all these vectors are real. The 2-particle basis is also very simple [10]:

$$\begin{aligned} \Psi_{i_1 i_2}^{(1)} &= \delta_{i_1-} \delta_{i_2-}, & \Psi_{i_1 i_2}^{(2)} &= \frac{1}{\sqrt{2}} (\delta_{i_1+} \delta_{i_2-} + \delta_{i_1-} \delta_{i_2+}), \\ \Psi_{i_1 i_2}^{(3)} &= \frac{1}{\sqrt{2}} (\delta_{i_1+} \delta_{i_2-} - \delta_{i_1-} \delta_{i_2+}), & \Psi_{i_1 i_2}^{(4)} &= \delta_{i_1+} \delta_{i_2+}. \end{aligned} \quad (8.2)$$

Here again the superscript indexes the basis vectors and the subscripts $i_1, i_2 = \pm$ denotes the vector indexes in the 2-particle vector space. Here we also emphasize that this 2-particle basis is real and rapidity independent. With this remark we would like to pay the attention, that the first numerical checks of the Pálmai-Takács conjecture in [10], which were performed upto 2-particle states, were not sensible to the difference between the two definitions (6.7) and (6.8).

It is worth to discuss a bit more on the meaning of the basis vectors of (8.2). The vectors $\Psi^{(4)}$ and $\Psi^{(1)}$ correspond to the two antisoliton and two soliton states, respectively. The vector $\Psi^{(2)}$ and $\Psi^{(3)}$ describe the symmetric and antisymmetric soliton-antisoliton states, respectively. At the level of the magnonic Bethe-equations (B.12), $\Psi^{(4)}$ and $\Psi^{(1)}$ are states without Bethe-roots, while $\Psi^{(2)}$ and $\Psi^{(3)}$ are described by a single Bethe-root. Using the terminology of appendix B.2 $\Psi^{(2)}$ is described by a real Bethe-root: $\lambda^{(2)} = \frac{\theta_1 + \theta_2}{2} + i\frac{\pi}{2}$ and $\Psi^{(3)}$ is given by a self-conjugate root: $\lambda^{(3)} = \frac{\theta_1 + \theta_2}{2} + i\frac{(1+p)\pi}{2}$, provided we are in the repulsive $1 < p$ regime of the theory.

¹²For arbitrary number of solitons and not only upto 3.

In the space of 3-particle states, we need the Bethe-eigenvectors only in the $Q = 1$ sector. It has also a simple form:

$$\Psi_{i_1 i_2 i_3} = \Psi_{+--\delta_{i_1+}\delta_{i_2-}\delta_{i_3-}} + \Psi_{-+-\delta_{i_1-}\delta_{i_2+}\delta_{i_3-}} + \Psi_{+--\delta_{i_1-}\delta_{i_2-}\delta_{i_3+}}, \quad (8.3)$$

where

$$\Psi_{i_1 i_2 i_3} = \frac{C^{i_1 i_2 i_3}}{N_\Psi}, \quad (8.4)$$

such that $C^{i_1 i_2 i_3}$ is given by (7.44) with (7.46) and N_Ψ is given by (B.31). Actually this vector stands for 3 eigenvectors, since it depends on a Bethe-root λ_1 , and the Bethe-equation (7.46) have 3 independent solutions in this sector: a real one and two self-conjugate ones. One can recognize that the vector Ψ in (8.3) differs by a complex conjugation with respect to the one enters the conjecture of [10]. The reason is that at the study of the diagonal limit of the kinematical pole axiom (7.39), we recognized that one should sandwich the form-factor with the left eigenvector of the transfer-matrix instead of its right eigenvector proposed earlier in [10]. Due to the hermiticity properties of the soliton transfer matrix (B.18), this is just a complex conjugation at the level of the eigenvectors. Here the 3-particle wave vector Ψ is complex and rapidity dependent, thus all these affairs matter. In the states upto 2-particles, which was studied in [10] to check the conjecture, this complex conjugation problem didnot arise.

Now, we know how the Bethe-eigenvectors we need are described by the roots of the magnonic Bethe-equations (B.12). This makes possible to write down the densities of the states (6.6) corresponding to the eigenstates under consideration. They can be simply read off from the formulas (4.3)-(4.8), which give the Bethe-Yang limit of the Gaudin-matrix. We will specialize these formulas for the zero and one root cases of the 1-, 2- and 3-particle eigenstates. First, we rewrite the matrix Φ (4.7) for the zero and one-root states by emphasizing the Bethe-root and particle number dependence better:

$$\begin{aligned} \Phi_{j,k}^{(n)}(\lambda) &= \left(\ell \cosh \theta_j + \sum_{s=1}^n \tilde{G}_{js}(\lambda) \right) \delta_{jk} - \tilde{G}_{jk}(\lambda), \\ \tilde{G}_{jk}(\lambda) &= G(\theta_j - \theta_k) + \frac{1}{i} \frac{V_j(\lambda) V_k(\lambda)}{\psi^{(n)}(\lambda)}, \quad \psi^{(n)}(\lambda) = \sum_{j=1}^n V_j(\lambda), \\ V_j(\lambda) &= (\ln B_0)'(\lambda - \theta_j). \end{aligned} \quad (8.5)$$

For the zero root case the $\sim \frac{V_j(\lambda) V_k(\lambda)}{\psi^{(n)}(\lambda)}$ term must be skipped¹³.

Then the densities of the states given by the Bethe-vectors (8.1), (8.2) and (8.3) can be given by determinants of this matrix. Now, we list the necessary densities below. The densities (6.6) for the 1-particle states (8.1) are the same as those of a free particle:

$$\rho_1^{(\pm)}(1) \equiv \rho_1(1) = \ell \cosh \theta_1 = \ell c_1, \quad (8.6)$$

¹³We denote this case by writing symbolically \emptyset instead of λ into the argument.

where for short we introduce the notations $c_j = \cosh \theta_j$ and $s_j = \sinh \theta_j$. The densities (6.6) for the 4-dimensional basis of 2-particle states (8.1) are given by the determinants as follows :

$$\begin{aligned}\rho_2^{(1)}(1, 2) &= \det_{2 \times 2} \Phi^{(2)}(\emptyset) = \ell^2 c_1 c_2 + \ell(c_1 + c_2)G(\theta_{12}), \\ \rho_2^{(2)}(1, 2) &= \det_{2 \times 2} \Phi^{(2)}(\lambda^{(2)}) = \ell^2 c_1 c_2 + \ell(c_1 + c_2)\tilde{G}_{12}(\lambda^{(2)}), \\ \rho_2^{(3)}(1, 2) &= \det_{2 \times 2} \Phi^{(2)}(\lambda^{(3)}) = \ell^2 c_1 c_2 + \ell(c_1 + c_2)\tilde{G}_{12}(\lambda^{(3)}), \\ \rho_2^{(4)}(1, 2) &= \det_{2 \times 2} \Phi^{(2)}(\emptyset) = \ell^2 c_1 c_2 + \ell(c_1 + c_2)G(\theta_{12}),\end{aligned}\tag{8.7}$$

with

$$\lambda^{(2)} = \frac{\theta_1 + \theta_2}{2} + i\frac{\pi}{2}, \quad \lambda^{(3)} = \frac{\theta_1 + \theta_2}{2} + i\frac{\pi(1+p)}{2}.\tag{8.8}$$

Finally the density corresponding to the $Q = 1$ sector of the 3-particle states is given by:

$$\rho_3^{(\Psi)}(1, 2, 3) = \det_{3 \times 3} \Phi^{(3)}(\lambda_1),\tag{8.9}$$

where λ_1 denotes the solution of the magnonic Bethe-equations (7.46). Now we are in the position to check the conjecture of [10] analytically in the 2-particle sector.

8.1 Checking the conjecture for 2-particle states

We make the test only in the $Q = 0$ sector of the 2-particle space, since in the purely solitonic $Q = \pm 2$ sectors the conjecture has been verified for the operators J_μ and Θ and for any number of solitons in papers [19] and [20]. In the sequel we compute the expectation values in the states described by the color wave functions $\Psi^{(2)}$ and $\Psi^{(3)}$ given in (8.2). The first step in the computation is to determine the branching coefficients of these wave-functions with respect to the 1-particle color wave-functions of (8.1). Due to the simple form of these vectors the branching coefficients of the decomposition (6.13) of $\Psi^{(2)}$ and $\Psi^{(3)}$ can be read off immediately:

$$\left. \begin{aligned} C_{+-}^{(s)} &= \frac{1}{\sqrt{2}}, & C_{+-}^{(s)} &= \frac{r_s}{\sqrt{2}}, \\ C_{--}^{(s)} &= 0, & C_{++}^{(s)} &= 0, \end{aligned} \right\} \text{ for } s = 2, 3 \quad \text{with } r_2 = 1, \quad r_3 = -1.\tag{8.10}$$

Now applying the conjectured formula (6.11) to the $Q = 0$ states of the 2-particle space, one obtains:

$$\begin{aligned} F_2^{\mathcal{O},(s)}(\theta_1, \theta_2) &= \langle \mathcal{O} \rangle_0 + \frac{1}{\rho_2^{(s)}(1, 2)} \left\{ F_{4, \text{symm}}^{\mathcal{O},(s)}(1, 2) + \frac{1}{2} F_{2, \text{symm}}^{\mathcal{O},(+)}(1) \rho_1^{(-)}(2) + \right. \\ &\quad \left. \frac{1}{2} F_{2, \text{symm}}^{\mathcal{O},(-)}(1) \rho_1^{(+)}(2) + \frac{1}{2} F_{2, \text{symm}}^{\mathcal{O},(+)}(2) \rho_1^{(-)}(1) + \frac{1}{2} F_{2, \text{symm}}^{\mathcal{O},(-)}(2) \rho_1^{(+)}(1) \right\}, \quad s = 2, 3, \end{aligned}\tag{8.11}$$

where $\langle \mathcal{O} \rangle_0$ denotes the vacuum expectation value, the densities are given by (8.6) and (8.7) and the symmetric diagonal form-factors in the states $\Psi^{(2)}$ and $\Psi^{(3)}$ can be determined from the formula (6.7) using the results (7.22)-(7.34).

It is easier to start testing the conjecture with the operator J_μ . Since J_μ is a charge conjugation negative operator all the symmetric form-factors entering (8.11) become zero. The vacuum expectation value is also zero because of the same reason $\langle J_\mu \rangle_0 = 0$. Thus, the conjecture of [10] suggests that $F_2^{J_\mu, (s)}(\theta_1, \theta_2) = 0$, for $s = 2, 3$. Due to the charge conjugation negativity of the current it is true exactly, as well. Thus for J_μ the conjecture gives the expected trivial result.

For the trace of the stress energy tensor, due to the charge conjugation positivity¹⁴ of the operator, (8.11) simplifies:

$$F_2^{\Theta, (s)}(\theta_1, \theta_2) = \langle \Theta \rangle_0 + \frac{1}{\rho_2^{(s)}(1, 2)} \left\{ F_{4, \text{symm}}^{\Theta, (s)}(1, 2) + F_{2, \text{symm}}^{\Theta, (+)}(1) \rho_1(2) + F_{2, \text{symm}}^{\Theta, (+)}(2) \rho_1(1) \right\}, \quad s = 2, 3, \quad (8.12)$$

where the symmetric diagonal 2-particle form-factor is a constant as it can be read off from (7.7):

$$F_{2, \text{symm}}^{\Theta, (\pm)}(1) = \mathcal{M}^2, \quad (8.13)$$

the necessary densities are listed in (8.6) and (8.7) and the symmetric diagonal 4-particle form-factors can be constructed from (7.24), (7.25) by the prescription (6.7) and exploiting the charge conjugation positivity:

$$F_{4, \text{symm}}^{\Theta, (s)}(1, 2) = F_{+-+ -}^{\Theta, \text{symm}}(\theta_1, \theta_2) + r_s F_{+ - - +}^{\Theta, \text{symm}}(\theta_1, \theta_2), \quad (8.14)$$

with r_s given in (8.10). Using the identity

$$\tilde{G}_{12}(\lambda^{(s)}) = \Omega(\theta_{12}) + r_s \varphi(\theta_{12}), \quad s = 2, 3, \quad (8.15)$$

and the formulas (7.24), (7.25) the concrete form of these form-factors can be written in the form as follows:

$$F_{4, \text{symm}}^{\Theta, (s)}(1, 2) = 2\mathcal{M}^2(1 + c_1 c_2 - s_1 s_2) \tilde{G}_{12}(\lambda^{(s)}), \quad s = 2, 3. \quad (8.16)$$

Putting everything together one ends up with the final formula for the expectation value as follows:

$$F_2^{\Theta, (s)}(\theta_1, \theta_2) = \langle \Theta \rangle_0 + \frac{1}{\rho_2^{(s)}(1, 2)} \left\{ 2\mathcal{M}^2(1 + c_1 c_2 - s_1 s_2) \tilde{G}_{12}(\lambda^{(s)}) + \mathcal{M}^2 \ell(c_1 + c_2) \right\}, \quad s = 2, 3, \quad (8.17)$$

which is exactly the same as the formula (4.16) coming from the exact result and specified to the single Bethe-root configurations $\lambda^{(2)}$ and $\lambda^{(3)}$.

The next step is to check the conjecture of [10] in the $Q = 1$ sector of the space of 3-particle states. Here the computations are much more involved, this is why we will write down only the main steps and list the ingredients of the necessary computations.

¹⁴Namely, in this case form-factors are invariant with respect to conjugating their indexes.

8.2 Checking the conjecture for 3-particle states

The first step in the computation is the decomposition (6.13) of the 3-particle wave function (8.3) in terms of 1- (8.1) and 2-particle (8.2) wave functions. For a subset $A = \{A_1\} \subset \{1, \dots, 3\}$ with a single element, the branching coefficients can be computed by the scalar product as follows:

$$C_{st}(A) = \sum_{i_1, i_2, i_3 = \pm} \varphi_{i_{A_1}}^{(s)*} \psi_{i_{\bar{A}_1} i_{\bar{A}_2}}^{(t)*} \Psi_{i_1 i_2 i_3}, \quad s \in \{+, -\}, \quad t \in \{1, 2, 3, 4\}. \quad (8.18)$$

Writing the analogous formula for the case, when $A \subset \{1, \dots, 3\}$ has two elements, one obtains the relation:

$$C_{st}(A) = C_{ts}(\bar{A}), \quad s \in \{+, -\}, \quad t \in \{1, 2, 3, 4\}, \quad A \subset \{1, \dots, 3\}. \quad (8.19)$$

Thus from (8.18) and from the "color" wave functions (8.1), (8.2) and (8.3), all the necessary branching coefficients can be determined:

$$\begin{aligned} C_{1+}(\{1, 2\}) &= C_{+1}(\{3\}) = \Psi_{--+}, & C_{2-}(\{1, 2\}) &= C_{-2}(\{3\}) = \frac{\Psi_{+--} + \Psi_{-+-}}{\sqrt{2}}, \\ C_{3-}(\{1, 2\}) &= C_{-3}(\{3\}) = \frac{\Psi_{+--} - \Psi_{-+-}}{\sqrt{2}}, & C_{1+}(\{1, 3\}) &= C_{+1}(\{2\}) = \Psi_{--+}, \\ C_{2-}(\{1, 3\}) &= C_{-2}(\{2\}) = \frac{\Psi_{--+} + \Psi_{+--}}{\sqrt{2}}, & C_{3-}(\{1, 3\}) &= C_{-3}(\{2\}) = \frac{\Psi_{+--} - \Psi_{--+}}{\sqrt{2}}, \\ C_{1+}(\{2, 3\}) &= C_{+1}(\{1\}) = \Psi_{+--}, & C_{2-}(\{2, 3\}) &= C_{-2}(\{1\}) = \frac{\Psi_{--+} + \Psi_{-+-}}{\sqrt{2}}, \\ C_{3-}(\{2, 3\}) &= C_{-3}(\{1\}) = \frac{\Psi_{-+-} - \Psi_{--+}}{\sqrt{2}}, \end{aligned} \quad (8.20)$$

where $\Psi_{i_1 i_2 i_3}$ is given in (8.4). In the main formula (6.11) for the diagonal matrix elements, the absolute value squared of these coefficients arise. They can be expressed in terms of the elements of a Hermitian matrix M :

$$M = \begin{pmatrix} \frac{V_1}{V_1+V_2+V_3} & \frac{C_1 C_2 B_3}{N_\Psi} & \frac{C_1 C_3}{N_\Psi} \\ \frac{C_1 C_2}{N_\Psi} & \frac{V_2}{V_1+V_2+V_3} & \frac{B_1 C_2 C_3}{N_\Psi} \\ \frac{C_1 C_3 B_2}{N_\Psi} & \frac{C_2 C_3}{N_\Psi} & \frac{V_3}{V_1+V_2+V_3} \end{pmatrix}, \quad (8.21)$$

in the following way¹⁵:

$$\begin{aligned} |C_{1+}(\{1, 2\})|^2 &= |C_{+1}(\{3\})|^2 = M_{33}, & |C_{2-}(\{1, 2\})|^2 &= |C_{-2}(\{3\})|^2 = L_{12}^{(+)}, \\ |C_{3-}(\{1, 2\})|^2 &= |C_{-3}(\{3\})|^2 = L_{12}^{(-)}, & |C_{1+}(\{1, 3\})|^2 &= |C_{+1}(\{2\})|^2 = M_{22}, \\ |C_{2-}(\{1, 3\})|^2 &= |C_{-2}(\{2\})|^2 = L_{13}^{(+)}, & |C_{3-}(\{1, 3\})|^2 &= |C_{-3}(\{2\})|^2 = L_{13}^{(-)}, \\ |C_{1+}(\{2, 3\})|^2 &= |C_{+1}(\{1\})|^2 = M_{11}, & |C_{2-}(\{2, 3\})|^2 &= |C_{-2}(\{1\})|^2 = L_{23}^{(+)}, \\ |C_{3-}(\{2, 3\})|^2 &= |C_{-3}(\{1\})|^2 = L_{23}^{(-)}, \end{aligned} \quad (8.22)$$

¹⁵We just recall: $B_j = B_0(\lambda_1 - \theta_j)$, $C_j = C_0(\lambda_1 - \theta_j)$, $V_j = (\ln B_0)'(\lambda_1 - \theta_j)$.

where we introduced the short notation: $L_{ij}^{(\pm)} = \frac{M_{ii} + M_{jj} \pm M_{ij} \pm M_{ji}}{2}$.

Now we have all the necessary ingredients to compare the conjectured formula (6.11) to the exact ones (4.16)-(4.18) for the operators Θ and J_μ . Now, the computations become quite involved, thus they were performed by the software *Mathematica*. We just write down in words the strategy of the comparison. The $\frac{1}{\rho_3^{(\Psi)}}$ term naturally arises in the exact formulas (4.16)-(4.18), if the inverse Gaudin-matrix is expressed by the co-factor matrix \mathcal{K} :

$$\Phi_{jk}^{-1} = \frac{\mathcal{K}_{kj}}{\det \Phi} = \frac{\mathcal{K}_{kj}}{\rho_3^{(\Psi)}(1, 2, 3)}. \quad (8.23)$$

Then only the numerator of the conjectured formula (6.11) remains to be checked. Inserting all previously computed form factors and branching coefficients, it turns out the numerator is a second order polynomial in ℓ , such that the coefficients are composed of elementary functions multiplied by $G(\theta_{ij})$ transcendental terms¹⁶. It is easy to see that this "transcendental" structure is the same for both the conjecture (6.11) and the exact results: (4.16)-(4.18). The coefficients of these transcendental terms are complicated combinations of elementary functions containing the single Bethe-root λ_1 in the argument. These coefficients do not seem to match for the first sight, but exploiting the Bethe-equations finally it turns out that they agree. Thus upto 3-particle states in the sine-Gordon model, the Pálmai-Takács conjecture [10] gives the correct result for Bethe-Yang limit of the diagonal matrix elements of the $U(1)$ current and the trace of the stress energy tensor, provided one modifies the definition of polarized form-factors from the original form (6.7) to (6.8). This simple modification corresponds to a $\Psi \rightarrow \Psi^*$ exchange in the original definition of [10].

9 Comments on some subtle points of the conjecture of [10]

In the previous section for the operators Θ and J_μ , we checked the conjecture of [10] for the Bethe-Yang limit of expectation values of local operators upto 3-particle states. The agreement found between the conjectured and the exact results seems to be a convincing evidence for the correctness of this conjecture. Nevertheless, in this section we would like to pay the attention to some delicate points of the conjecture, which require further work to be confirmed. These two subtle points are the existence of the symmetric diagonal limit of form-factors and the order of rapidities in the Bethe-wave functions.

9.1 Existence of symmetric diagonal limit of form-factors in a non-diagonally scattering theory

In this section we argue, that the existence of symmetric diagonal limit of form-factors in a non-diagonally scattering theory is not obvious at all. Apparently, we cannot prove the existence of this limit in general, thus it cannot be excluded, that this limit is divergent in most of the cases.

¹⁶The same structure arose in the 2-particle case. See (8.17).

We start our argument by writing down the kinematical singularity axiom (5.7) in the limit, when only the rapidities of the sandwiching states are close to each other. Similarly to (7.39) we formulate the axiom on the basis of the Bethe-eigenvectors. Let Ψ and $\Phi^{(\epsilon)}$ be the color wave functions corresponding to the two states sandwiching the operator. Thus, Ψ is a left eigenvector of the soliton transfer-matrix (B.4) with rapidity parameters: $\vec{\theta} = \{\theta_1, \dots, \theta_n\}$ and $\Phi^{(\epsilon)*}$ is a right eigenvector of the soliton transfer-matrix with rapidity parameters¹⁷: $\vec{\theta}^\epsilon = \{\theta_1^\epsilon, \dots, \theta_n^\epsilon\}$. Then they satisfy the eigenvalue equations:

$$\begin{aligned} \Psi^{i_1 \dots i_n} \tau(\theta_1 | \vec{\theta})_{i_1 i_2 \dots i_n}^{l \alpha_2 \dots \alpha_n} &= \Lambda_\Psi(\theta_1 | \vec{\theta}) \Psi^{l \alpha_2 \dots \alpha_n}, \\ \tau(\theta_1^\epsilon | \vec{\theta}^\epsilon)_{l \bar{\beta}_2 \dots \bar{\beta}_n}^{j_1 j_2 \dots j_n} \Phi_{j_1 \dots j_n}^{(\epsilon)*} &= \Lambda_\Phi(\theta_1^\epsilon | \vec{\theta}^\epsilon) \Phi_{l \bar{\beta}_2 \dots \bar{\beta}_n}^{(\epsilon)*}. \end{aligned} \quad (9.1)$$

With these sandwiching states the kinematical singularity axiom takes the form:

$$\begin{aligned} F_{\Phi\Psi}(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) &= \frac{i}{\epsilon_1} \left(1 - \frac{\Lambda_\Psi(\theta_1 | \vec{\theta})}{\Lambda_\Phi(\theta_1^\epsilon | \vec{\theta}^\epsilon)} \right) \Phi_{k \bar{\beta}_2 \dots \bar{\beta}_n}^{(\epsilon)*} \Psi^{k \alpha_2 \dots \alpha_n} F_{\beta_n \dots \beta_2 \alpha_2 \dots \alpha_n}(\hat{\theta}_n, \dots, \hat{\theta}_2, \theta_2, \dots, \theta_n) \\ &+ O(1)_{\epsilon_1} \end{aligned} \quad (9.2)$$

where in accordance with the definition (6.7), $F_{\Phi\Psi}$ denotes the form-factor polarized with the Bethe-vectors Φ and Ψ :

$$\begin{aligned} F_{\Phi\Psi}(\hat{\theta}_n, \dots, \hat{\theta}_1, \theta_1, \dots, \theta_n) &= \sum_{b_1, \dots, b_n = \pm} \sum_{a_1, \dots, a_n = \pm} \Phi_{b_1 \dots b_n}^{(\epsilon)*}(\theta_1^\epsilon, \dots, \theta_n^\epsilon) \times \\ &F_{\bar{b}_n \dots \bar{b}_1 a_1 \dots a_n}(\theta_n^\epsilon + i\pi, \dots, \theta_1^\epsilon + i\pi, \theta_1, \dots, \theta_n) \Psi_{a_1 \dots a_n}(\theta_1, \dots, \theta_n). \end{aligned} \quad (9.3)$$

Again, we used the short notation: $\hat{\theta}_j = \theta_j + \epsilon_j + i\pi$. Formula (9.2) has serious implications on the existence of the symmetric diagonal limit of form-factors in a non-diagonally scattering theory.

If the theory is of purely elastic scattering, than there is no index structure in (9.2). Thus $\Phi = \Psi = 1$ and $\Lambda_\Phi(\theta_1^\epsilon | \vec{\theta}^\epsilon) \rightarrow \Lambda_\Psi(\theta_1 | \vec{\theta})$ can be written. In this case the prefactor $\frac{i}{\epsilon_1} \left(1 - \frac{\Lambda_\Psi(\theta_1 | \vec{\theta})}{\Lambda_\Psi(\theta_1^\epsilon | \vec{\theta}^\epsilon)} \right)$ becomes $O(1)$ in ϵ , which implies that the symmetric diagonal limit always exist (finite). Moreover the exchange axiom (5.5) ensures, that the limiting form-factor is a symmetric function of the rapidities.

If the theory is of non-diagonally scattering the prefactor $\frac{i}{\epsilon_1} \left(1 - \frac{\Lambda_\Psi(\theta_1 | \vec{\theta})}{\Lambda_\Phi(\theta_1^\epsilon | \vec{\theta}^\epsilon)} \right)$ in (9.2) is not always $O(1)$ in ϵ ! Moreover it is always divergent if the vectors Ψ and $\Phi^{(\epsilon)}|_{\epsilon=0}$ are not equal. This means, that the near diagonal in rapidity limit of the form-factors is divergent if the matrix element is nondiagonal in the color space. On the other hand, the prefactor $\frac{i}{\epsilon_1} \left(1 - \frac{\Lambda_\Psi(\theta_1 | \vec{\theta})}{\Lambda_\Phi(\theta_1^\epsilon | \vec{\theta}^\epsilon)} \right)$ in (9.2) has a finite value in $\epsilon \rightarrow 0$ limit, if¹⁸ both sandwiching states correspond to the same eigenstate of the soliton transfer-matrix. Nevertheless, this fact

¹⁷We just recall the notation used in the preceding sections: $\theta_j^\epsilon = \theta_j + \epsilon_j$, $j = 1, \dots, n$.

¹⁸In the $\epsilon \rightarrow 0$ limit.

alone does not guarantee, that the symmetric diagonal limit of the form-factors would be finite. This is so, because apart from the prefactor we analyzed, there is another term in (9.2), a sum of the near diagonal in rapidity limit of form-factors with all polarizations, weighted by the color wave-functions. We argued in the previous lines, that the near diagonal in rapidity limit of form-factors is divergent in general, which means that this sum is composed of divergent terms in the $\epsilon \rightarrow 0$ limit. Actually, the degree of divergence in ϵ increases with the number of sandwiching particles. To get finite result for these matrix elements very nontrivial cancellations must occur! Actually such nontrivial cancellations happened, when we computed the symmetric diagonal limit of the 3-particle form-factors of the current in section 7.

Nevertheless, our conclusion is that the symmetric diagonal limit of form-factors in a non-diagonally scattering theory is not obviously finite. Thus, to trust the conjecture of [10] beyond 3-particle states, it would be necessary to prove that the symmetric diagonal limit of form-factors exists for generic states, as well.

9.2 The order of rapidities

The next delicate point in the conjecture of [10] is the matter of the order of rapidities. Here we will not state, that there might be problems with the conjecture, but rather we would like to shed light on the fact, that the so far achieved analytical tests of this paper are still not enough to confirm certain parts of the conjecture. This unconfirmed part is how to do correctly the color-wave function decomposition (6.13). The issue here is that in general these wave functions do depend on the particle's rapidities. What's more they do depend on their orderings, as well. In this paper we did computations upto 3-particle matrix elements. Thus 3-particle wave functions must have been decomposed with respect to 1- and 2-particle color wave functions. But, as it is emphasized in section 8, incidentally the 1- and 2-particle color wave functions are independent of the rapidities. Consequently, our computations cannot confirm, whether the ordering of rapidities in the arguments of the wave-functions in the right hand side of the decomposition formula (6.13) is correct if more than three particle states are considered.

A possible reassuring solution to this problem could be, if one could prove that the conjectured formula of [10] is invariant under any permutations of the rapidities of the sandwiching state.

10 Summary and conclusion

In this paper we consider two important local operators of the sine-Gordon theory; the trace of the stress energy tensor and the $U(1)$ current.

We showed, that the finite volume expectation values of these operators in any eigenstate of the Hamiltonian of the model, can be expressed in terms of solutions of sets of linear integral equations (A.1)-(A.11). The large volume solution of these equations allowed us to get analytical formulas in the repulsive regime for the Bethe-Yang limit of these diagonal matrix elements. These formulas are expressed in terms of the Bethe-roots

characterizing the corresponding eigenstate of the soliton transfer matrix (B.4). This analytical formula allowed us to check a former conjecture [10] for the Bethe-Yang limit of expectation values of local operators in a non-diagonally scattering theory. We computed all expectation values upto 3-particle states both from our analytical formulas and from the conjectured formula of [10], and we found perfect agreement between the results of the two different computations. To be more precise to get agreement we had to make a tiny modification in the conjectured formula of [10]. Namely, we had to change slightly the definition of the symmetric diagonal form-factors, which are basic building blocks of the formula. In the conjecture of [10] they are defined as appropriately (6.7) polarized sandwiches of the form-factors with right eigenvectors of the soliton transfer matrix. However, from our computations it turns out that they should be defined as polarized sandwiches of the form-factors with left eigenvectors of the soliton transfer matrix. Since upto 2-particle states the left and right eigenvectors are the same, this issue arises first at the level of 3-particle states, which were not tested in the original paper [10].

Despite the success of the 3-particle checks, there are still some subtle points of the conjecture, which could not be confirmed by our analytical computations. First of all, the finiteness of the symmetric diagonal limit of form-factors for a generic state in a non-diagonally scattering theory is still unproven. Second, the hereby performed analytical tests were still not sensible to some details of the conjectured formula. Namely, upto 3-particle states our computations could not check the correctness of the rapidity dependence of the eigenvectors entering the right hand side of the decomposition rule (6.13), since all 1- and 2-particle wave-functions are incidentally independent of the rapidities.

Thus our final conclusions are as follows. Our analytical checks gave very strong support for the validity of the conjectured formula of [10] for the Bethe-Yang limit of expectation values in non-diagonally scattering theories. Our computations suggest, that the conjecture is well established upto 3-particle states, but to firmly trust it beyond 3-particle states, two further statements should be proven. First, it should be proven that the symmetric diagonal limit of form-factors is finite in a non-diagonally scattering theory, as well. Second, to get some more confidence about whether the rapidity dependence of wave functions is correctly embedded into the conjectured formula, one should also prove that the conjectured formula is invariant with respect to the permutations of the rapidities of the sandwiching state.

Nevertheless, the fact that the conjecture of [10] was found to be correct at least upto 3-particle states, opens the door to safely apply it to compute finite temperature correlators [45], and various one-point functions [46, 47, 48, 49], by their form-factor series representations upto 3-particle contributions.

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A Integral equations for the derivatives of the counting-function

In this appendix we write down the linear integral equations satisfied by the θ - and ℓ -derivatives of the counting-function. The equations we list below can be obtained by differentiating the NLIE (3.2)-(3.15). The equations related to the derivative of $Z(\theta|\ell)$ with respect to θ (3.19) and ℓ (3.20) can be written in an incorporated way, because the equations for the two different derivatives differ only in a single source term. To have a more compact representation of the equations it is useful to pack all complex roots into a single set:

$$\{u_j\}_{j=1}^{m_K} = \{c_j\}_{j=1}^{m_C} \cup \{w_k\}_{k=1}^{m_W}, \quad m_K = m_C + m_W, \quad (\text{A.1})$$

such that

$$\begin{aligned} u_j &= c_j, & j &= 1, \dots, m_C, \\ u_{m_C+j} &= w_j, & j &= 1, \dots, m_W. \end{aligned} \quad (\text{A.2})$$

In accordance with (A.2), from (3.19) and (3.20) we define the corresponding X variables as well:

$$\begin{aligned} X_{\nu,j}^{(u)} &= X_j^{(c)}, & \nu &\in \{d, \ell\}, \quad j = 1, \dots, m_C, \\ X_{\nu, m_C+j}^{(u)} &= X_j^{(w)}, & \nu &\in \{d, \ell\}, \quad j = 1, \dots, m_W. \end{aligned} \quad (\text{A.3})$$

Using this notation the linear integral equations take the form:

$$\begin{aligned} \mathcal{G}_\nu(\theta) &= f_\nu(\theta) + \sum_{j=1}^{m_H} \mathbf{G}(\theta, h_j) X_{\nu,j}^{(h)} - \sum_{j=1}^{m_S} (\mathbf{G}(\theta, y_j - i\eta) + \mathbf{G}(\theta, y_j + i\eta)) X_{\nu,j}^{(y)} - \\ &\quad \sum_{j=1}^{m_K} \mathbf{G}(\theta, u_j) X_{\nu,j}^{(u)} + \sum_{\alpha=\pm_\infty} \int \frac{d\theta'}{2\pi} \mathbf{G}(\theta, \theta' - i\alpha\eta) \mathcal{G}_\nu(\theta' + i\alpha\eta) \mathcal{F}_\alpha(\theta' + i\alpha\eta), \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \sum_{k=1}^{m_H} [Z'(h_j) \delta_{jk} - \mathbf{G}(h_j, h_k)] X_{\nu,k}^{(h)} &= f_\nu(h_j) - \sum_{k=1}^{m_S} (\mathbf{G}(h_j, y_k + i\eta) + \mathbf{G}(h_j, y_k - i\eta)) X_{\nu,k}^{(y)} - \\ \sum_{k=1}^{m_K} \mathbf{G}(h_j, u_k) X_{\nu,k}^{(u)} &+ \sum_{\alpha=\pm_\infty} \int \frac{d\theta'}{2\pi} \mathbf{G}(h_j, \theta' - i\alpha\eta) \mathcal{G}_\nu(\theta' + i\alpha\eta) \mathcal{F}_\alpha(\theta' + i\alpha\eta), \quad j=1, \dots, m_H, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \sum_{k=1}^{m_K} [Z'(u_j) \delta_{jk} + \mathbf{G}(u_j, u_k)] X_{\nu,k}^{(u)} &= f_\nu(u_j) - \sum_{k=1}^{m_S} (\mathbf{G}(u_j, y_k + i\eta) + \mathbf{G}(u_j, y_k - i\eta)) X_{\nu,k}^{(y)} + \\ \sum_{k=1}^{m_H} \mathbf{G}(u_j, h_k) X_{\nu,k}^{(h)} &+ \sum_{\alpha=\pm_\infty} \int \frac{d\theta'}{2\pi} \mathbf{G}(u_j, \theta' - i\alpha\eta) \mathcal{G}_\nu(\theta' + i\alpha\eta) \mathcal{F}_\alpha(\theta' + i\alpha\eta), \quad j=1, \dots, m_K, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned}
& \sum_{k=1}^{m_S} [Z'(y_j) \delta_{jk} + \mathbf{G}(y_j, y_k + i\eta) + \mathbf{G}(y_j, y_k - i\eta)] X_{\nu,k}^{(y)} = f_\nu(y_j) + \sum_{k=1}^{m_H} \mathbf{G}(y_j, h_k) X_{\nu,k}^{(h)} - \\
& \sum_{k=1}^{m_K} \mathbf{G}(y_j, u_k) X_{\nu,k}^{(u)} + \sum_{\alpha=\pm\infty}^{\infty} \int \frac{d\theta'}{2\pi} \mathbf{G}(y_j, \theta' - i\alpha\eta) \mathcal{G}_\nu(\theta' + i\alpha\eta) \mathcal{F}_\alpha(\theta' + i\alpha\eta), \quad j=1, \dots, m_S,
\end{aligned} \tag{A.7}$$

where η is a positive contour deformation parameter¹⁹ such that $\eta < \min(p, p\pi, |\operatorname{Im} u_j|)$, $\mathcal{F}_\pm(\theta)$ is defined in (3.24), the index ν can be either d or ℓ telling us which derivative of $Z(\theta)$ is considered. The source term $f_\nu(\theta)$ for the two choices of the index ν is given by the formulas:

$$f_d(\theta) = \begin{cases} \ell \cosh(\theta), & |\operatorname{Im}\theta| \leq \min(\pi, p\pi), \\ \ell \cosh_{II}(\theta) & \min(\pi, p\pi) < |\operatorname{Im}\theta| \leq \frac{\pi}{2}(1+p), \end{cases} \tag{A.8}$$

$$f_\ell(\theta) = \begin{cases} \sinh(\theta), & |\operatorname{Im}\theta| \leq \min(\pi, p\pi), \\ \sinh_{II}(\theta) & \min(\pi, p\pi) < |\operatorname{Im}\theta| \leq \frac{\pi}{2}(1+p), \end{cases} \tag{A.9}$$

where the second determination of a function is defined by (3.5). The function $\mathbf{G}(\theta, \theta')$ in the equations (A.4)-(A.7) agrees with $G(\theta - \theta')$ of (3.4) in the fundamental domain and it is equal to the appropriate second determination of $G(\theta)$ if either of its arguments goes out of the fundamental domain $|\operatorname{Im}(\theta - \theta')| \leq \min(\pi, p\pi)$. In the sequel we give the precise prescription, how to compute $\mathbf{G}(\theta, \theta')$ for any pair of values of its arguments. In this way we can get rid of the possible errors which can be easily committed when multiple second determination of a function should be done. The function $\mathbf{G}(\theta, \theta')$ will be defined as the solution of a linear integral equation. Let:

$$K(\theta) = \frac{1}{p+1} \frac{\sin \frac{2\pi}{p+1}}{\sinh \frac{\theta-i\pi}{p+1} \sinh \frac{\theta+i\pi}{p+1}}. \tag{A.10}$$

This function is the derivative of the scattering-phase of the elementary magnon excitations of the 6-vertex model with anisotropy parameter $\gamma = \frac{\pi}{p+1}$. Then $\mathbf{G}(\theta, \theta')$ for arbitrary values of θ and θ' can be determined by solving the linear integral equation as follows:

$$\mathbf{G}(\theta, \theta') + \int_{-\infty}^{\infty} \frac{d\theta''}{2\pi} K(\theta - \theta'') \mathbf{G}(\theta'', \theta') = K(\theta - \theta'). \tag{A.11}$$

This equation can be solved by means of Fourier transformation along any horizontal lines of the complex plane. When both arguments are in the fundamental domain: $\max\{|\operatorname{Im}(\theta)|, |\operatorname{Im}(\theta')|\} \leq \min(\pi, p\pi)$, then the solution of (A.11) gives the well known kernel of the NLIE of the sine-Gordon theory. Namely, $\mathbf{G}(\theta, \theta') = G(\theta - \theta')$ with $G(\theta)$ given in (3.4). The linear integral equation (A.11) tells us how to continue analytically $\mathbf{G}(\theta, \theta')$ out of this fundamental regime. For example, if one continues one of the variables

¹⁹If $m_S \neq 0$ it is preferable to consider η to be a positive infinitesimal parameter.

of $\mathbf{G}(\theta, \theta')$ out of the fundamental domain, then one gets the second determination of $G(\theta - \theta')$ defined by (3.5) etc. Thus the function $\mathbf{G}(\theta, \theta')$ incorporates all possible second determinations which appear in the NLIE (3.2)-(3.15) of the model. This means that one does not need to take care of the subtle rules of second determination, but the solution of (A.11) will automatically give the functional form of \mathbf{G} in any regime of the complex plane.

B Algebraic Bethe Ansatz for the soliton transfer matrix

The monodromy and transfer matrices made out of the S-matrix (2.5) of the sine-Gordon model are of central importance in this paper. They enter the form-factor axiom (5.7) and play an important role in the conjecture of [10] for the diagonal matrix elements of local operators of the theory.

In this appendix we summarize the most important properties of the monodromy matrix and recall the Algebraic Bethe Ansatz [43] diagonalization of the transfer matrix.

The basic object is the n -particle monodromy matrix built from the S-matrix of the model (2.5):

$$\mathcal{T}_a^b(\theta|\theta_1, \dots, \theta_n)_{a_1 a_2 \dots a_n}^{b_1 b_2 \dots b_n} = \mathcal{S}_{a a_1}^{k_1 b_1}(\theta - \theta_1) \mathcal{S}_{k_1 a_2}^{k_2 b_2}(\theta - \theta_2) \dots \mathcal{S}_{k_{n-1} a_n}^{b_{n-1} b_n}(\theta - \theta_n). \quad (\text{B.1})$$

For the algebraic Bethe Ansatz techniques, it is generally written as a 2 by 2 matrix in the auxiliary space:

$$\mathcal{T}(\theta|\vec{\theta}) = \begin{pmatrix} \mathcal{T}_-^-(\theta|\vec{\theta}) & \mathcal{T}_-^+(\theta|\vec{\theta}) \\ \mathcal{T}_+^-(\theta|\vec{\theta}) & \mathcal{T}_+^+(\theta|\vec{\theta}) \end{pmatrix} = \begin{pmatrix} A(\theta|\vec{\theta}) & B(\theta|\vec{\theta}) \\ C(\theta|\vec{\theta}) & D(\theta|\vec{\theta}) \end{pmatrix}, \quad (\text{B.2})$$

such that the entries act on the 2^n dimensional vector space spanned by n soliton doublets $\mathcal{V}_n = (\mathbb{C}^2)^{\otimes n}$. Here for short we introduced the notation $\vec{\theta} = \{\theta_1, \theta_2, \dots, \theta_n\}$.

As a consequence of the Yang-Baxter equation (2.10), the entries of the monodromy matrix satisfy the Yang-Baxter algebra relations:

$$\mathcal{S}_{a_1 a_2}^{k_1 k_2}(\theta - \theta') \mathcal{T}_{k_1}^{b_1}(\theta|\vec{\theta}) \mathcal{T}_{k_2}^{b_2}(\theta'|\vec{\theta}) = \mathcal{T}_{a_2}^{k_1}(\theta'|\vec{\theta}) \mathcal{T}_{a_1}^{k_2}(\theta|\vec{\theta}) \mathcal{S}_{k_1 k_2}^{b_2 b_1}(\theta - \theta'). \quad (\text{B.3})$$

The transfer matrix is defined as the trace of the monodromy matrix over the auxiliary space:

$$\tau(\theta|\vec{\theta}) = \sum_{a=\pm} \mathcal{T}_a^a(\theta|\vec{\theta}). \quad (\text{B.4})$$

As a consequence of (B.3) the transfer matrices form a commuting family of operators on \mathcal{V}_n :

$$\tau(\theta|\vec{\theta}) \tau(\theta'|\vec{\theta}) = \tau(\theta'|\vec{\theta}) \tau(\theta|\vec{\theta}). \quad (\text{B.5})$$

This means that the eigenvectors of the transfer matrices are independent of the spectral parameter θ , but they do depend on the inhomogeneity vector $\vec{\theta}$, such that the order

of rapidities within this vector matters, as well! The transfer matrix commutes with the solitonic charge \mathcal{Q} and the charge parity \mathcal{C} operators, which act on a vector $V \in \mathcal{V}_n$ as follows:

$$(\mathcal{Q}V)_{i_1 i_2 \dots i_n} = \mathcal{Q} V_{i_1 i_2 \dots i_n} \quad \mathcal{Q} = \sum_{k=1}^n i_k, \quad (\text{B.6})$$

$$(\mathcal{C}V)_{i_1 i_2 \dots i_n} = V_{\bar{i}_1 \bar{i}_2 \dots \bar{i}_n}, \quad \text{with} \quad \bar{i}_k = -i_k, \quad k = 1, \dots, n. \quad (\text{B.7})$$

The $B(\theta|\vec{\theta})$ and $C(\theta|\vec{\theta})$ elements of the monodromy matrix act as charge raising and lowering operators:

$$[\mathcal{Q}, B(\theta|\vec{\theta})] = 2 B(\theta|\vec{\theta}), \quad (\text{B.8})$$

$$[\mathcal{Q}, C(\theta|\vec{\theta})] = -2 C(\theta|\vec{\theta}). \quad (\text{B.9})$$

The diagonalization of the transfer matrix can be done using the usual procedure of the Algebraic Bethe Ansatz [43]. There exist a trivial eigenvector of $\tau(\lambda|\vec{\theta})$, the pure antisoliton state:

$$|0\rangle_{a_1 a_2 \dots a_n} = \prod_{j=1}^n \delta_{a_j}^-. \quad (\text{B.10})$$

Then the eigenvectors of the transfer matrix are given by acting a sequence of B -operators on this trivial eigenstate:

$$\Psi(\{\lambda_j\}|\vec{\theta}) = \frac{1}{\mathcal{N}_\Psi} B(\lambda_1|\vec{\theta}) B(\lambda_2|\vec{\theta}) \dots B(\lambda_r|\vec{\theta}) |0\rangle, \quad (\text{B.11})$$

such that the λ_j spectral parameters of the B -operators satisfy the Bethe-equations as follows:

$$\prod_{k=1}^n B_0(\lambda_j - \theta_k) = \prod_{k \neq j}^r \frac{B_0(\lambda_k - \lambda_j)}{B_0(\lambda_j - \lambda_k)}, \quad j = 1, \dots, r. \quad (\text{B.12})$$

The term \mathcal{N}_Ψ in (B.11) is to fix the norm of the state to the required value. In our computations the normalization condition for \mathcal{N}_Ψ is that the norm of Ψ should be 1. The eigenvalue of the transfer matrix on the state $\Psi(\{\lambda_j\}|\vec{\theta})$;

$$\tau(\lambda|\vec{\theta}) \Psi(\{\lambda_j\}|\vec{\theta}) = \Lambda(\lambda, \{\lambda_j\}|\vec{\theta}) \Psi(\{\lambda_j\}|\vec{\theta}), \quad (\text{B.13})$$

is given by the formula:

$$\Lambda(\lambda, \{\lambda_j\}|\vec{\theta}) = \prod_{j=1}^n S_0(\lambda - \theta_k) \Lambda_0(\lambda, \{\lambda_j\}|\vec{\theta}), \quad (\text{B.14})$$

where $S_0(\theta)$ is given in (2.6) and $\Lambda_0(\lambda, \{\lambda_j\}|\vec{\theta})$ is the eigenvalue of the transfer matrix made out of $S_{ab}^{cd}(\theta)$ and it is given by:

$$\Lambda_0(\lambda, \{\lambda_j\}|\vec{\theta}) = \prod_{j=1}^r \frac{1}{B_0(\lambda_j - \lambda)} + \prod_{k=1}^n B_0(\lambda - \theta_k) \prod_{j=1}^r \frac{1}{B_0(\lambda - \lambda_j)}. \quad (\text{B.15})$$

In the computations of the paper we need an analogous to (B.11) expression for the complex conjugate vector of $\Psi(\{\lambda_j\}|\vec{\theta})$, too. For this reason we need the properties of the monodromy and transfer matrices under hermitian conjugation. From the properties (2.11)-(2.15) of the S-matrix and from the definition (B.1) one can prove the following hermitian conjugation rule for the monodromy matrix:

$$\mathcal{T}_a^b(\lambda|\vec{\theta})^\dagger = \mathcal{T}_a^{\bar{b}}(\lambda^* + i\pi|\vec{\theta}), \quad (\text{B.16})$$

which implies for the components the following rules:

$$\begin{aligned} A^\dagger(\lambda|\vec{\theta}) &= D(\lambda^* + i\pi|\vec{\theta}), & D^\dagger(\lambda|\vec{\theta}) &= A(\lambda^* + i\pi|\vec{\theta}), \\ B^\dagger(\lambda|\vec{\theta}) &= C(\lambda^* + i\pi|\vec{\theta}), & C^\dagger(\lambda|\vec{\theta}) &= B(\lambda^* + i\pi|\vec{\theta}). \end{aligned} \quad (\text{B.17})$$

It follows for the transfer matrix that:

$$\tau^\dagger(\lambda|\vec{\theta}) = \tau(\lambda^* + i\pi|\vec{\theta}). \quad (\text{B.18})$$

Thus, the transfer matrix is a hermitian operator along the line: $\lambda = \rho + i\frac{\pi}{2}$, with $\rho \in \mathbb{R}$. The hermitian conjugation relations (B.17) imply that the complex conjugate vector $\Psi^*(\{\lambda_j\}|\vec{\theta})$ can be represented as follows:

$$\Psi^*(\{\lambda_j\}|\vec{\theta}) = \langle 0|C(\lambda_1^* + i\pi|\vec{\theta})C(\lambda_2^* + i\pi|\vec{\theta})\dots C(\lambda_r^* + i\pi|\vec{\theta})\frac{1}{\mathcal{N}_\Psi}. \quad (\text{B.19})$$

It can be seen that if a set $\{\lambda_j\}_{j=1}^r$ is a solution of the Bethe-equations (B.12), then the set $\{\lambda_j^* + i\pi\}_{j=1}^r$ is also a solution of (B.12). Thus for solutions which are invariant under this transformation the complex conjugate vector can be written in a simpler form:

$$\Psi^*(\{\lambda_j\}|\vec{\theta}) = \langle 0|C(\lambda_1|\vec{\theta})C(\lambda_2|\vec{\theta})\dots C(\lambda_r|\vec{\theta})\frac{1}{\mathcal{N}_\Psi}. \quad (\text{B.20})$$

Now it is easy to determine the normalization constant \mathcal{N}_Ψ , because it is nothing but the Gaudin-norm [36, 37, 38] of the Bethe-state $B(\lambda_1|\vec{\theta})B(\lambda_2|\vec{\theta})\dots B(\lambda_r|\vec{\theta})|0\rangle$:

$$\mathcal{N}_\Psi^2 = \langle 0|C(\lambda_1|\vec{\theta})C(\lambda_2|\vec{\theta})\dots C(\lambda_n|\vec{\theta})B(\lambda_1|\vec{\theta})B(\lambda_2|\vec{\theta})\dots B(\lambda_r|\vec{\theta})|0\rangle. \quad (\text{B.21})$$

If one would like to apply the Algebraic Bethe Ansatz technique directly to $\tau(\lambda|\vec{\theta})$, one should carry unnecessarily a lot of $S_0(\theta)$ factors. This can be avoided, if one diagonalizes the transfer matrix constructed out of the S_0 removed part of the S-matrix. To be more

concrete analogously to (B.1) one should define the "reduced" monodromy matrix by the formula:

$$T_a^b(\theta|\theta_1, \dots, \theta_n)_{a_1 a_2 \dots a_n}^{b_1 b_2 \dots b_n} = S_{a a_1}^{k_1 b_1}(\theta - \theta_1) S_{k_1 a_2}^{k_2 b_2}(\theta - \theta_2) \dots S_{k_{n-1} a_n}^{b_{n-1} b_n}(\theta - \theta_n), \quad (\text{B.22})$$

where $S_{ab}^{cd}(\theta)$ is the matrix part of the S-matrix (2.5) given by (2.7)-(2.9). Analogously to (B.2) it can be written as a 2 by 2 matrix in the auxiliary space:

$$T(\lambda|\vec{\theta}) = \begin{pmatrix} T_-^-(\lambda|\vec{\theta}) & T_-^+(\lambda|\vec{\theta}) \\ T_+^-(\lambda|\vec{\theta}) & T_+^+(\lambda|\vec{\theta}) \end{pmatrix} = \begin{pmatrix} \mathcal{A}(\lambda|\vec{\theta}) & \mathcal{B}(\lambda|\vec{\theta}) \\ \mathcal{C}(\lambda|\vec{\theta}) & \mathcal{D}(\lambda|\vec{\theta}) \end{pmatrix}. \quad (\text{B.23})$$

Its matrix elements satisfy the same Yang-Baxter algebra (B.3) as those of $\mathcal{T}(\lambda|\vec{\theta})$. The "reduced" transfer matrix $t(\lambda|\vec{\theta})$ is defined by taking the trace in the auxiliary space:

$$t(\lambda|\vec{\theta}) = \sum_{a=\pm} T_a^a(\lambda|\vec{\theta}). \quad (\text{B.24})$$

It differs from $\tau(\lambda|\vec{\theta})$ in only a trivial scalar factor:

$$\tau(\lambda|\vec{\theta}) = \prod_{k=1}^n S_0(\lambda - \theta_k) t(\lambda|\vec{\theta}). \quad (\text{B.25})$$

Thus their common eigenvector $\Psi(\lambda, \{\lambda_j\}|\vec{\theta})$ (B.11) and its complex conjugate (B.20) can be expressed in terms of the elements of the "reduced" monodromy matrix completely analogously to the formulas (B.11) and (B.20):

$$\Psi(\{\lambda_j\}|\vec{\theta}) = \frac{1}{N_\Psi} \mathcal{B}(\lambda_1|\vec{\theta}) \mathcal{B}(\lambda_2|\vec{\theta}) \dots \mathcal{B}(\lambda_r|\vec{\theta})|0\rangle, \quad (\text{B.26})$$

$$\Psi^*(\{\lambda_j\}|\vec{\theta}) = \langle 0|\mathcal{C}(\lambda_1|\vec{\theta}) \mathcal{C}(\lambda_2|\vec{\theta}) \dots \mathcal{C}(\lambda_n|\vec{\theta}) \frac{1}{N_\Psi}. \quad (\text{B.27})$$

Certainly the normalization factor is also changed compared to (B.11) and (B.20):

$$N_\Psi^2 = \langle 0|\mathcal{C}(\lambda_1|\vec{\theta}) \mathcal{C}(\lambda_2|\vec{\theta}) \dots \mathcal{C}(\lambda_n|\vec{\theta}) \mathcal{B}(\lambda_1|\vec{\theta}) \mathcal{B}(\lambda_2|\vec{\theta}) \dots \mathcal{B}(\lambda_r|\vec{\theta})|0\rangle, \quad (\text{B.28})$$

which can be written as a Slavnov-determinant [42]. The eigenvalue of $t(\lambda|\vec{\theta})$ on $\Psi(\{\lambda_j\}|\vec{\theta})$ is exactly $\Lambda_0(\lambda, \{\lambda_j\}|\vec{\theta})$ given in (B.15).

We continue this appendix by specializing the main formulas of this appendix to the 3-particle case.

B.1 Formulas for the 3-particle case

Due to the charge conjugation symmetry, in the 3-particle case the number of Bethe-roots in (B.12) can be either zero or one. The zero root case corresponds to the trivial pure solitonic eigenvector (B.10). Here we do not deal with this trivial case, but we are

interested in the state described by a single Bethe-root. In this case the Bethe-equations take the simple form:

$$\prod_{j=1}^3 B_0(\lambda_1 - \theta_j) = 1. \quad (\text{B.29})$$

The eigenvalue of the soliton transfer matrix (B.14), when its spectral parameter takes the value of one of the rapidities, is given²⁰ by:

$$\Lambda(\theta_j|\vec{\theta}) = \prod_{k=1}^3 S_0(\theta_j - \theta_k) \frac{1}{B_0(\lambda_1 - \theta_j)}, \quad j = 1, 2, 3. \quad (\text{B.30})$$

In this one-root case the normalization factor N_Ψ in (B.28) takes the form:

$$N_\Psi^2 = \langle 0|\mathcal{C}(\lambda_1|\vec{\theta}) \mathcal{B}(\lambda_1|\vec{\theta})|0\rangle = p \sinh\left(\frac{i\pi}{p}\right) \sum_{j=1}^3 (\ln B_0)'(\lambda_1 - \theta_j). \quad (\text{B.31})$$

In the computation of the symmetric form-factors some derivatives with respect to the particle's rapidities will be important. Differentiating (B.30) with respect to θ_q one obtains:

$$\partial_q \log \Lambda(\theta_s|\vec{\theta}) = -i G(\theta_s - \theta_q) - (\ln B_0)'(\lambda_1 - \theta_s) \frac{\partial \lambda_1}{\partial \theta_q}, \quad s \neq q, \quad (\text{B.32})$$

with $G(\theta)$ given in (3.4). The derivative $\frac{\partial \lambda_1}{\partial \theta_q}$ can be obtained by differentiating the Bethe-equation (B.29):

$$\frac{\partial \lambda_1}{\partial \theta_q} = \frac{(\ln B_0)'(\lambda_1 - \theta_q)}{\sum_{k=1}^3 (\ln B_0)'(\lambda_1 - \theta_k)}, \quad q = 1, 2, 3. \quad (\text{B.33})$$

If we have one single root then due to the $\lambda \rightarrow \lambda^* + i\pi$ symmetry of the Bethe-equation the single root of the equation can be either "real" or "self-conjugate". A solution λ_1 is called real, if it is a fixed point of the symmetry $\lambda \rightarrow \lambda^* + i\pi$, i.e. $\lambda_1 = \lambda_1^* + i\pi$. Here we use the term "real", because in a more convenient parameterization this type of roots would be actually a real numbers. Namely, if it is parameterized as $\lambda_1 = \rho_1 + i\frac{\pi}{2}$ then the fix point equation restricts ρ_1 to be real.

Due to the $ip\pi$ symmetry of the functions entering the the Bethe-equations (B.12), they have another symmetry, as well. Namely if λ_j is a solution of the equations then $\lambda_j + i\pi p$ is also a solution. This means that the solutions can be restricted to a fundamental domain given by the strip of width $ip\pi$. By definition a "self-conjugate" root satisfies the combination of symmetries: $\lambda \rightarrow \lambda^* + i\pi$ and $\lambda \rightarrow \lambda \pm ip\pi$, namely

$$\lambda_1 = \lambda_1^* + i\pi \pm ip\pi. \quad (\text{B.34})$$

If it is parameterized again as $\lambda_1 = \rho_1 + i\frac{\pi}{2}$, then ρ_1 has a fixed imaginary part: $\text{Im}\rho_1 = \frac{p\pi}{2}$.

The numerical solution of the equation (B.29) shows that in the repulsive regime ($1 < p$) from the 3 different solutions, two ones are self-conjugated and one is real.

²⁰Since here we have only one Bethe-root, for short we skipped it from the list of arguments of the eigenvalue.

B.2 Classification of the magnonic Bethe-roots

As the simple discussion at the end of the previous subsection shows, there are two symmetries of the magnonic Bethe-equations (B.12):

$$\begin{aligned} \bullet \quad & \{\lambda_j\}_{j=1}^r = \{\lambda_j^* + i\pi\}_{j=1}^r, \\ \bullet \quad & \{\lambda_j\}_{j=1}^r = \{\lambda_j + ip\pi\}_{j=1}^r. \end{aligned} \tag{B.35}$$

They imply the following classification of the roots.

$$\begin{aligned} \bullet \quad \text{Real-roots:} \quad & \text{Im}(\lambda_j - i\frac{\pi}{2}) = 0, & j = 1, \dots, n_r, \\ \bullet \quad \text{Close-roots:} \quad & |\text{Im}(\lambda_j - i\frac{\pi}{2})| \leq \min(\frac{\pi}{2}, \frac{(2p-1)\pi}{2}), & j = 1, \dots, n_c, \\ \bullet \quad \text{Wide-roots:} \quad & \min(\frac{\pi}{2}, \frac{(2p-1)\pi}{2}) < |\text{Im}(\lambda_j - i\frac{\pi}{2})| \leq \frac{p\pi}{2}, & j = 1, \dots, n_w. \end{aligned} \tag{B.36}$$

A special type of wide-root is the self-conjugate root, whose imaginary part is exactly $i\frac{(1+p)\pi}{2}$. From the symmetries (B.35) of the asymptotic Bethe-equations it also follows that all roots, which are neither real nor self-conjugate appear in pairs being symmetric to the line $\text{Im}z = \frac{\pi}{2}$. In this way we can speak about close-and wide-pairs similarly to the Bethe-roots entering the NLIE (3.2), which describes the exact finite volume spectrum of the sine-Gordon model.

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